We present some results allowing to solve convex quadratic programs by the partial entropic perturbation method applied to the objective functions. Standard quadratic programming and quadratic programming with entropic perturbation are obtained as special cases of this approach.

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1. INTRODUCTION

The entropic perturbation method developed by S.C. Fang, J.R. Rajasekera and H.-S.J. Tsao [4, 5, 6, 7, 8, 9, 10, 15, 16], is an efficient method for solving linear or quadratic programming problems. This method consists of adding an entropic perturbation of the form

$$\mu \sum_{j=1}^{n} x_j \ln x_j, \quad \mu > 0,$$

to the objective function, where $x_1, \ldots, x_n$ are the variables of the problem. Apparently, the problems obtained in this way are more complicated than the initial ones, however they have the advantage that they can be solved using the geometric entropic programming method, introduced by Erlander [3] to construct some dual problems. These dual problems are convex programming problems, usually without restrictions, and solving them (through one of the different known methods) leads to an $\varepsilon$-optimal solution of the initial linear or quadratic programming problem. Interesting algorithmic aspects can be found in [19].

In this paper we study the standard quadratic programming problem with partial entropic perturbation of the form

$$\sum_{j \in J} x_j \ln x_j, \quad \text{where } J \subseteq \{1, \ldots, n\},$$
that uses only part of the variables. We mention that the partial entropic perturbation method was introduced by Kas and Klafszky [11] for linear programming problems.

In Section 2 we introduce the primal quadratic programming problem with partial entropic perturbation and construct a geometric dual problem. In Section 3 we describe the duality between them using Fenchel’s Theorem from Convex Analysis [17]. As special cases of our results we recover duality from standard quadratic programming, from quadratic programming with entropic perturbation [7], and from linear programming with partial entropic perturbation [11], [1].

We would like to point out that the hypothesis for the partial entropic perturbation is weaker than the corresponding hypothesis for the total entropic perturbation of all variables, and this fact allows one to use the partial entropic perturbation method in situations in which the total perturbation cannot be applied. Besides these generalizations, the partial entropic perturbation occurs directly for estimation problems when only some variables are entropic.

2. THE CONSTRUCTION OF THE DUAL

Consider the standard quadratic programming problem

\[(P): \quad \min f(x) = \frac{1}{2} x^T D x + c^T x \quad \text{s.t.} \quad A x = b; \quad x \geq 0, \]

where \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) is the variable vector, \( n \in \mathbb{N}^* \), \( c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n \) and \( b = (b_1, \ldots, b_m)^T \in \mathbb{R}^m \) are known vectors, \( A = (a_{ij})_{i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}} \in \mathbb{R}^{m \times n} \) is a known matrix, \( m \in \mathbb{N}^* \), \( m \leq n \) and \( D \in \mathbb{R}^{n \times n} \) is a known diagonal matrix with diagonal entries \( d_i \geq 0 \), \( i \in \{1, \ldots, n\} \).

Let \( J \subseteq \{1, \ldots, n\} \) be a subset of subscripts and \( \mu > 0 \) an arbitrary scalar called the partial entropic perturbation factor. For \( \varepsilon \)-optimal solving of problem \((P)\) we consider the quadratic programming problem with partial entropic perturbation

\[(P(\mu)): \quad \min f_{\mu}(x) = \frac{1}{2} x^T D x + c^T x + \mu \sum_{j \in J} x_j \ln x_j \quad \text{s.t.} \quad A x = b; \quad x \geq 0, \]

where we agree that \( 0 \ln 0 = 0 \). Let \( \bar{J} = \{1, \ldots, n\} \setminus J \).

We will assume that problem \((P)\) has at least one feasible solution

\[(1) \quad x \in \{ u \in \mathbb{R}^n / A u = b, \ u \geq 0 \} \text{ such that } x_j > 0, \ \forall j \in J. \]
This assumption is called the $J$-interior point assumption.

Remark 2.1. Under the $J$-interior point assumption (1) the problem $(P(\mu))$ is consistent. Moreover,

$$\lim_{x_j \to 0} x_j \ln x_j = 0, \quad \lim_{x_j \to \infty} \left( \frac{1}{2} d_j x_j^2 + c_j x_j + \mu x_j \ln x_j \right) = +\infty$$

and the function $x_j \ln x_j$ is strictly decreasing on the interval $[0, \frac{1}{e}]$ and strictly convex for each $j \in J$, the problem $(P(\mu))$ has a finite optimum and every optimal solution $x^*(\mu)$ of this problem is $J$-interior, i.e., $x^*_j(\mu) > 0$, $\forall j \in J$.

Because of the entropic perturbation, problem $(P(\mu))$ seems more complicated than problem $(P)$. However, this perturbation allows to construct a dual convex programming problem with linear restrictions.

Let $j \in J$ be arbitrarily and fixed. For each positive function $\varphi_j(y)$, $y \in \mathbb{R}^m$, and for each $x_j > 0$, applying the logarithmic inequality

$$\ln \frac{\varphi_j(y)}{x_j} \leq \frac{\varphi_j(y)}{x_j} - 1,$$

with equality if and only if

(2) \hspace{1cm} \varphi_j(y) = x_j,

we deduce that

(3) \hspace{1cm} \mu \ln \frac{\varphi_j(y)}{x_j} - \frac{d_j [x_j - \varphi_j(y)]^2}{2 x_j} \leq \mu \frac{\varphi_j(y)}{x_j} - \mu,

with equality if and only if equation (2) holds.

Inequality (3) can be rewritten as

(4) \hspace{1cm} -\frac{d_j x_j}{2} - \frac{d_j \varphi_j^2(y)}{2 x_j} + \left[ d_j \varphi_j(y) + \mu \ln \varphi_j(y) \right] - \mu \ln x_j \leq \mu \frac{\varphi_j(y)}{x_j} - \mu.

As the function

$$F : (0, +\infty) \to \mathbb{R}, \hspace{1cm} F(t) = d_j t + \mu \ln t$$

is bijective, the equation

(5) \hspace{1cm} d_j \varphi_j(y) + \mu \ln \varphi_j(y) = \sum_{i=1}^{m} a_{ij} y_i - c_j - \mu

has a unique positive solution $\varphi_j(y)$ for every $y \in \mathbb{R}^m$ and $j \in J$. Now, inequality (4) becomes

$$-\frac{d_j x_j}{2} - \frac{d_j \varphi_j^2(y)}{2 x_j} + \sum_{i=1}^{m} a_{ij} y_i - c_j - \mu - \mu \ln x_j \leq \mu \frac{\varphi_j(y)}{x_j} - \mu.$$
Multiplying both members by $x_j > 0$ and summing after $j \in J$ yield
\[
\sum_{j \in J} \frac{1}{2} d_j x_j^2 + \sum_{j \in J} c_j x_j + \mu \sum_{j \in J} x_j \ln x_j \geq \sum_{j \in J} \frac{1}{2} d_j \varphi_j^2(y) - \mu \sum_{j \in J} \varphi_j(y) + \sum_{i=1}^m y_i \sum_{j \in J} a_{ij} x_j.
\]
Letting $x_j \downarrow 0$, we deduce that this inequality is true for every $x_j > 0$, $j \in J$.

For every $j \in \bar{J}$, let $z_j \geq 0$ such that
\[
\sum_{i=1}^m a_{ij} y_i - d_j z_j \leq c_j.
\]
Multiplying both members by $x_j \geq 0$ and summing after $j \in \bar{J}$ yield
\[
\sum_{j \in \bar{J}} \frac{1}{2} d_j x_j^2 + \sum_{j \in \bar{J}} c_j x_j \geq -\frac{1}{2} \sum_{j \in \bar{J}} d_j z_j^2 + \sum_{i=1}^m y_i \sum_{j \in \bar{J}} a_{ij} x_j.
\]
Using the obvious inequality $(z_j - x_j)^2 \geq 0$, with equality if and only if $z_j = x_j$, we deduce that
\[
\sum_{j \in \bar{J}} \frac{1}{2} d_j x_j^2 + \sum_{j \in \bar{J}} c_j x_j \geq -\frac{1}{2} \sum_{j \in \bar{J}} d_j z_j^2 + \sum_{i=1}^m y_i \sum_{j \in \bar{J}} a_{ij} x_j,
\]
for every $x_j \geq 0$, $j \in \bar{J}$.

Adding inequalities (6) and (7) yields
\[
\frac{1}{2} \sum_{j=1}^n d_j x_j^2 + \sum_{j=1}^n c_j x_j + \mu \sum_{j \in J} x_j \ln x_j \geq \frac{1}{2} \sum_{j=1}^n d_j \varphi_j^2(y) - \mu \sum_{j \in J} \varphi_j(y) + \sum_{i=1}^m y_i \sum_{j=1}^n a_{ij} x_j.
\]
If the vector $x = (x_1, \ldots, x_n)\top$ satisfies the restriction $Ax = b$, then inequality (8) can be rewritten as
\[
\frac{1}{2} x^\top Dx + c^\top x + \mu \sum_{j \in J} x_j \ln x_j \geq -\frac{1}{2} \sum_{j \in J} d_j \varphi_j^2(y) - \mu \sum_{j \in J} \varphi_j(y) + \sum_{i=1}^m b_i y_i - \frac{1}{2} \sum_{j \in \bar{J}} d_j z_j^2.
\]
Note that the left-hand side of this inequality is exactly the objective function $f_\mu(x)$ of problem $(P(\mu))$. In this way, we can define the geometric
dual problem of problem \((P(\mu))\), namely,

\[
(D(\mu)) : \quad \begin{align*}
\max g_\mu(y, z) &= -\frac{1}{2} \sum_{j \in J} d_j \varphi_j^2(y) - \mu \sum_{j \in J} \varphi_j(y) + \\
&\quad + \sum_{i=1}^m b_i y_i - \frac{1}{2} \sum_{j \in \bar{J}} d_j z_j^2 \\
&\text{s.t.} \\
&\sum_{i=1}^m a_{ij} y_i - d_j z_j \leq c_j, \quad \forall j \in \bar{J}; \\
&z_j \geq 0, \quad \forall j \in \bar{J},
\end{align*}
\]

where \(y = (y_1, \ldots, y_m) \in \mathbb{R}^m\) and \(z = (z_j)_{j \in J}\).

Remark 2.2. The restriction \(z_j \geq 0, \forall j \in \bar{J}\), can be eliminated from problem \((D(\mu))\) because if \((y, z)\) is a feasible solution of problem \((\hat{D}(\mu))\) which is obtained from problem \((D(\mu))\) by eliminating this restriction, then taking

\[
\tilde{z}_j = \begin{cases} 
  z_j, & \text{if } z_j \geq 0 \\
  -z_j, & \text{if } z_j < 0
\end{cases}, \quad \forall j \in \bar{J}
\]

shows that \((y, \tilde{z})\) is a feasible solution of problem \((D(\mu))\) and

\[
g_\mu(y, \tilde{z}) = g_\mu(y, z).
\]

3. DUALITY RESULTS

In this section we establish duality theorems between problems \((P(\mu))\) and \((D(\mu))\). The next result follows directly from (9).

Theorem 3.1 (Weak duality). If \(x\) and \((y, z)\) are feasible solutions of problems \((P(\mu))\) and \((D(\mu))\), respectively, then \(f_\mu(x) \geq g_\mu(y, z)\).

Because equality in (6) holds if and only if \(x_j = \varphi_j(y), \forall j \in J\), and in (7) if and only if

\[
z_j = x_j \quad \text{and} \quad \left( \sum_{i=1}^m a_{ij} y_i - d_j z_j = c_j \text{ or } x_j = 0 \right), \quad \forall j \in \bar{J},
\]

we obtain the following result.
Theorem 3.2. Let \( y^* = (y^*_1, \ldots, y^*_m) \in \mathbb{R}^m \) and \( x^* = (x^*_1, \ldots, x^*_n) \in \mathbb{R}^n \), \( x^* \geq 0 \) be such that
\[
d_jx^*_j + \mu \ln x^*_j = \sum_{i=1}^{m} a_{ij}y^*_i - c_j - \mu, \quad \forall j \in J,
\]
\[
\sum_{i=1}^{m} a_{ij}y^*_i - d_jx^*_j - c_j \leq 0, \quad \forall j \in \bar{J},
\]
\[
\left(\sum_{i=1}^{m} a_{ij}y^*_i - d_jx^*_j - c_j\right)x^*_j = 0, \quad \forall j \in \bar{J}.
\]
If \( x^* \) satisfies the restriction \( Ax^* = b \), then \( x^* \) is an optimal solution of problem \( (P(\mu)) \), and \((y^*, x^*)\) is an optimal solution of problem \( (D(\mu)) \). Moreover, in this case
\[
f_{\mu}(x^*) = g_{\mu}(y^*, x^*),
\]
i.e. problems \( (P(\mu)) \) and \( (D(\mu)) \) have the same optimal value.

To prove the strong duality theorem we will need the following result.

Lemma 3.1. The function \( g_{\mu}(y, z) \) is concave.

Proof. We have \( g_{\mu}(y, z) = \sum_{j \in J} \Psi_j(y) + \Theta(y) + \Lambda(z) \), where
\[
\Psi_j(y) = -\frac{1}{2}d_j\varphi^2_j(y) - \mu\varphi_j(y), \quad \forall y \in \mathbb{R}^m,
\]
\[
\Theta(y) = \sum_{i=1}^{m} b_iy_i, \quad \forall y \in \mathbb{R}^m, \quad \Lambda(z) = -\frac{1}{2}\sum_{j \in \bar{J}} d_jz^2_j, \quad \forall z = (z_j)_{j \in \bar{J}}.
\]
Obviously, the function \( \Theta(y) \) is linear, and the function \( \Lambda(z) \) is concave. Hence it is enough to prove that the function \( \Psi_j(y) \) is concave for every \( j \in J \), whence it will follow that \( g_{\mu}(y, z) \) is concave as a sum of concave functions.

Let \( j \in J \) be arbitrarily fixed. We have
\[
\frac{\partial \Psi_j(y)}{\partial y_k}(y) = -(d_j\varphi_j(y) + \mu)\frac{\partial \varphi_j}{\partial y_k}(y), \quad \forall k \in \{1, \ldots, m\}.
\]
For each \( j \in J \), define the function \( \phi_j : (0, +\infty) \times \mathbb{R}^m \rightarrow \mathbb{R} \) by
\[
\phi_j(x_j, y) = d_jx_j + \mu \ln x_j - \sum_{i=1}^{m} a_{ij}y_i + c_j + \mu.
\]
It follows from (5) that \( \phi_j(\varphi_j(y), y) = 0, \) \( \forall y \in \mathbb{R}^m \). Therefore
\[
\frac{\partial \phi_j}{\partial y_k}(\varphi_j(y), y) = 0, \quad \forall k \in \{1, \ldots, m\}, \forall y \in \mathbb{R}^m.
\]
We obtain
\[ 0 = \frac{\partial \phi_j}{\partial x_j}(\varphi_j(y), y) \frac{\partial \varphi_j}{\partial y_k}(y) + \frac{\partial \phi_j}{\partial y_k}(\varphi_j(y), y) = \left[ d_j + \frac{\mu}{\varphi_j(y)} \right] \frac{\partial \varphi_j}{\partial y_k}(y) - a_{kj}, \]
hence
\[ \frac{\partial \varphi_j}{\partial y_k}(y) = \frac{a_{kj}\varphi_j(y)}{d_j\varphi_j(y) + \mu}, \quad \forall k \in \{1, \ldots, m\}. \] (11)

It follows from (10) and (11) that
\[ \frac{\partial \Psi_j}{\partial y_k}(y) = -a_{kj}\varphi_j(y), \quad \forall k \in \{1, \ldots, m\}. \]

Therefore
\[ \frac{\partial^2 \Psi_j}{\partial y_{k_1} \partial y_{k_2}}(y) = -\frac{\varphi_j(y)}{d_j\varphi_j(y) + \mu} a_{k_1j} a_{k_2j}, \quad \forall k_1, k_2 \in \{1, \ldots, m\}. \]

We deduce that the Hessian of \( \Psi_j(y) \) is
\[ H_{\Psi_j}(y) = r_j(y) A^{(j)} A^{(j)\top}, \]
where
\[ r_j(y) = -\frac{\varphi_j(y)}{d_j\varphi_j(y) + \mu}, \]
\[ A^{(j)} = (a_{ij}^{(j)})_{i \in \{1, \ldots, m\}}, \quad a_{ij}^{(j)} = \begin{cases} a_{ij} & \text{if } l = j \\ 0 & \text{if } l \neq j \end{cases}. \]

Obviously, \( r_j(y) < 0 \). Therefore, the Hessian \( H_{\Psi_j}(y) \) is negative semi-definite, which means that \( \Psi_j(y) \) is concave. \( \square \)

Remark 3.1. It follows from the proof of Lemma 3.1 that if \( \text{rank}(a_{ij})_{i \in \{1, \ldots, m\}, j \in J} = m \) or \( \exists j \in \bar{J} \) such that \( d_j > 0 \),
then \( g_{\mu}(y, z) \) is strictly concave, which guarantees the uniqueness of the optimal solution of the dual problem \((D(\mu))\).

We can now prove the strong duality between problems \((P(\mu))\) and \((D(\mu))\).

Theorem 3.3 (Strong duality). Assume that the primal problem \((P(\mu))\) satisfies the \( J \)-interior point assumption (1) and that the dual problem \((D(\mu))\) has feasible solutions. Then the two problems have the same optimal value. Moreover, if \((y^*(\mu), z^*(\mu))\) is an optimal solution of the dual problem \((D(\mu))\), then a vector \( x^*(\mu) \) given by the relations
\begin{align*}
(12) & \quad d_j x_j^*(\mu) + \mu \ln x_j^*(\mu) = \sum_{i=1}^m a_{ij} y_i^*(\mu) - c_j - \mu, \quad \forall j \in J, \\
(13) & \quad x_j^*(\mu) = z_j^*(\mu), \quad \forall j \in \bar{J},
\end{align*}
is an optimal solution of the primal problem \((P(\mu))\) and
\[ f_\mu(x^*(\mu)) = g_\mu(y^*(\mu), z^*(\mu)). \]

**Proof.** The primal problem \((P(\mu))\) can be written in the form
\[ \inf_{x \in \mathbb{R}^n} \left\{ \tilde{f}(x) - h(Ax) \right\}, \]
where
\[ \tilde{f}(x) = \begin{cases} f_\mu(x), & \text{if } x \geq 0 \\ +\infty, & \text{otherwise} \end{cases}, \quad \forall x \in \mathbb{R}^n, \]
and
\[ h(w) = -\delta(w|\{b\}) = \begin{cases} 0, & \text{if } w = b \\ -\infty, & \text{if } w \neq b \end{cases}, \quad \forall w \in \mathbb{R}^m, \]
\(\delta(\cdot|\{b\})\) being the indicator function of the set \(\{b\}\).

Obviously, \(\tilde{f}\) is a closed proper convex function and \(h\) is a closed proper concave function. By Fenchel Duality Theorem ([17], p. 332) the Fenchel dual of the problem \((P(\mu))\) is
\[ \sup_{y \in \mathbb{R}^m} \left\{ h^*(y) - \tilde{f}^*(A^\top y) \right\}, \]
where the functions \(h^*\) and \(\tilde{f}^*\) are the Fenchel conjugates of \(h\) and \(\tilde{f}\), respectively. According to the definition of the Fenchel conjugate, we have
\[ \tilde{f}^*(u) = \sup_{x \geq 0} \left\{ x^\top u - \tilde{f}(x) \right\} = \sup_{x \geq 0} \left\{ \sum_{j \in J} \left[ -\frac{1}{2} d_j x_j^2 + (u_j - c_j) x_j + \mu x_j \ln x_j \right] + \sum_{j \notin \bar{J}} \left[ -\frac{1}{2} d_j x_j^2 + (u_j - c_j) x_j \right] \right\}. \]

Obviously, for each \(j \in \bar{J}\) we have
\[ \sup_{x_j \geq 0} \left\{ -\frac{1}{2} d_j x_j^2 + (u_j - c_j) x_j \right\} = \begin{cases} 0, & \text{if } u_j \leq c_j, \\ \frac{(u_j - c_j)^2}{2d_j}, & \text{if } u_j > c_j \text{ and } d_j > 0, \\ +\infty, & \text{if } u_j > c_j \text{ and } d_j = 0. \end{cases} \]

For every \(j \in J\), consider the function \(F_j : [0, \infty) \to \mathbb{R}\),
\[ F_j(x_j) = -\frac{1}{2} d_j x_j^2 + (u_j - c_j) x_j - \mu x_j \ln x_j. \]
The equation \(F_j(x_j) = 0\) is equivalent to
\[ d_j x_j + \mu \ln x_j = u_j - c_j - \mu. \]
Therefore, according to (5), this equation has a unique solution $x_j^* > 0$. It follows that

$$\sup_{x_j \geq 0} F_j(x_j) = F_j(x_j^*) = \frac{1}{2}d_j(x_j^*)^2 + \mu x_j^*.$$  

Equations (15), (16), (17) and (19) imply that

$$\tilde{f}^*(u) = \sum_{j \in J} F_j(x_j^*) + \sum_{j \in J_+} \frac{(u_j - c_j)^2}{2d_j},$$

with the restriction

$$u_j \leq c_j, \quad \forall j \in J_0,$$

where $x_j^*$ is the unique solution of equation (18), and

$$\bar{J}_+ = \{ j \in \bar{J} / u_j \geq c_j, \ d_j > 0 \}, \quad \bar{J}_0 = \{ j \in \bar{J} / d_j = 0 \}.$$

For $u = A^Ty$, equation (18) becomes

$$d_j x_j + \mu \ln x_j = \sum_{i=1}^m a_{ij} y_i - c_j - \mu,$$

hence, according to (5), this equation has a unique solution $x_j^* = \varphi_j(y)$. Equations (20), (21), (22) and (19) imply that

$$\tilde{f}^*(A^Ty) = \sum_{j \in J} F_j(\varphi_j(y)) + \sum_{j \in J_+} \frac{1}{2d_j} \left( \sum_{i=1}^m a_{ij} y_i - c_j \right)^2 =$$

$$\sum_{j \in J} \left[ \frac{1}{2}d_j \varphi_j^2(y) + \mu \varphi_j(y) \right] + \frac{1}{2} \sum_{j \in J_{d > 0}} d_j z_j^2,$$

with the restrictions

$$\sum_{i=1}^m a_{ij} y_i \leq c_j, \quad \forall j \in \bar{J} \text{ s.t. } d_j = 0,$$

where

$$z_j = \frac{1}{d_j} \left( \sum_{i=1}^m a_{ij} y_i - c_j \right), \quad \forall j \in \bar{J} \text{ s.t. } d_j > 0.$$

Obviously,

$$h^*(y) = -\delta^*(-y|\{b\}) = b^Ty, \quad \forall y \in \mathbb{R}^m.$$
Using (14), (23), (24), (25) and (26) we deduce that the Fenchel dual of problem \((P(\mu))\) can be written as

\[
\sup \left( -\frac{1}{2} \sum_{j \in J} d_j \varphi_j^2(y) - \mu \sum_{j \in J} \varphi_j(y) + \sum_{i=1}^{m} b_i y_i - \frac{1}{2} \sum_{j \in J \setminus J^*} d_j z_j^2 \right) \quad \text{s.t.}
\]

\[
\sum_{i=1}^{m} a_{ij} y_i \leq c_j, \quad \forall j \in \bar{J} \text{ s.t. } d_j = 0;
\]

\[
\sum_{i=1}^{m} a_{ij} y_i - d_j z_j \leq c_j, \quad \forall j \in \bar{J} \text{ s.t. } d_j > 0;
\]

\[
z_j \geq 0, \quad \forall j \in \bar{J} \text{ s.t. } d_j > 0,
\]

which means that it is equivalent to the geometric dual problem \((D(\mu))\).

It follows from our hypothesis and the Fenchel Duality Theorem that problems \((P(\mu))\) and \((D(\mu))\) have optimal solutions and the same optimal value.

Let now \((y^*(\mu), z^*(\mu))\) be an optimal solution of problem \((D(\mu))\) and consider the vector \(x^*(\mu)\) given by equations (12), (13). As the function \(\alpha : [0, \infty) \to \mathbb{R}, \alpha(t) = -\frac{1}{2} d_j t^2\) is decreasing, it follows from the optimality of the dual solution \((y^*(\mu), z^*(\mu))\) that

\[
z_j^*(\mu) = 0 \text{ or } \sum_{i=1}^{m} a_{ij} y_i^*(\mu) - d_j z_j^*(\mu) = c_j, \quad \forall j \in \bar{J}.
\]

Equations (13) and (27) imply that \(x^*(\mu)\) satisfies the equations

\[
\left( \sum_{i=1}^{m} a_{ij} y_i^*(\mu) - d_j x_j^*(\mu) - c_j \right) x_j^*(\mu) = 0, \quad \forall j \in \bar{J}.
\]

It follows from (27) and Remark 2.2 that \(y^*(\mu)\) is a maximum point of the function \(h_\mu : \mathbb{R}^m \to \mathbb{R}\) defined by

\[
h_\mu(y) = -\frac{1}{2} \sum_{j \in J} d_j \varphi_j^2(y) - \mu \sum_{j \in J} \varphi_j(y) + \sum_{i=1}^{m} b_i y_i - \frac{1}{2} \sum_{j \in J} d_j \zeta_j^2(y),
\]

where

\[
\zeta_j(y) = \begin{cases} 
\frac{1}{d_j} \left( \sum_{i=1}^{m} a_{ij} y_i - c_j \right), & \text{if } d_j > 0, \\
0, & \text{if } d_j = 0
\end{cases}, \quad \forall y \in \mathbb{R}^m, \forall j \in \bar{J}.
\]

It follows that

\[
\frac{\partial h_\mu}{\partial y_k}(y^*(\mu)) = 0, \quad \forall k \in \{1, \ldots, m\}.
\]
For every $k \in \{1, \ldots, m\}$, we use (29), (30) and (11) from the proof of Lemma 3.1 to get

$$\frac{\partial h_\mu}{\partial y_k}(y) = \sum_{j \in J} a_{kj} \varphi_j(y) - \sum_{d_j > 0} \left( \sum_{i=1}^m a_{ij} y_i - c_j \right). \quad (32)$$

Then, according to (12) and (5), we have

$$x_j^*(\mu) = \varphi_j(y^*(\mu)) > 0, \quad \forall j \in J. \quad (33)$$

From (13) and (27) we get

$$x_j^*(\mu) = \frac{1}{d_j} \left( \sum_{i=1}^m a_{ij} y_i^*(\mu) - c_j \right) \geq 0, \quad \forall j \in \bar{J} \text{ s.t. } d_j > 0, \quad (34)$$

$$x_j^*(\mu) = 0, \quad \forall j \in \bar{J} \text{ s.t. } d_j = 0. \quad (35)$$

It follows from (33), (34) and (35) that

$$x^*(\mu) \geq 0, \quad (36)$$

and from (31), (32), (33), (34) and (35) that

$$0 = b_k - \sum_{j=1}^n a_{kj} x_j^*(\mu), \quad \forall k \in \{1, \ldots, m\}, \quad (37)$$

which means that

$$Ax^*(\mu) = b. \quad \square$$

By Theorem 3.2, equations (12), (28), (36) and (37) imply that $x^*(\mu)$ is an optimal solution of the primal problem $(P(\mu))$ and $f_\mu(x^*(\mu)) = g_\mu(y^*(\mu), z^*(\mu)).$

Remark 3.2. For $J = \emptyset$ one recovers duality from standard quadratic programming, and for $J = \{1, \ldots, n\}$ one recovers duality from quadratic programming with total entropic perturbation [7].

Remark 3.3. In the absence of the quadratic term, that is for $D = 0$, one can solve equation (5) and get explicitly the function

$$\varphi_j(y) = e^{\frac{1}{\mu} \left( \sum_{i=1}^m a_{ij} y_i - c_j \right) - 1}, \quad \forall j \in J.$$

In this way, one recovers duality from linear programming with partial entropic perturbation [1].
REFERENCES


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