SOME EXAMPLES OF 1-CONVEX
NON-EMBEDDABLE THREEFOLDS

GIOVANNI BASSANELLI and MARCO LEONI

We construct a family of 1-convex threefolds with exceptional curve $C$ of type $(0, -2)$, which are not embeddable in $\mathbb{C}^m \times \mathbb{C}P_n$. In order to show that they are not Kähler, we exhibit a real 3-dimensional chain $A$ whose boundary is the complex curve $C$.

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1. INTRODUCTION

In general, a 1-convex manifold with 1-dimensional exceptional set is embeddable, that is, it can be realized as an embedded subvariety of $\mathbb{C}^m \times \mathbb{C}P_n$, for suitable $m$ and $n$. The only possible exceptions are given by the following result.

**Theorem 1.1** (see Theorem 3 in [C1]). *Let $X$ be a non-embeddable, 1-convex manifold whose exceptional set $C$ has dimension 1. Then $\dim \mathbb{C}X = 3$ and $C$ has an irreducible component which is a rational curve of type $(-1, -1)$, $(0, -2)$ or $(1, -3)$.*

As regards the existence of the quoted exceptions, in [C1] there is an example of type $(-1, -1)$. In [V1], p. 242 B there is an example of type $(0, -2)$, but the argument is dubious (see [C1], Remark 4, but see also [V2] and [C2]). For the case $(1, -3)$ nothing is known.

Indeed, the first two cases are easier and we shall show

**Theorem 1.2.** *For every integer $k \geq 1$ there is a non-embeddable 1-convex threefold $\tilde{X}_k$ having as exceptional set a smooth rational curve $C$ whose sequence of normal bundles is

$$(0, -2), \ldots, (0, -2), (1, -1).$$

In particular, $N_{C_{\tilde{X}_1}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $N_{C_{\tilde{X}_k}} = \mathcal{O}(0) \oplus \mathcal{O}(-2)$ for $k \geq 2$."

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For \( k = 1 \) we retrieve Coltţoiu’s example. We point out that our construction is explicit and elementary. In order to see that \( \tilde{X}_k \) is not Kähler, we shall exhibit a real 3-chain \( A \) whose boundary is the exceptional curve \( C \).

2. THE PROOF OF THEOREM 1.2

Definition 2.1. Let \( k \geq 1 \) be an integer. The equations

\[
\begin{align*}
w &= \frac{1}{z} \\
y_1 &= z^2 x_1 + zx^k \\
y_2 &= x_2
\end{align*}
\]

in the coordinates \((z, x_1, x_2)\) and \((w, y_1, y_2)\), define a fiber bundle \( W_k \) on \( \mathbb{CP}_1 \), with fiber \( C^2 \).

The following results holds.

Proposition 2.2 (see [L], p. 263 and [P], p. 234). The null section of \( W_k \) is an exceptional rational curve \( C \) of type \((-1, -1)\) for \( k = 1 \), or \((0, -2)\) for \( k \geq 2 \).

There is a geometrical description of these threefolds.

Proposition 2.3. Let \( k \geq 1 \) be an integer. Let \( N_k \xrightarrow{f_k} \mathbb{C}^4 \) be the blow-up with center at the complex smooth surface

\[
S_k := \{ z \in \mathbb{C}^4; z_1 - iz_2 = z_3 - z_4^k = 0 \}.
\]

Then \( W_k \) is the strict transform of the hypersurface

\[
Y_k := \{ z \in \mathbb{C}^4; z_1^2 + z_2^2 + z_3^2 - z_4^{2k} = 0 \},
\]

which is singular only at the origin \( P_k = 0 \); the null section of \( W_k \) is \( C_k = f_k^{-1}(P_k) \).

Proof. It is enough to follow the outline of [P], Example 2.14. \( \square \)

Now, we shall investigate in more detail this geometric construction and build a suitable commutative diagram

\[
\begin{array}{ccccccc}
N_0 & \xrightarrow{h_1} & N_1 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_{k-1}} & N_{k-1} & \xrightarrow{h_k} & N_k \\
\downarrow f_0 & & \downarrow f_1 & & \cdots & & \downarrow f_{k-1} & & \downarrow f_k \\
M_0 & \xrightarrow{g_1} & M_1 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{k-1}} & M_{k-1} & \xrightarrow{g_k} & M_k = \mathbb{C}^4.
\end{array}
\]

Step 1. Applying the desingularization process to the hypersurface \( Y_k \subset \mathbb{C}^4 := M_k \) we get the sequence

\[
M_0 \xrightarrow{g_1} M_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{k-1}} M_{k-1} \xrightarrow{g_k} M_k = \mathbb{C}^4.
\]
More precisely, this sequence is defined by induction: $M_{k-1} \xrightarrow{g_k} M_k = \mathbb{C}^4$ is the blow-up at $P_k := 0$. Then we define the chart $(U_{k-1}; u_1, \ldots, u_4)$ of $M_{k-1}$ by requiring that in these coordinates the map $g_k$ has the equations

\begin{equation}
\begin{aligned}
z_j &= u_j u_4, & \text{for } j = 1, 2, 3 \\
z_4 &= u_4.
\end{aligned}
\end{equation}

Let $Y_{k-1} \subset M_{k-1}$ be the strict transform of $Y_k$: we get that the only singular point of $Y_{k-1}$ is $P_{k-1} := 0 \in U_{k-1}$ and

\begin{equation}
Y_{k-1} \cap U_{k-1} = \{ u_1^2 + u_2^2 + u_3^2 - u_4^{2(k-1)} = 0 \}.
\end{equation}

Comparing (2.6) with (2.3), we notice that we can iterate the process: the process ends.

Next, we need the following result, which is probably well known. Nevertheless, for lack of reference, we shall give the proof.

**Proposition 2.4.** Let $M$ be a complex manifold. Assume that $S$ is a complex submanifold of $M$ and $P$ is a complex submanifold of $S$. The diagram

\[
\begin{array}{ccc}
N' & \xrightarrow{h} & N \\
\downarrow f' & & \downarrow f \\
M' & \xrightarrow{g} & M
\end{array}
\]

is commutative. Here, $g$ is the blow-up with center $P$, $S'$ the strict transform of $S$ in $M'$, $f$ (resp. $f'$) the blow-up with center $S$ (resp. $S'$), $h$ the blow-up with center $f^{-1}(P)$.

**Proof.** The problem is local, therefore we may assume $P = \mathbb{C}^l \times \{0\} \subset S = \mathbb{C}^{l+m} \times \{0\} \subset M = \mathbb{C}^{l+m+n}$. Moreover, it is enough to consider the case $l = 0$, thus $P$ is a point of $S$, which is a $k$-plane of $M = \mathbb{C}^n$. Now, it suffices to prove the following

**Claim.** Proposition 2.4 holds if $P$ is a point of a $k$-plane $S$ of $M = \mathbb{CP}^n$.

**Proof of the claim.** Assume $2 \leq k \leq n - 2$, otherwise the statement is obvious. Fix a $(n - k - 1)$-plane $L$ of $\mathbb{CP}^n$ such that $L \cap S = \emptyset$. Thus, $L$ parametrizes the family $\{ R_x \}_{x \in L}$ of all $(k + 1)$-planes containing $S$.

Let $R'_x \subset N'$ be the strict transform of $R_x$ under $g \circ f'$. Then $N'$ is the disjoint union $\bigcup_{x \in L} R'_x$ and an holomorphic projection $\tau : N' \to L$ is naturally defined.

Let $E'_g \subset N'$ be the strict transform of the exceptional set $E_g \simeq \mathbb{CP}_{n-1}^{k}$ of $M' \xrightarrow{g} M = \mathbb{CP}^n$. The map $\tau$ induces a projection $E'_g \xrightarrow{\tau} L$ whose fibers $R_x \cap E'_g$ are biholomorphic to $\mathbb{CP}^k$. Moreover, for each line $r$ of $R'_x \cap E'_g$,
we get \( r_{N'} E'_g = -1 \). It follows that \( N' \) can be contracted to a manifold \( Q \) which contains \( L \) and this contraction \( N' \xrightarrow{\alpha} Q \) is the blow-up of center \( L \). Moreover, \( \alpha \) and \( \tau \) agree on \( E'_g \).

Let \( E'_f \subset N' \) be the exceptional set of \( N' \xrightarrow{\tau'} M' \). Thus \( \tau \otimes f' : E'_f \to L \times S' \) is a biholomorphism (since each fiber of \( E'_f \xrightarrow{\tau'} L \) intersects each fiber of \( E'_f \xrightarrow{\tau'} S' \) in only one point). But \( (\tau \otimes f')(E'_g \cap E'_f) = L \times \mathcal{E} \), where \( \mathcal{E} \simeq \mathbb{CP}^{k-1} \) is the exceptional set of \( S' \xrightarrow{g} S \). Therefore, the map \( \alpha \) acts on \( E'_f \) as the projection \( L \times S' \to L \times S \). Thus \( E := \alpha(E'_f) \simeq L \times S \) is a complex manifold.

As before, for every \( q \in S \) and every line \( s \) of \( L \times \{q\} \subset E \) we have \( s \cdot q E = -1 \). Therefore, \( Q \) can be contracted to a manifold \( T \) which contains \( S \) and this contraction is the blow-up \( Q \to T \). But now \( T \) and \( M = \mathbb{CP}_n \) are bimeromorphic and biholomorphic outside of \( S \), therefore they are biholomorphic. \( \Box \)

Now, recalling Proposition 2.3, we complete our construction.

**Step 2.** Define \( S_{j-1} \) as the strict transform of \( S_j \) by means of the map \( M_{j-1} \xrightarrow{g_j} M_j \), \( 1 \leq j \leq k \). Let \( N_j \xrightarrow{f_j} M_j \) be the blow-up of center \( S_j \), \( 0 \leq j \leq k - 1 \). Moreover, let \( C_j := f_j^{-1}(P_j) \) and \( N_{j-1} \xrightarrow{h_j} N_j \) be the blow-up of center \( C_j \), \( 1 \leq j \leq k \). Then diagram (2.4) is commutative.

**Proof.** By Proposition 2.4, it is enough to check that \( P_j \in S_j, j = 0, \ldots, k \). But, as noted above, \( P_j = 0 \in U_j \) and, using the chart \( U_j \), it is straighforward to check that

\[
(2.7) \quad S_j \cap U_j = \{ u_1 - iu_2 = u_3 - u_4 \}. \quad \Box
\]

**Corollary 2.5.** Let \( X_j \) be the strict transform of \( Y_j \) under the map \( N_j \xrightarrow{f_j} M_j \), \( 0 \leq j \leq k \). Considering restrictions of maps, from (2.4) we get the commutative diagram

\[
(2.8) \quad \begin{array}{ccccccccc}
X_0 & \xrightarrow{h_1} & X_1 & \xrightarrow{b_2} & \cdots & \xrightarrow{b_{k-1}} & X_{k-1} & \xrightarrow{h_k} & X_k = W_k \\
\downarrow f_0 & & \downarrow f_1 & & & & \downarrow f_{k-1} & & \downarrow f_k \\
Y_0 & \xrightarrow{g_1} & Y_1 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{k-1}} & Y_{k-1} & \xrightarrow{g_k} & Y_k.
\end{array}
\]

Moreover,

(i) \( X_j \) is smooth, the rational curve \( C_j \) (which is the center of \( h_j \)) is contained in \( X_j \) and there is a neighbourhood of \( C_j \) in \( X_j \) biholomorphic to a neighbourhood of the null section of \( W_j \), \( 1 \leq j \leq k \);

(ii) \( C_{j+1} = h_{j+1}(C_j) \), \( j \geq 1 \);
(iii) $X_0 \xrightarrow{f_0} Y_0$ is a biholomorphism;
(iv) the exceptional divisor $E_0$ of $h_1$ is biholomorphic to $\mathbb{CP}_1 \times \mathbb{CP}_1$ and the induced map $E_0 \simeq \mathbb{CP}_1 \times \mathbb{CP}_1 \xrightarrow{h_1} C_1 \simeq \mathbb{CP}_1$ is one of the canonical projections.

Proof. Diagram (2.8) is well defined, because diagram (2.4) is commutative. Comparing equations (2.3) and (2.6), (2.2) and (2.7), we can apply Proposition 2.3 for $j = 1, \ldots, k$. Thus, we get that $W_k = X_k$, $W_j = X_j \cap f_j^{-1}(U_j)$ and $C_j$ is its null section, for $1 \leq j \leq k - 1$. By Proposition 2.2, $N_{C_1|X_1} = (-1, -1)$, while $N_{C_j|X_j} = (0, -2)$ for $j \geq 2$. Thus, the exceptional divisor $E_{j-1}$ of $h_j$ is a rational ruled surface: $E_0 = F_0$ (this proves (iv)) and $E_j = F_j$ for $j \geq 1$. Now, the curve $C_j$ is not a fiber of $F_j$, otherwise $N_{C_j|X_j} = (0, -1)$, thus $C_j$ is a section of $E_j \simeq F_2$. This shows (ii). Finally, since $S_0$ and $Y_0$ are smooth, $X_0 \xrightarrow{f_0} Y_0$ is a biholomorphism. □

Remark 2.6. In the rational ruled surface $F_2$, there is only one curve $C$ with negative self-intersection: $C.F_2 = -2$. Since $C_j$ is not a fiber of $E_j$, it follows easily from the exact sequence

$$0 \rightarrow \mathcal{O}(C_j.E_j C_j) \rightarrow N_{C_j|X_j} \rightarrow \mathcal{O}(C_j.X_j E_j) \rightarrow 0$$

that $C_j$ is the curve of $E_j$ with negative self-intersection. This means that the sequence $X_0 \rightarrow \cdots \rightarrow X_k$ is the sequence of the blow-ups associated with the curve $C_k$.

Let us now state the following elementary result.

Lemma 2.7. Let $Q := \{z \in \mathbb{CP}_3; z_0^2 + z_1^2 + z_2^2 - z_3^2 = 0\}$. Every line of $Q$ has a real point.

Proof. Let $r \subset Q$ be a line and let $P \in r$. If $P$ is not a real point, consider the line $s$ passing through $P$ and $\overline{P}$. Now, $s$ is a real line and $s \cap \mathbb{RP}_3$ is external to the real sphere $Q \cap \mathbb{RP}_3$, therefore there are exactly two planes passing through $s$ and tangent to $Q$, and the tangent points are real. One of these planes must be the plane $\alpha$ defined by the lines $r$ and $s$ (in fact, $\alpha \cap Q$ contains $r$ and is thus a degenerate conic), therefore it is tangent to $Q$ at a real point $Q$, which must belong to $r$. □

By the detailed description of the map $X_k \xrightarrow{f_k} Y_k$ given in (2.8), the following statement is a simple consequence.

Corollary 2.8. Let $B := \{z \in Y_k \cap \mathbb{RP}_4; z_4 > 0\}$ and $A := f_k^{-1}(B)$. Then $A$ is a real threefold with boundary $\partial A = C_k$.

Proof. Let $D := (g_k \circ \cdots \circ g_1)^{-1}(B)$. Since diagram (2.8) is commutative and $h_j$ is surjective, for every $j$, $A = h_k \circ \cdots \circ h_1(f_0^{-1}(D))$. It follows from
(2.5) that \( g_k^{-1}(B) \subset U_{k-1} \), and iterating this argument we get that \( D \subset U_0 \), more precisely,

\[
D = \{ x \in \mathbb{R}^4; x_1^2 + x_2^2 + x_3^2 - 1 = 0, x_4 > 0 \}.
\]

Therefore, the boundary \( \partial D = \{ x \in \mathbb{R}^4; x_1^2 + x_2^2 + x_3^2 - 1 = x_4 = 0 \} \) is contained in \( \{ z \in \mathbb{C}^4; z_1^2 + z_2^2 + z_3^2 - 4z_4 = 0 \} = E_0 \cap U_0 \). By Corollary 2.5(iv), the fibers of the map \( E_0 \simeq f_0^{-1}(E_0) \rightarrow C_1 \) are one of the two families of lines on the quadric \( E_0 \). By Lemma 2.7, each of these lines intersect \( \partial D \), therefore \( h(f_0^{-1}(\partial D)) = C_1 \). Thus, by Corollary 2.5(ii), we have \( \partial A = C_k \). □

In order to obtain our example \( \tilde{X}_k \), we must perturb \( Y_k \) outside the origin.

**Lemma 2.9.** For every fixed integer \( k \geq 1 \) there exist an integer \( N > k \) and \( 0 < \varepsilon \leq 1 \) such that the origin is the only singular point of the hypersurface

\[
(2.9) \quad \tilde{Y}_k := \{ z_1^2 + z_2^2 + z_3^2 - z_4^2 + \varepsilon(z_1^{2N} + z_2^{2N} + z_3^{2N} + z_4^{2N}) = 0 \}.
\]

The equations \( w_j := z_j(1+\varepsilon z_j^{2N})^{1/2} \), \( 1 \leq j \leq 3 \) and \( w_4 = z_4(1-\varepsilon z_4^{2N})^{1/2} \) define a biholomorphic map \( V \xrightarrow{\Phi} \tilde{V} \) between two neighbourhoods of the origin of \( \mathbb{C}^4 \). Thus, we can define \( \tilde{X}_k \) gluing \( f_k^{-1}(V) \cap X_k \) and \( \tilde{Y}_k \setminus \{ 0 \} \) by means of \( \Phi \). The maps \( \tilde{X}_{j-1} \to \tilde{X}_j \) and \( \tilde{Y}_{j-1} \to \tilde{Y}_j \) are defined as above since nothing is changed near \( P_j \) and \( C_j \) while the maps \( \tilde{X}_j \to \tilde{Y}_j \) are defined by a gluing process (these maps are not blow-ups).

**Proposition 2.10.** \( \tilde{X}_k \) is not embeddable.

**Proof.** Let \( \tilde{B} := \mathbb{R}^4 \cap \tilde{Y}_k \cap \{ x_4 > 0 \} \). It follows from (2.9) that \( \tilde{B} \) is relatively compact and from the definition of \( \Phi \) that \( B \cap \tilde{V} = \Phi^{-1}(B \cap V) \). Thus, the preimage \( \tilde{A} \) of \( \tilde{B} \) is again a 3-chain with boundary \( C_k \). Hence the exceptional curve \( C_k = \partial \tilde{A} \) is a boundary in \( \tilde{X}_k \), which is not Kähler. □

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Dipartimento di Matematica
Università degli Studi di Parma
Parco Area delle Scienze 53/A
I-43100 Parma, Italy
giovanni.bassanelli@unipr.it

and

Collège Louis Nucéra
171 Route de Turin
06300 Nice, France
marco.leoni2003@libero.it