INVEX SETS
AND NONSMOOTH INVEX FUNCTIONS
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We give conditions for invexity of sets and show that the family of some invex subsets of $\mathbb{R}^n$ is a nontrivial vector subspace of $\mathbb{R}^n$. Also, a representation form for an invex set is given. We show that if for a nonsmooth function of $\mathbb{R}^n$ every stationary point is a global minimum, then this function is pseudoinvex on the set of its stationary points. The domains of the invex, pseudoinvex and quasiinvex functions are invex sets. We extend the notions of invexity, pseudoinvexity and quasiinvexity for real functions defined on arbitrary sets.

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1. INTRODUCTION

Pini [13] introduced in 1991 the notion of $\eta$-invex set and Mititelu [8] defined in 1994 the invex sets. The invex, pseudoinvex and quasiinvex functions were introduced by Hanson [5] in 1981 in the differentiable framework. A differentiable function is invex if and only if every stationary point is a global minimum point [2]. If a function is pseudoinvex, then every stationary point of it is a global minimum [9]. For a quasiinvex function, every local minimum point is a global one [9]. For enjoying these properties, the functions mentioned are used in optimization theory. We recall that the domains of differentiable invex functions are invex sets [8].

Craven [3] defined in 1986 the nonsmooth invex functions. All types of nonsmooth invex and generalized invex functions at a point, grouped in $\rho$-classes, were introduced by Giorgi and Mititelu [4], using the upper Dini directional derivative, and by Mititelu and Stancu-Minasian [12], using the Clarke directional derivative.

This paper establishes some properties of invex sets and nonsmooth [generalized] invex functions on open and arbitrary (nonempty) sets. It is divided into four sections. Section 1 is an introduction. In Section 2 we establish necessary and sufficient conditions for the invexity of the sets using the cone of feasible directions [1]. In addition, there is given a representation form of an
invex set. Also, it is shown that the family of all subsets of \( \mathbb{R}^n \) that are \( \eta \)-invex with respect to vector functions of the form \( \eta(x, u) \) only, and are homogeneous in \( u \), is a nontrivial vector subspace of \( \mathbb{R}^n \). In Section 3 we show that if for a nonsmooth function on \( \mathbb{R}^n \) every stationary point is a global minimum, then this function is pseudoinvex on the set of its stationary points. Also, it is shown that the domains of the invex, pseudoinvex and quasiinvex nonsmooth functions are invex sets. In Section 4 we consider invex functions defined on arbitrary (nonempty) sets.

2. INVEX SETS

**Definition 2.1 [8].**

1) A nonempty set \( X \subseteq \mathbb{R}^n \) is said to be an invex set at \( u \in X \) if there is a vector function \( \eta : X \times X \to \mathbb{R}^n \) such that

\[
(2.1) \quad \forall x \in X, \forall \lambda \in [0,1] \Rightarrow u + \lambda \eta(x, u) \in X.
\]

2) A set \( X \) is called invex if (2.1) holds for all \( u \in X \). (There is excluded the situation \( \eta = 0 \) [\( \eta \) is the zero function], when \( X \) is a trivial invex set.)

**2.1. Invexity conditions for sets**

We recall that the invex sets generalize the convex sets (property that results for \( \eta(x, u) = x - u \)). If \( X \) is invex with respect to \( \eta \) then it is called \( \eta \)-invex for short. From (2.1) we get that \( \eta(x, u) \) is a feasible direction to \( X \) at the point \( u \). Therefore,

\[
\eta(x, u) \in F_X(u) = \{ v \in \mathbb{R}^n \mid \forall \lambda \in [0,1] : u + \lambda \eta(v, u) \in X \},
\]

where \( F_X(u) \) is the cone of feasible directions to \( X \) at \( u \). Then the invexity of \( X \) at \( u \) amounts to

\[
(2.2) \quad \eta(x, u) \in F_X(u) \Rightarrow u + \lambda \eta(x, u) \in X, \forall \lambda \in [0,1] (\eta \neq 0).
\]

**Example 2.1.** The graph of the function \( f : [-1,1] \to \mathbb{R}, f(x) = x^3 \), namely \( G_f = \{ (x, y) \in \mathbb{R}^2 \mid y = x^3, x \in [-1,1] \} \), is an invex set at no point of it. We remark that at the current point \( (u, u^3) \in G_f \) we have \( F_{G_f}(u, u^3) = \{ (0,0) \} \). For this reason (it results \( \eta = 0 \)) the set \( G_f \) is some invex at no of its points.

**Theorem 2.1.** If a nonempty set \( X \) is an \( \eta \)-invex set at \( u \), then \( F_X(u) \supset \{0\} \).

We denote by \( \text{ri} F_X(u) \) the relative interior [1] of \( F_X(u) \). We remark that \( \text{ri} G_f = \emptyset \). If \( \text{ri} X \neq \emptyset \) and \( F_X(u) = \{0\} \), then \( u \) is an extremal point [1] of \( X \).
In general, if $F_X(u) \ni \{0\}$ then $F_X(u)$ is not a convex cone. If $X$ is a convex set, then $F_X(u)$ is a convex cone [1].

**Theorem 2.2.** Suppose that $X$ is a nonempty set and $\text{ri} F_X(u) \neq \emptyset$. Then $X$ is an invex set at $u$.

*Proof.* Obviously, $F_X(u) \ni \{0\}$. Consider the points $u, x \in X$. If $[u, x] \subseteq X$ then $X$ is invex at $u$ with respect to $\eta$ defined by $\eta(x, u) = x - u$. Now, assume that $[u, x] \nsubseteq X$. Since $\text{ri} F_X(u) \neq \emptyset$, we can consider a point $x' \in X$ such that $[u, x'] \subseteq X$ and $x' - u \in F_X(u)$. In this case, the correspondence $(u, x) \rightarrow x'$ defines an application $\eta$ by $x' = u + \eta(x, u)$, for which we have $u + \eta(x, u) \in X$ and $u + \lambda \eta(x, u) \in X$, $\forall \lambda \in [0, 1]$. With this $\eta$, $X$ is an $\eta$-invex set. $\square$

**Remark 2.1.** If $X$ is $\eta$-invex at $u$ then, according to (2.2), it is sufficient that $\eta : X \times X \rightarrow F_X(u)$ ($\eta \neq 0$). If $X$ is invex (at each $u \in X$), then $\eta : X \times X \rightarrow \mathbb{R}^n$.

**Remark 2.2.** Any nonempty open set is invex.

### 2.2. A representation of an invex set

We have the following result.

**Theorem 2.3.** If a set $X$ is invex with respect to a vector function $\eta : X \times X \rightarrow \mathbb{R}^n$, then it admits the representation

$$X = \bigcup_{u \in X} \bigcup_{x \in X} \{u + \lambda \eta(x, u) \in X \mid \forall \lambda \in [0, 1]\}.$$ 

*Proof.* We have

$$\{u + \lambda \eta(x, u) \in X \mid \forall \lambda \in [0, 1]\} \subseteq X, \quad \bigcup_{x \in X} \{u + \lambda \eta(x, u) \in X \mid \forall \lambda \in [0, 1]\} \subseteq X.$$ 

Denote

$$X_u = \bigcup_{x \in X} \{u + \lambda \eta(x, u) \in X \mid \forall \lambda \in [0, 1]\}$$

to deduce that $X_u \subseteq X$ and

$$\bigcup_{u \in X} X_u \subseteq X.$$ 

(2.3)

Let us that the equality sign holds in (2.3). Assume on the contrary that $\bigcup_{u \in X} X_u \subseteq X$. Then there exists a point $z \in X \setminus \bigcup_{u \in X} X_u$. Consequently,
z \notin \bigcup_{u \in X} X_u. For z \in X we have z \in X_z \subseteq \bigcup_{u \in X} X_u, that is, a contradiction. Therefore, X = \bigcup_{u \in X} X_u. \square

2.3. The vector subspace of some invex subsets of \( \mathbb{R}^n \)

Let arbitrary subsets \( X \) and \( Y \) of \( \mathbb{R}^n \) and vector functions \( \eta : X \times X \to \mathbb{R}^n \) and \( \mu : Y \times Y \to \mathbb{R}^n \) be given. We define the operations below.

(O1) If \( X \) is invex at \( u \in X \) with respect to \( \eta \) and \( Y \) is invex at \( v \in Y \) with respect to \( \mu \) then

\[
\forall (x + y) \in X + Y, \forall \lambda \in [0, 1] \to (u + v) + \lambda[\eta(x, u) + \mu(y, v)] \in X + Y.
\]

It follows that \( X + Y \) is an invex set with respect to the vector function

\[
\sigma : (X + Y) \times (X + Y) \to \mathbb{R}^n, \quad \sigma(x + y, u + v) = \eta(x, u) + \mu(y, v).
\]

(O2) If \( X \) is invex at \( u \in X \) with respect to \( \eta \), \( \eta(x, u) \) is homogeneous with respect to \( u \) and \( \alpha \in \mathbb{R} \), then \( \forall \alpha \in \mathbb{R}, \forall x \in X, \forall \lambda \in [0, 1] \to \alpha u + \lambda \eta(x, \alpha u) \in \alpha X \). Therefore, \( \alpha X \) is an invex set with respect to this \( \eta \).

We denote by \( H(\mathbb{R}^n) \) the family of all subsets of \( \mathbb{R}^n \) which are invex with respect to vector functions \( \eta(x, u) \) that are homogeneous in the variable \( u \). Then we have

Theorem 2.4. The family \( H(\mathbb{R}^n) \) is a vector subspace of \( \mathbb{R}^n \) with respect to the operations (O1) and (O2).

3. NONSMOOTH INVEX FUNCTIONS ON OPEN SETS

Let \( A \subseteq \mathbb{R}^n \) be a nonempty open set and a function \( f : A \to \mathbb{R} \). The upper Dini directional derivative of \( f \) at the point \( x \in A \) in the direction \( v \in \mathbb{R}^n \) is defined by

\[
f'_+(x; v) = \limsup_{\lambda \downarrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda}.
\]

Theorem 3.1. (a) If \( f'_+ \) is finite near \((x, v)\), then the function \( f'_+(x; \cdot) \) is positively homogeneous [9]. (b) Assume that \( f \) is continuous near \( x \) and \( f'_+ \) is finite near \((x; v)\). Then the function \( f'_+(\cdot; v) \) is upper semicontinuous at \( x \) while the function \( f'_+(x; \cdot) \) is upper semicontinuous at \( v \).

Proof. (b) Let a sequence \( (x_k) \subset A, x_k \to x \) and \( \lambda_k \downarrow 0; f'_+(x_k; v) \) being a upper limit at \((x, 0)\), there exists a sequence \( \delta \left( \frac{1}{k} \right) \downarrow 0 \) such that if \( \lambda_k < \delta \left( \frac{1}{k} \right) \)
then
\[ f'_+(x_k; v) < \frac{f(x_k + \lambda_k v) - f(x_k)}{\lambda_k} + \frac{1}{k}. \]

We have
\[
\limsup_{k \to \infty} f'_+(x_k; v) \leq \limsup_{k \to \infty} \frac{f(x_k + \lambda_k v) - f(x_k)}{\lambda_k} = \lim_{k \to \infty} \frac{f(x_k + \lambda_k v) - f(x_k)}{\lambda_k} (= l),
\]
where we used the continuity of \( f \). There also exists the iterated limit
\[
l_1 = \lim_{\lambda_k \downarrow 0} \lim_{x_k \to x} \frac{f(x_k + \lambda_k v) - f(x_k)}{\lambda_k} = \lim_{\lambda_k \downarrow 0} \frac{f(x + \lambda_k v) - f(x)}{\lambda_k} = f'(x; v) = f'_+(x; v).
\]

We have \( l = l_1 \), hence \( \limsup_{k \to \infty} f'_+(x_k; v) = f'_+(x; v) \) [6].

The upper semicontinuity of \( f'_+(x; \cdot) \) at \( v \) can be established in a similar manner. □

In what follows we consider vector functions of the form \( \eta : A \times A \to \mathbb{R}^n \), where \( \eta \neq 0 \).

**Definition 3.1** [9]. 1) A function \( f \) is said to be **invex at** \( u \in A \) if there exists a vector function \( \eta \) such that
\[
\forall x \in A : \quad f(x) - f(u) \geq f'_+(u; \eta(x, u)).
\]

2) A function \( f \) is said to be **invex on** \( A \) (**invex** for short) if it is invex at each point of \( A \).

(If \( f \) is invex with respect to \( \eta \), then \( f \) is called **\( \eta \)-invex** for short.)

**Definition 3.2** [9]. 1) A function \( f \) is said to be **pseudoinvex at** \( u \in A \) if there exists a vector function \( \eta \) such that
\[
\forall x \in A : \quad f'_+(u; \eta(x, u)) \geq 0 \Rightarrow f(x) \geq f(u).
\]

2) A function \( f \) is said to be **pseudoinvex on** \( A \) if it is pseudoinvex at each of \( A \).

(If \( f \) is pseudoinvex with respect to \( \eta \), then \( f \) is called **\( \eta \)-pseudoinvex** for short.)

**Definition 3.3** [9]. 1) A function \( f \) is said to be **quasiinvex at** \( u \in A \) if there exists a vector function \( \eta \) such that
\[
\forall x \in A : \quad f(x) \leq f(u) \Rightarrow f'_+(u; \eta(x, u)) \leq 0.
\]

2) A function \( f \) is said to be **quasiinvex on** \( A \) if it is quasiinvex at each point of \( A \).
**Theorem 3.2.** The domain of an invex, preinvex or quasiinvex function is an invex set.

**Proof.** We establish the pseudoinvex variant of the theorem. The function $f'_+(u; \cdot)$ is positively defined. Then, for any $t > 0$, it follows from (3.1) that

$$f'_+(u; t\eta(x, u)) = \limsup_{\lambda \downarrow 0} \frac{f(u + \lambda t \eta(x, u)) - f(u)}{\lambda} \geq 0.$$ 

Therefore, for all $x \in A$, $t > 0$ for which $0 \leq f'_+(u; t \eta(x, u)) < \infty$ we have $u + \lambda t \eta(x, u) \in A$. Hence $u + \lambda t \eta(x, u) \in A$, $\forall \lambda t > 0$. Putting $\mu = \lambda t$, we obtain in particular that $u + \mu \eta(x, u) \in A$, $\forall \mu \in [0, 1]$, that is, $A$ is an $\eta$-invex set. □

**Definition 3.4 (Komlós, [7]).** A point $x^0 \in A$ is said to be a stationary point of a function $f$ if $f'_+(x^0; v) \geq 0$, $\forall v \in \mathbb{R}^n$.

The main property that characterizes the differentiable invex functions is that every stationary point of it is a global minimum and conversely [2].

In a nonsmooth framework, the necessity of this property only holds, even for pseudoinvex functions (see Theorem 2.9 of [11]). In what follows, we show that the sufficiency of this property also holds under appropriate restricted conditions.

**Lemma 3.3.** Let $x^0 \in A$ and functions $f : A \rightarrow \mathbb{R}$ and $\eta : A \times A \rightarrow \mathbb{R}^n$. Assume that the function $\eta(\cdot, x^0)$ is surjective on $A$. Then $x^0$ is a stationary point of $f$ if and only if $f'_+(x^0; \eta(x, x^0)) \geq 0$, $\forall x \in A$.

**Proof.** Equivalently, we should establish the relation

$$f'_+(x^0; v) \geq 0, \forall v \in \mathbb{R}^n \Leftrightarrow f'_+(x^0; \eta(x, x^0)) \geq 0, \forall x \in A.$$ 

If $\eta(\cdot, x^0)$ is surjective on $A$, then it is obvious that $v \in \mathbb{R}^n$ if and only if there exists $\eta(x, x^0) \in \mathbb{R}^n$ (where $x \in A$) such that $v = \eta(x, x^0)$ and so, (3.2) holds. □

**Theorem 3.4.** Assume that for a nonsmooth function $f : A \rightarrow \mathbb{R}$ every stationary point is a global minimum. Then $f$ is pseudoinvex on the set of its stationary points.

**Proof.** If $x^0$ is a stationary point of $f$, then we have

$$f'_+(x^0; v) \geq 0, \forall v \in \mathbb{R}^n \Rightarrow f(x) \geq f(x^0), \forall x \in A.$$ 

According to Lemma 3.3, if for a vector function $\eta : A \times A \rightarrow \mathbb{R}^n$ the restriction $\eta(\cdot, x^0)$ is a surjective function, then relation (3.2) holds. Moreover, it follows from (3.2) and (3.3) that

$$\forall x \in A : f'_+(x^0; \eta(x, x^0)) \geq 0 \Rightarrow f(x) \geq f(x^0).$$
that is $f$ is $\eta$-pseudoinvex at $x^0$. Such a function $\eta$ always exists [K. Kuratowski]. □

4. NONSMOOTH INVEX FUNCTIONS ON ARBITRARY SETS

Consider now a function $f : X \to \mathbb{R}$, where $X$ is a nonempty set in $\mathbb{R}^n$, and let $D$ be an open set in $\mathbb{R}^n$ that includes $X$. Relative to the invexity of $f$ on $X$, comparatively with invexity on open sets, a difference appears on the boundary of $X$ and it refers to $\eta$, which must be a homeomorphism. (We denote the boundary of $X$ by $\partial X$.)

Theorem 4.1. Let $D$ be a nonempty open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and functions $f : D \to \mathbb{R}$ and $\eta : X \times X \to \mathbb{R}^n$. Moreover,

1) $f$ is continuous and $f'_+ \text{ is finite on } D \times \mathbb{R}^n$;
2) $\eta$ is a homeomorphism;
3) $f$ is invex on $\partial X$ with respect to the restriction
$$\eta|_{\partial X}[\eta : \partial X \times \partial X \to \mathbb{R}^n].$$

Then $f$ is invex on $X$ with respect to $\eta$.

Proof. As $f$ is $\eta$-invex on $\partial X$, we have
$$\forall x, u \in \partial X : f(x) - f(u) \geq f'_+(u, \eta(x, u)). \tag{4.1}$$

Let $y, b \in \partial X$. Because $\eta$ is a homeomorphism we have $(x \to y, u \to b) \Leftrightarrow (u \to b, \eta(x, u) \to \eta(y, b))$. Taking lim sup as $x \to y$, $u \to b$ in (4.1), we obtain
$$f(y) - f(b) \geq \limsup_{u \to b} f'_+(u; \eta(x, u)) \tag{4.2}$$

Also, there exists the iterated limit
$$l_1 = \limsup_{u \to b} \limsup_{\eta(x, u) \to \eta(y, b)} f'_+(u; \eta(x, u)) = \limsup_{u \to b} f'_+(u; \eta(y, b)) = f'_+(b; \eta(y, b)), \tag{4.3}$$

where we took into account that $f'_+(x, v)$ is upper semicontinuous with respect to each variable (see Theorem 3.1). Then $l = l_1$ and (4.2) becomes

$$f(y) - f(b) \geq f'_+(b, \eta(y, b)).$$

Finally, considering all the variants of $x \to y$, $u \to b$, we can derive (4.3) for $\forall y, b \in X$. □

Theorem 4.2. Let $D$ be a nonempty open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and functions $f : D \to \mathbb{R}$ and $\eta : X \times X \to \mathbb{R}^n$. Assume conditions
1) and 2) of Theorem 4.1 are satisfied and also the that $f$ is pseudoinveze on $\text{ri } X$ with respect to restriction $\eta|_{\text{ri } X}$. Then $f$ is pseudoinveze on $X$ with respect to $\eta$.

**Theorem 4.3.** Let $D$ be a nonempty open set in $\mathbb{R}^n$, $X \subset D$ a nonempty subset and functions $f : D \rightarrow \mathbb{R}$ and $\eta : X \times X \rightarrow \mathbb{R}^n$. Assume conditions 1) and 2) of Theorem 4.1 are satisfied and also that $f$ is quasiinvex on $\text{ri } X$ with respect to restriction $\eta|_{\text{ri } X}$. Then $f$ is quasiinvex on $X$ with respect to $\eta$.

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