USING MONOGENOUS SURFACES IN SOLVING PLANE BOUNDARY PROBLEMS

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The Dirichlet and Neumann boundary problems are formulated using some surfaces associated with monogenous functions. The spatial image of a monogenous surface associated with the solution of a boundary problem provides new methods of solving such a problem. The transformation of a Neumann boundary problem into a Dirichlet boundary problem is done using monogenous quaternions. The transformation of surfaces with negative Gauss curvature into monogenous surfaces with the same total curvature, is also discussed.

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1. INTRODUCTION

Monogenous quaternions are the subject of many papers and books. The topic we are dealing with is related to quaternions of the form

$$K = x + iy + ju(x, y) + kv(x, y), \tag{1.1}$$

where $1, i, j, k$ are elements of an associative but noncommutative A_4 algebra, with

$$ij = -ji = k; \quad jk = -kj = i; \quad ki = -ik = j; \quad i^2 = j^2 = k^2 = -1.$$

Quaternions of the form (1.1) comprise the complex functions

$$f(z) = f(x, y) = u(x, y) + iv(x, y); \quad i^2 = -1, \quad z = x + iy.$$

With each quaternion (1.1) a real surface

$$(S): \quad \overline{r} = \overline{i}y + \overline{j}u(x, y) + \overline{k}v(x, y), \tag{1.2}$$

is associated, where $\overline{i}, \overline{j}, \overline{k}$ are the Euclidean versors, and y, $u(x, y)$, $v(x, y)$ are as in (1.1).

In our papers [1], [2], [3] some geometrical properties of a surface (S) were given namely:

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T_1. A surface $(S) : \overline{r} = \overline{i}y + \overline{j}u(x, y) + \overline{k}v(x, y)$ is monogenous if and only if the function $f(x, y) = u(x, y) + iv(x, y)$ is monogenous.

T_2. The mean curvature of a monogenous surface is zero, that is, the surface is minimal.

T_3. The total (Gauss) curvature of a monogenous surface is negative.

In [4] we showed that the solution to the Neumann plane problem for the disc $|z| < a$ can be obtained using the solution of the Dirichlet problem for the same disc. Between the quaternions corresponding to these solutions there is a linear relation.

For the Dirichlet problem the function $u(x, y)$ is continuous on the boundary C, that is supposed to be sufficiently smooth.

We recall (see [6]) that if $u(x, y)$ is the solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in a simply connected domain } D \subset \mathbb{R}^2, \\ u\,|_C = u_0(\zeta),\ \zeta \in C(\text{ boundary of } D),\ \zeta = x + iy, \end{cases} \tag{1.3}$$

then

$$K = x + iy + ju(x, y) + kv(x, y) \tag{1.4}$$

is the quaternion associated with the solution $u(x, y)$, where $v(x, y)$ is the harmonic conjugate of $u(x, y)$. The monogenous surface

$$(S): \quad \overline{r} = \overline{i}y + \overline{j}u(x, y) + \overline{k}v(x, y)$$

is then associated to the Dirichlet problem (1.3).

The Neumann problem

$$\begin{cases} \Delta u = 0 \text{ in a simply connected domain } D \subset \mathbb{R}^2, \\ \frac{\partial u}{\partial n}\Big|_C = u_1(\zeta),\ \zeta \in C \text{ (boundary of } D) \end{cases} \tag{1.5}$$

can be reduced to a Dirichlet problem for the function $v(x, y) : D \subset \mathbb{R}^2 \to \mathbb{R}$, namely,

$$\begin{cases} \Delta v = 0 \text{ in a simply connected domain } D \subset \mathbb{R}^2, \\ v|_C = v_0(\delta) = \int_0^\delta u_1(\zeta)\,\mathrm{d}\zeta, \end{cases} \tag{1.6}$$

where $v(x, y)$ is the harmonic conjugate of $u(x, y)$.

In the particular case when the domain D is the disc $|z| < a$, the solution to the Neumann problem with $u^*(x, y)$ and $v^*(x, y)$ is associated (see [4]) with the monogenous surface

$$(S^*): \quad \overline{r} = \overline{i}y + \overline{j}u^*(x, y) + \overline{k}v^*(x, y) \tag{1.7}$$

and the quaternion

$$K^* = x + iy + ju^*(x, y) + kv^*(x, y). \tag{1.8}$$

Let K be the quaternion (1.4) associated with the solution to the Dirichlet problem (and with the surface (S)), and let K^* be the quaternion (1.8) associated with the solution to the Neumann problem (and with the surface (S^*)). Between these quaternions there exists ([4]) a relation of the form

$$K^* = K \cdot A, \tag{1.9}$$

where A is an unknown quaternion of the form

$$A = a + ib + jc + kd.$$

If K and K^* are known, then we have

$$\begin{aligned} a &= \frac{x^2+y^2+uu^*+vv^*}{x^2+y^2+u^2+v^2}, \quad & c &= \frac{xu^*+yv^*-yv-xu}{x^2+y^2+u^2+v^2}, \\ b &= \frac{u^*v-v^*u}{x^2+y^2+u^2+v^2}, \quad & d &= \frac{x(v^*-v)+y(u-u^*)}{x^2+y^2+u^2+v^2}. \end{aligned} \tag{1.10}$$

Relation (1.9) of multiplication of two quaternions can be regarded ([5]) as the multiplication of two space rotations given by square matrices of order 2.

If $K = x + iy + ju + kv$ is the quaternion associated with the solution to the Dirichlet problem, then it can identified with a rotation matrix $K = [\mathbb{R}_{\bar{t}}^{\psi}]$, with

$$\mathbb{R}_{\bar{t}}^{\psi} = \mathbf{1}\cos\frac{\psi}{2} + [l\mathbf{I} + m\mathbf{J} + n\mathbf{K}]\sin\frac{\psi}{2}, \tag{1.11}$$

where $\bar{t} = l\bar{i} + m\bar{j} + n\bar{k}$ and

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

and ψ is the rotation angle around the vector $\bar{t}$ satisfying

$$\cos\frac{\psi}{2} = x, \quad l\sin\frac{\psi}{2} = y, \quad m\sin\frac{\psi}{2} = u, \quad n\sin\frac{\psi}{2} = v, \quad |x| \le 1. \tag{1.12}$$

Explicitly, (1.11) can be written as

$$[\mathbb{R}_{\bar{t}}^{\psi}] = \begin{bmatrix} \cos\frac{\psi}{2} + in\sin\frac{\psi}{2} & (-m+il)\sin\frac{\psi}{2} \\ -(m+il)\sin\frac{\psi}{2} & \cos\frac{\psi}{2} - in\sin\frac{\psi}{2} \end{bmatrix}. \tag{1.13}$$

For the quaternion $K^* = x + iy + ju^* + kv^*$ associated with the solution to the Neumann problem, of rotation matrix $K^* = [\mathbb{R}_{\bar{t}^*}^{\psi^*}]$, we have ([4])

$$\begin{cases} \cos\frac{\psi^*}{2} = x, \; l^*\sin\frac{\psi^*}{2} = y, \; m^*\sin\frac{\psi^*}{2} = u^*, \; n^*\sin\frac{\psi^*}{2} = v^*, \\ \psi^* = \psi, \; l^* = l, \; m^* = \frac{u^*}{\sin\frac{\psi}{2}}, \; n^* = \frac{v^*}{\sin\frac{\psi}{2}}, \; \bar{t}^* = l^*\bar{i} + m^*\bar{j} + n^*\bar{k}. \end{cases} \tag{1.14}$$

2. MAIN RESULTS

We have

THEOREM 2.1. *If* $K_1 = \left[\mathbb{R}_{\bar{t}_1}^{\psi_1}\right]$ *and* $K_2 = \left[\mathbb{R}_{\bar{t}_2}^{\psi_2}\right]$ *are the quaternions of the form* (1.13) *associated with the solutions to the Dirichlet problems*

$$\begin{cases} \Delta u = 0 \text{ in } D \subset \mathbb{R}^2, \\ u|_C = u_1(x,y) \text{ on the boundary } C \text{ of } D, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \Delta u = 0 \text{ in } D \subset \mathbb{R}^2, \\ u|_C = u_2(x,y) \text{ on the boundary } C \text{ of } D, \end{cases} \tag{2.2}$$

with rotation angle ψ_1 *around the vector* $\bar{t}_1 = l_1\bar{i} + m_1\bar{j} + n_1\bar{k}$, *respectively with rotation angle* ψ_2 *around the vector* $\bar{t}_2 = l_2\bar{i} + m_2\bar{j} + n_2\bar{k}$, *satisfying* (1.12), (1.14), *then the solution to the Dirichlet problem*

$$\begin{cases} \Delta u = 0 \text{ in } D \subset \mathbb{R}^2, \\ u|_C = u_1(x,y) + u_2(x,y) \text{ on } C, \end{cases} \tag{2.3}$$

has an associated quaternion of the form

$$K = \left[\mathbb{R}_{\bar{t}_1+\bar{t}_2}^{\psi}\right]. \tag{2.4}$$

This quaternion represents a rotation of the same angle as in the first rotations, thus $\psi = \psi_1 = \psi_2$, *around the vector*

$$\bar{t} = \bar{t}_1 + \bar{t}_2 = (l_1 + l_2)\bar{i} + (m_1 + m_2)\bar{j} + (n_1 + n_2)\bar{k}, \tag{2.5}$$

with $l_1 = l_2$.

Proof. The rotation angle ψ associated with the quaternion K in problem (2.3) is the same as the rotation angles ψ_1 and ψ_2 because of (1.12) and (1.14) that yield

$$x = \cos\frac{\psi_1}{2} = \cos\frac{\psi_2}{2}, \quad y = l_1 \sin\frac{\psi_1}{2} = l_2 \sin\frac{\psi_2}{2}.$$

For the rest of the conclusion we use the superposition principle and equations (1.12) and (1.14). $\square$

3. THE CONNECTION WITH MONOGENOUS SURFACES

I. Let $u(x,y)$ be the solution to the Dirichlet problem (1.3) and

$$(S): \quad \bar{r} = \bar{i}y + \bar{j}u(x,y) + \bar{k}v(x,y) \tag{3.1}$$

the monogenous surface associated with it.

Theorem 3.1. *All points of the monogenous surface* (3.1) *are ordinary.*

Proof. The normal vector at an arbitrary point is given by

$$\overline{r}_x \times \overline{r}_y = \overline{i}(u_x^2 + u_y^2) + \overline{j}u_y - \overline{k}u_x, \tag{3.2}$$

hence $\overline{r}_x \times \overline{r}_y \neq \overline{0}$ for any non-constant solution $u(x, y)$. Here we used the monogenity conditions $u_x = v_y$, $u_y = -v_x$. $\square$

II. We consider again the Dirichlet problem (1.3) and suppose that the solution $u(x, y) \in C^2(\overline{D})$ and the associated monogenous surface is (3.1). Let

$$\phi(x, y) = 0 \tag{3.3}$$

be the equation of the boundary C of the domain D and suppose that ϕ satisfies the conditions of the implicit function theorem.

Let $y = \varphi(x)$ be the implicit function defined by equation (3.3). Then on the boundary C we have

$$\begin{cases} u = u_0(x, \varphi(x)) \\ v = v_0(x, \varphi(x)) \end{cases} \tag{3.4}$$

and the surface (S) becomes the curve

$$(S_D): \quad \overline{r} = \overline{i}\varphi(x) + \overline{j}u_0(x, \varphi(x)) + \overline{k}v_0(x, \varphi(x)). \tag{3.5}$$

Now, the Dirichlet problem (1.3) can be reformulated in the form: determine the monogenous surface (S) given by (3.1), that passes through the curve (S_D) given by (3.5).

Example 3.1. Consider the Dirichlet problem for the unit disc with constant data on the boundary, where $\phi(x, y) = 0$ is the circle $x^2 + y^2 - 1 = 0$, $u = u_0 =$ constant and $v = v_0 =$ constant. By (3.5), in this case the curve (S_D) will be

$$\overline{r}_D = \overline{i}\sqrt{1 - x^2} + \overline{j}u_0 + \overline{k}(v_0 + C)$$

which is a plane curve. When $x \in (-1, 1)$, the vector $\overline{r}_D$ describes a segment.

According to the Meusnier-Catalan theorem, among the minimal surfaces with total Gauss negative curvature, the only ruled surfaces (different from a plane) are the straight helicoids with director plane. As a result, the surface associated with the solution to our Dirichlet problem is either a plane or a straight helicoid with director plane of the form $\overline{r} = (u \cos v, u \sin v, av)$.

III. Recall the Identity Principle for Holomorphic Functions: a function $w(z)$ holomorphic in a domain $D \subset \mathbb{R}^2$ is uniquely determined in D by its values at the points of a sequence converging to a point in D.

We can now extend this result as follows.

THEOREM 3.2. *Let $D \subset \mathbb{C}$ be a complex domain and let $\mathcal{H}(D)$ be the set of holomorphic functions on D. If $f, \varphi \in \mathcal{H}(D)$, $f = u(x,y) + iv(x,y)$, $\varphi = \widetilde{u}(x,y) + i\widetilde{v}(x,y)$ and*

$$(S_f): \quad \overline{r}_f = \overline{i}y + \overline{j}u(x,y) + \overline{k}v(x,y),$$

$$(S_\varphi): \quad \overline{r}_\varphi = \overline{i}y + \overline{j}\widetilde{u}(x,y) + \overline{k}\widetilde{v}(x,y)$$

are the monogenous surfaces associated with the holomorphic functions f and φ, then for a sequence of points $(x_n, y_n) \in D$ converging to the point $(x_0, y_0) \in D$ such that $\overline{r}_f(x_n, y_n) = \overline{r}_\varphi(x_n, y_n)$ and $\overline{r}_f(x_0, y_0) = \overline{r}_\varphi(x_0, y_0)$, the surfaces (S_f) and (S_φ) coincide on D, that is,

$$\overline{r}_f(x,y) = \overline{r}_\varphi(x,y), \quad (x,y) \in D \subset \mathbb{C}.$$

Proof. The proof is immediate because the bijectivity between the functions $f, \varphi \in \mathcal{H}(D)$ and the corresponding associated monogenous surfaces, and the Identity Principle for Holomorphic Functions that holds for f and φ. □

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