ON LIMIT DISTRIBUTIONS OF TRIGONOMETRIC SUMS

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Any variance mixture of Gaussian distributions can be a limit distribution of trigonometric sums.

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1. INTRODUCTION

It is known that a lacunary sequence $\{\cos 2\pi n_j \omega\}$ of trigonometric functions has similar property to those of independent random variables when n_j grows rapidly. One of the most famous classical results concerning this phenomenon is probably the following central limit theorem of Salem-Zygmund [1].

THEOREM A. *Let $\big(\Omega, \mathcal{F}, P\big)$ be the Lebesgue probability space, i.e., Ω is the unit interval $[\,0, 1\,]$, $\mathcal{F}$ the σ-field consisting of Lebesgue measurable sets, and P the Lebesgue measure on the unit interval. Suppose that a sequence $\{n_j\}$ of integers satisfies Hadamard's gap condition $n_{j+1}/n_j > q > 1$, that a sequence $\{a_j\}$ of real numbers satisfies*

$$A_j^2 = (a_1^2 + \cdots + a_j^2)/2 \to \infty \quad \textit{and} \quad a_j = o(A_j) \quad \textit{as } j \to \infty,$$

and that $\{\alpha_j\}$ is an arbitrary sequence of real numbers. Then the law of the normalized sum

$$\frac{1}{A_N} \sum_{j=1}^{N} a_j \cos 2\pi n_j (\omega + \alpha_j)$$

with respect to the conditioned measure $P(\,\cdot \mid B) = P(\,\cdot \cap B)/P(B)$ converges to the standard Gaussian distribution provided that $P(B) > 0$.

By the above result, we see that the limit distribution of a normalized sum of trigonometric functions is always Gaussian under Hadamard's gap condition and moderate behaviour of $\{a_n\}$. There have been various attempts to prove the central limit theorem under weaker gap conditions. We refer, for example, to Erdös [2], Takahashi [9], and Fukuyama and Takahashi [3].

By the way, if we do not assume any lacunarity, the limit distribution is not necessarily Gaussian. Actually, we have the following example of Erdös-Fortet.

Example B. If $\{n_j\}$ is the set theoretical union $\{2^j-2\}_{j\in\mathbf{N}}\cup\{2^j-1\}_{j\in\mathbf{N}}$, then the law of

$$\frac{1}{\sqrt{N/2}}\sum_{j=1}^{N}\cos 2\pi n_j\omega$$

converges to the mixture

$$Q(\,\cdot\,)=\int_0^1 N_{0,(\cos\pi t)^2}(\,\cdot\,)\,\mathrm{d}t,$$

of Gaussian distributions, where $N_{0,v}$ denotes the Gaussian distribution with mean 0 and variance v and, in case $v=0$, the delta distribution δ_0 with unit mass at 0.

One can find historical remarks on this example in Salem–Zygmund [8, II, page 61] Kac [4, pages 646 and 664] and Kac [5].

In this note we prove that the class of limit distributions of non-lacunary trigonometric sums is rather wide and includes important distributions. To state this result precisely, we introduce here the notion of *variance mixture of Gaussian distributions.*

A probability measure Q is said to be a variance mixture of Gaussian distributions if there exists a probability distribution F on $[\,0,\infty)$ such that

$$Q(\,\cdot\,)=\int_0^\infty N_{0,t}(\,\cdot\,)\,F(\mathrm{d}t). \tag{1.1}$$

It is known (cf. Kelker [6]) that Q is infinitely divisible if F is infinitely divisible, and that all symmetric α-stable distribution, the Cauchy distribution and all Laplace distributions are variance mixtures of Gaussian distributions.

We are now in a position to state our result.

THEOREM 1. *For any variance mixture Q of Gaussian distributions, there exist $\{a_j\}$, $A_j\to\infty$ and $\{\alpha_j\}$ such that the law of*

$$\frac{1}{A_N}\sum_{j=1}^{N}a_j\cos 2\pi j(\omega+\alpha_j) \tag{1.2}$$

on Lebesgue probability space converges weakly to Q. In particular, any symmetric stable distribution, including the Cauchy distribution, can be a weak limit.

2. MIXING CONVERGENCE

We first recall a notion and results concerning mixing convergence of random variables.

Suppose that the law of a random variable X_n on some probability space $(\Omega, \mathcal{F}, P)$ converges to some distribution Q. This convergence is said to be mixing if the law of X_n under the conditioned measure $P(\cdot \mid E)$ converges weakly to Q for any $E \in \mathcal{F}$ with $P(E) > 0$.

Note that Theorem A states mixing convergence.

The following result (cf. Theorem 4.5 of Billingsley [1]) is essentially due to Mogyoródi [7].

Theorem C. *Suppose that the law of X_n converges to Q and this convergence is mixing. If Y_n converges in probability to some random variable Y, then the law of (X_n, Y_n) converges weakly to the law of (X, Y), where X is a random variable independent of Y and distributed as Q.*

3. PROOF OF THEOREM 1

We shall construct a_j, A_j and α_j such that Q defined by (1.1) is the limit of (1.2).

Let $g(t)$ be the generalized inverse of $F(t)$ and let $f(t) = \sqrt{g(t)}$. Then $f^2(t)$ is a non-negative measurable function satisfying

$$\left| \left\{ t \in [0,1] \mid f^2(t) \leq x \right\} \right| = F(x),$$

i.e., the law of $f^2(t)$ is F. Let us consider a sequence $\{\varepsilon_k\}$ satisfying

$$\varepsilon_k \downarrow 0 \quad \text{and} \quad \varepsilon_k \sqrt{k} \uparrow \infty \quad \text{as } k \to \infty$$

and put

$$E_k = \left\{ t \in [0,1] \mid f(t) > \varepsilon_k \sqrt{k} \right\} \quad \text{and} \quad f^{(k)}(t) = \begin{cases} f(t) & \text{if } t \notin E_k, \\ \varepsilon_k \sqrt{k} & \text{if } t \in E_k. \end{cases}$$

If we denote by $f_n^{(k)}$ the nth partial sum of the Fourier series of $f^{(k)}$, we can find a sequence $\{m_k\}$ of integers such that

$$\left\| f^{(k)} - f_{m_k}^{(k)} \right\|_2 \leq \varepsilon_k / \sqrt{k},$$

where $\|\cdot\|_2$ denotes the $L^2[0,1]$-norm. We can also find a sequence $\{n_k\}$ of integers such that

$$n_k + m_k < n_{k+1} - m_{k+1} \quad \text{and} \quad n_{k+1} \geq 2n_k, \quad k \geq 1.$$

Since $f^{(k)}(t)$ converges to $f(t)$ for all t, and because of the mixing convergence of

$$\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}\cos 2\pi n_k t$$

to the standard normal distribution (cf. Theorem A), by Theorem C the law of

$$f(t)\times\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}\cos 2\pi n_k t \tag{3.1}$$

on Lebesgue probability space converges weakly to that of $f\times X$, where X is a standard normal random variable,independent of f. Since the law of $f\times X$ is Q, the law of (3.1) converges weakly to Q, too.

We have $f^{(k)}(t)=f(t)$ except for finitely many k, and thereby we have

$$\left|\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}\big\{f(t)-f^{(k)}(t)\big\}\cos 2\pi n_k t\right|\le\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}\big|f(t)-f^{(k)}(t)\big|\to 0.$$

Thus, the law of

$$\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}f^{(k)}(t)\cos 2\pi n_k t \tag{3.2}$$

also converges weakly to Q. Hence, by the estimate

$$\left\|\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}\big\{f^{(k)}(t)-f^{(k)}_{m_k}(t)\big\}\cos 2\pi n_k t\right\|_2$$

$$\le\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}\big\|\big\{f^{(k)}(t)-f^{(k)}_{m_k}(t)\big\}\cos 2\pi n_k t\big\|_2\le\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}\frac{\varepsilon_k}{\sqrt{k}}\to 0,$$

the law of

$$\frac{1}{\sqrt{N/2}}\sum_{k=1}^{N}f^{(k)}_{m_k}(t)\cos 2\pi n_k t \tag{3.3}$$

converges weakly to Q.

Since $f^{(k)}_{m_k}(t)$ is an m_kth partial sum of the Fourier series of $f^{(k)}$, we can express it as

$$f^{(k)}_{m_k}(t)=\sum_{j=0}^{m_k}\cos 2\pi(jt+\gamma_j).$$

Therefore we have

$$f_{m_k}^{(k)}(t)\cos 2\pi n_k t = \frac{1}{2}\sum_{j=0}^{m_k}(\cos 2\pi(n_k+j)t+\gamma_j)+\cos 2\pi(n_k-j)t+\gamma_j)$$

and all frequencies appearing above belong to $I_k = [\,n_k - m_k, n_k + m_k\,] \subset [\,n_k - m_k, n_{k+1} - m_{k+1} - 1\,]$. Note that the I_k are disjoint. Thus we can define a_j and α_j for $j = n_k - m_k, \ldots, n_{k+1} - m_{k+1} - 1$ by the trigonometric polynomial expansion

$$f_{m_k}^{(k)}(t)\cos 2\pi n_k t = \sum_{m=n_k-m_k}^{n_{k+1}-m_{k+1}-1} a_m \cos 2\pi m(t+\alpha_m).$$

Then (3.3) can be written as

$$\frac{1}{\sqrt{N/2}}\sum_{j=1}^{n_{N+1}-m_{N+1}-1} a_j \cos 2\pi j t,$$

and we have checked that this converges weakly to Q. Since we have the estimate

$$\left\|\left(\sum_{j=1}^{m} - \sum_{j=1}^{n_k-m_k}\right) a_j \cos 2\pi j t\right\|_2 \le \left\|f^{(k)}\right\|_2 \le \varepsilon_k\sqrt{k}$$

for $n_k - m_k \le m < n_{k+1} - m_{k+1}$, with $A_m = \sqrt{k/2}$ if $n_k - m_k \le m < n_{k+1} - m_{k+1}$, (1.2) should also converge weakly to Q, thus completing the proof. $\square$

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