GENERAL Δ-ERGODIC THEORY: AN EXTENSION

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A first version of the general Δ-ergodic theory was given in [6]. It has applications to the determination of basis of a strongly Δ-ergodic Markov chain (see [6] and references therein), the perturbed Markov chains (see [7]), the design and analysis of simulated annealing type algorithms (see [8]; for the simulated annealing see also [4], [5], and [9]), the asymptotic behaviour of reliability (see [8]; see also [3]) etc. In this paper we set forth an extension of the general Δ-ergodic theory of finite Markov chains.

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1. Δ-ERGODIC THEORY

A first version of the general Δ-ergodic theory was given in [6] (see also [2] on the beginnings of ergodic theory and for some basic results of it). For some applications of it see [6], [7], and [8] (see also [3], [4], [5], and [9]). In this paper we set forth an extension of the general Δ-ergodic theory of finite Markov chains. This more general theory is also called *general Δ-ergodic theory*. It contains: 1) Δ-ergodic theory; 2) limit Δ-ergodic theory; 3) relations between 1) and 2). In this section we deal with Δ-ergodic theory.

Consider a finite Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{1, 2, \ldots, r\}$, initial distribution p_0, and transition matrices $(P_n)_{n\geq 1}$. We frequently shall refer to it as the (finite) Markov chain $(P_n)_{n\geq 1}$. For all integers $m \geq 0$, $n > m$, define $P_{m,n} = P_{m+1}P_{m+2}\ldots P_n = ((P_{m,n})_{ij})_{i,j\in S}$. (The entries of a matrix Z will be denoted Z_{ij}.)

Set

$$\mathrm{Par}(E) = \{\Delta \mid \Delta \text{ is a partition of } E\},$$

where E is a nonempty set. We shall agree that the partitions do not contain the empty set, except for some cases (if needed) where this will be specified.

Definition 1.1. Let $\Delta_1, \Delta_2 \in \mathrm{Par}(E)$. We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

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In Δ-ergodic theory the natural space is $S \times \mathbf{N}$, called *state-time space*. Let $\emptyset \neq A \subseteq S$ and $\emptyset \neq B \subseteq \mathbf{N}$. Let $\Sigma \in \operatorname{Par}(A)$ (equivalently, we can consider a σ-algebra on A (it is known that for any finite σ-algebra $\mathcal{F}$ there exists a finite partition Δ such that $\mathcal{F} = \sigma(\Delta)$, where $\sigma(\Delta)$ is the σ-algebra generated by Δ)). Frequently, when we only use a partition Σ of A, we omit to say this. The three definitions below generalize Definitions 1.2, 1.3, and 1.4 from [6] ([6] corresponds to $\Sigma = (\{i\})_{i \in A}$), respectively.

Definition 1.2. Let $i, j \in S$. We say that i and j are in the same *weakly ergodic class on* $A \times B$ (or *on* $A \times B$ *with respect to* Σ, or *on* $(A \times B, \Sigma)$ when confusion can arise) if $\forall K \in \Sigma$, $\forall m \in B$ we have

$$\lim_{n \to \infty} \sum_{k \in K} \left[(P_{m,n})_{ik} - (P_{m,n})_{jk} \right] = 0.$$

Write $i \overset{A \times B}{\sim} j$ (with respect to Σ) (or $i \overset{(A \times B, \Sigma)}{\sim} j$) when i and j are in the same weakly ergodic class on $A \times B$. Then $\overset{A \times B}{\sim}$ is an equivalence relation and determines a partition $\Delta = \Delta(A \times B, \Sigma) = (C_1, C_2, \ldots, C_s)$ of S. The sets $C_1, C_2, \ldots, C_s$ are called *weakly ergodic classes on* $A \times B$.

Definition 1.3. Let $\Delta = (C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes on $A \times B$ of a Markov chain. We say that the chain is *weakly* Δ*-ergodic on* $A \times B$. In particular, a weakly (S)-ergodic chain on $A \times B$ is called *weakly ergodic on* $A \times B$ for short.

Definition 1.4. Let $(C_1, C_2, \ldots, C_s)$ be the partition of weakly ergodic classes on $A \times B$ of a Markov chain with state space S and $\Delta \in \operatorname{Par}(S)$. We say that the chain is *weakly* $[\Delta]$*-ergodic on* $A \times B$ if $\Delta \preceq (C_1, C_2, \ldots, C_s)$.

In connection with the above notions and notation we mention some special cases ($\Sigma \in \operatorname{Par}(A)$):

1. $A \times B = S \times \mathbf{N}$. In this case we can write $\sim$ instead of $\overset{S \times \mathbf{N}}{\sim}$ (or $\overset{\Sigma}{\sim}$ instead of $\overset{(S \times \mathbf{N}, \Sigma)}{\sim}$) and can omit '*on* $S \times \mathbf{N}$' in Definitions 1.2, 1.3, and 1.4.

2. $A = S$. In this case we can write $\overset{B}{\sim}$ instead of $\overset{S \times B}{\sim}$ (or $\overset{(B,\Sigma)}{\sim}$ instead of $\overset{(S \times B, \Sigma)}{\sim}$) and can replace '$S \times B$' by '(*time set*) B (*with respect to* Σ)' (or by '(B, Σ)') in Definitions 1.2, 1.3, and 1.4. A special subcase is $B = \{m\}$ $(m \geq 0)$; in this case we can write $\overset{m}{\sim}$ (or $\overset{(m,\Sigma)}{\sim}$) and can replace '*on* (*time set*) $\{m\}$' by '*at time* m' in Definitions 1.2, 1.3, and 1.4.

3. $B = \mathbf{N}$. In this case we can set $\overset{A}{\sim}$ instead of $\overset{A \times \mathbf{N}}{\sim}$ (or $\overset{(A,\Sigma)}{\sim}$ instead of $\overset{(A \times \mathbf{N}, \Sigma)}{\sim}$) and can replace '$A \times \mathbf{N}$' by '(*state set*) A (*with respect to* Σ)' (or by '(A, Σ)') in Definitions 1.2, 1.3, and 1.4.

PROPOSITION 1.5. *Let* $\Sigma_1, \Sigma_2 \in \mathrm{Par}(A)$ *with* $\Sigma_1 \preceq \Sigma_2$.

(i) *If* $i \overset{(A\times B,\Sigma_1)}{\sim} j$, *then* $i \overset{(A\times B,\Sigma_2)}{\sim} j$.

(ii) *If the Markov chain* $(P_n)_{n\geq 1}$ *is weakly* $[\Delta]$*- or* Δ*-ergodic on* $(A \times B, \Sigma_1)$, *then it is weakly* $[\Delta]$*-ergodic on* $(A \times B, \Sigma_2)$.

Proof. Obvious. □

Remark 1.6. For Proposition 1.5 an important case is $\Sigma_1 = (\{i\})_{i\in A}$ and $\Sigma_2 = (A)$. As to (ii) we show that weak Δ-ergodicity on $(A \times B, \Sigma_1)$ does not imply weak Δ-ergodicity on $(A \times B, \Sigma_2)$. For this, let

$$P_n = P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \forall n \geq 1.$$

We take $A = S = \{1, 2\}$, $\Sigma_1 = (\{1\}, \{2\})$, and $\Sigma_2 = (\{1, 2\})$. Then $(P_n)_{n\geq 1}$ is weakly $(\{1\}, \{2\})$-ergodic ($A \times B = S \times \mathbf{N}$) with respect to Σ_1 and weakly ergodic ($\Delta = (S)$ and $A \times B = S \times \mathbf{N}$) with respect to Σ_2.

The three definitions below generalize Definitions 1.5, 1.6, and 1.7 from [6], respectively.

Definition 1.7. Let $i, j \in S$. We say that i and j are in the same *uniformly weakly ergodic class on* $A\times B$ (or *on* $A\times B$ *with respect to* Σ, or *on* $(A \times B, \Sigma)$ when confusion can arise) if $\forall K \in \Sigma$ we have

$$\lim_{n\to\infty} \sum_{k\in K} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0$$

uniformly with respect to $m \in B$.

Write $i \overset{u,A\times B}{\sim} j$ (with respect to Σ) (or $i \overset{u,(A\times B,\Sigma)}{\sim} j$) when i and j are in the same uniformly weakly ergodic class on $A\times B$. Then $\overset{u,A\times B}{\sim}$ is an equivalence relation and determines a partition $\Delta = \Delta(A \times B, \Sigma) = (U_1, U_2, \ldots, U_t)$ of S. The sets $U_1, U_2, \ldots, U_t$ are called *uniformly weakly ergodic classes on* $A \times B$.

Definition 1.8. Let $\Delta = (U_1, U_2, \ldots, U_t)$ be the partition of uniformly weakly ergodic classes on $A \times B$ of a Markov chain. We say that the chain is *uniformly weakly* Δ*-ergodic on* $A \times B$. In particular, a uniformly weakly (S)-ergodic chain on $A \times B$ is called *uniformly weakly ergodic on* $A \times B$ for short.

Definition 1.9. Let $(U_1, U_2, \ldots, U_t)$ be the partition of uniformly weakly ergodic classes on $A \times B$ of a Markov chain with state space S and $\Delta \in \mathrm{Par}(S)$. We say that the chain is *uniformly weakly* $[\Delta]$*-ergodic on* $A \times B$ if $\Delta \preceq (U_1, U_2, \ldots, U_t)$.

As for weak Δ-ergodicity we mention some special cases ($\Sigma \in \mathrm{Par}(A)$):

1. $A \times B = S \times \mathbf{N}$. In this case we can write $\overset{u}{\sim}$ instead of $\overset{u,S\times\mathbf{N}}{\sim}$ (or $\overset{u,\Sigma}{\sim}$ instead of $\overset{u,(S\times\mathbf{N},\Sigma)}{\sim}$) and can omit '*on* $S \times \mathbf{N}$' in Definitions 1.7, 1.8, and 1.9.

2. $A = S$. In this case we can write $\overset{u,B}{\sim}$ instead of $\overset{u,S\times B}{\sim}$ (or $\overset{u,(B,\Sigma)}{\sim}$ instead of $\overset{u,(S\times B,\Sigma)}{\sim}$) and can replace '$S \times B$' by '(*time set*) B (*with respect to* Σ)' (or by '(B, Σ)') in Definitions 1.7, 1.8, and 1.9.

3. $B = \mathbf{N}$. In this case we can write $\overset{u,A}{\sim}$ instead of $\overset{u,A\times \mathbf{N}}{\sim}$ (or $\overset{u,(A,\Sigma)}{\sim}$ instead of $\overset{u,(A\times \mathbf{N},\Sigma)}{\sim}$) and can replace '$A \times \mathbf{N}$' by '(*state set*) A (*with respect to* Σ)' (or by '(A, Σ)') in Definitions 1.7, 1.8, and 1.9.

Proposition 1.10. *Let* $\Sigma_1, \Sigma_2 \in \operatorname{Par}(A)$ *with* $\Sigma_1 \preceq \Sigma_2$.

(i) *If* $i \overset{u,(A\times B,\Sigma_1)}{\sim} j$, *then* $i \overset{u,(A\times B,\Sigma_2)}{\sim} j$.

(ii) *If the Markov chain* $(P_n)_{n\geq 1}$ *is uniformly weakly* $[\Delta]$- *or* Δ-*ergodic on* $(A \times B, \Sigma_1)$, *then it is uniformly weakly* $[\Delta]$-*ergodic on* $(A \times B, \Sigma_2)$.

Proof. Obvious. □

The result below generalizes Proposition 1.8 from [6].

Proposition 1.11. *The following statements hold* (*here we only use a partition* $\Sigma \in \operatorname{Par}(A)$).

(i) *If* $i \overset{u,A\times B}{\sim} j$, *then* $i \overset{A\times B}{\sim} j$.

(ii) *If the chain is uniformly weakly* $[\Delta]$- *or* Δ-*ergodic on* $A \times B$, *then it is weakly* $[\Delta]$-*ergodic on* $A \times B$.

Proof. Obvious. □

If B is finite this result can be strengthened (the result below is a generalization of Proposition 1.9 from [6]).

Proposition 1.12. *Suppose that* B *is finite.*

(i) $i \overset{u,A\times B}{\sim} j$ *if and only if* $i \overset{A\times B}{\sim} j$.

(ii) *The chain is uniformly weakly* $[\Delta]$-*ergodic on* $A \times B$ *if and only if it is weakly* $[\Delta]$-*ergodic on* $A \times B$.

(iii) *The chain is uniformly weakly* Δ-*ergodic on* $A \times B$ *if and only if it is weakly* Δ-*ergodic on* $A \times B$.

Proof. Obvious. □

The above result implies that the case where B is finite is not important.

The two definitions below generalize Definitions 1.10 and 1.11 from [6], respectively.

Definition 1.13. Let C be a weakly ergodic class on $A \times B$. Let $\emptyset \neq A_0 \subseteq A$ for which $\exists K_1, K_2, \ldots, K_p \in \Sigma$ such that $A_0 = \bigcup_{u=1}^{p} K_u$. Let $\emptyset \neq B_0 \subseteq B$. We say that C is a *strongly ergodic class on* $A_0 \times B_0$ *with respect to* $A \times B$

(*and* Σ) if $\forall i \in C$, $\forall K \in \Sigma$ with $K \subseteq A_0$, $\forall m \in B_0$ the limit

$$\lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij} := \sigma_{m,K} = \sigma_{m,K}(C)$$

exists and does not depend on i.

Definition 1.14. Let C be a uniformly weakly ergodic class on $A \times B$. Let $\emptyset \neq A_0 \subseteq A$ for which $\exists K_1, K_2, \ldots, K_p \in \Sigma$ such that $A_0 = \bigcup_{u=1}^{p} K_u$. Let $\emptyset \neq B_0 \subseteq B$. We say that C is a *uniformly strongly ergodic class on* $A_0 \times B_0$ *with respect to* $A \times B$ (*and* Σ) if $\forall i \in C$, $\forall K \in \Sigma$ with $K \subseteq A_0$ the limit

$$\lim_{n\to\infty} \sum_{j\in K} (P_{m,m+n})_{ij} := \sigma_{m,K} = \sigma_{m,K}(C)$$

exists uniformly with respect to $m \in B_0$ and does not depend on i.

In connection with the last two definitions we mention some special cases:

1. $A \times B = A_0 \times B_0$. In this case we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* $A \times B$. A special subcase is $A \times B = A_0 \times B_0 = S \times \mathbf{N}$ and $C = S$ when we can say that the Markov chain itself is *strongly* (respectively, *uniformly strongly*) *ergodic.*

2. $A = A_0 = S$. In this case we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* (*time set*) B_0 *with respect to* (*time set*) B. If $B = B_0$, then we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* (*time set*) B. A special subcase of the case $A = A_0 = S$ and $B = B_0$ is $B = B_0 = \{m\}$ when we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class at time* m.

3. $B = B_0 = \mathbf{N}$. In this case we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* (*state set*) A_0 *with respect to* (*state set*) A. If $A = A_0$, then we can say that C is a *strongly* (respectively, *uniformly strongly*) *ergodic class on* (*state set*) A.

The result below generalizes Theorem 1.12 from [6].

THEOREM 1.15. *The following statements hold* (*we only use a partition* $\Sigma \in \mathrm{Par}(A)$).

(i) *If* U *is a uniformly strongly ergodic class on* $A_0 \times B_0$ *with respect to* $A \times B$, *then there exists a* (*unique*) *strongly ergodic class* C *on* $A_0 \times B_0$ (*with respect to* $A_0 \times B_0$) *and* $\Sigma \cap A_0$ *such that* $U \subseteq C$.

(ii) *If* U *is a uniformly strongly ergodic class on* $A \times B$, *then there exists a* (*unique*) *strongly ergodic class* C *on* $A \times B$ *such that* $U \subseteq C$. *Moreover, the class* C *cannot include another uniformly strongly ergodic class on* $A \times B$. *In other words, a strongly ergodic class on* $A \times B$ *includes at most a uniformly strongly ergodic class on* $A \times B$. *If* B *is finite, then* $U = C$.

Proof. (i) As U is included in a uniformly weakly ergodic class on $A_0 \times B_0$, there exists a weakly ergodic class C on $A_0 \times B_0$ (see Proposition 1.11(i)) such that $U \subseteq C$. Obviously, C is unique since it belongs to a unique partition of S. But because $\forall K \in \Sigma$ with $K \subseteq A_0$, $\forall m \in B_0$, $\exists i \in U$ such that the limit

$$\lim_{n \to \infty} \sum_{j \in K} (P_{m,m+n})_{ij} := \sigma_{m,K}$$

exists, we get that C is a strongly ergodic class on $A_0 \times B_0$.

(ii) The first half is as in (i) with the only difference that U is also a uniformly weakly ergodic class on $A \times B$. Further, suppose that there exists another uniformly strongly ergodic class U_1 on $A \times B$ such that $U_1 \subseteq C$ $(U \cap U_1 = \emptyset)$. Let $i \in U$ and $i_1 \in U_1$. From

$$\left| \sum_{j \in K} (P_{m,m+n})_{ij} - \sum_{j \in K} (P_{m,m+n})_{i_1 j} \right| \leq \left| \sum_{j \in K} (P_{m,m+n})_{ij} - \sigma_{m,K} \right| +$$

$$+ \left| \sigma_{m,K} - \sum_{j \in K} (P_{m,m+n})_{i_1 j} \right|, \quad \forall K \in \Sigma, \ \forall m \in B, \ \forall n \geq 1,$$

we get that $i \overset{u,A\times B}{\sim} i_1$. Hence there exists a uniformly strongly ergodic class V on $A \times B$ such that $U \cup U_1 \subseteq V$, and we have reached a contradiction. Obviously, we have $U = C$ when B is finite because of Proposition 1.12(i). □

The two definitions below generalize Definitions 1.13 and 1.14 from [6], respectively.

Definition 1.16. Consider a weakly (respectively, uniformly weakly) Δ-ergodic chain on $A \times B$ (with respect to Σ). We say that the chain is *strongly* (respectively, *uniformly strongly*) Δ-*ergodic on* $A \times B$ if any $C \in \Delta$ is a strongly (respectively, uniformly strongly) ergodic class on $A \times B$. In particular, a strongly (respectively, uniformly strongly) (S)-ergodic chain on $A \times B$ is called *strongly* (respectively, *uniformly strongly*) *ergodic on* $A \times B$ for short.

Definition 1.17. Consider a weakly (respectively, uniformly weakly) $[\Delta]$-ergodic chain on $A \times B$. We say that the chain is *strongly* (respectively, *uniformly strongly*) $[\Delta]$-*ergodic on* $A \times B$ if any $C \in \Delta$ is included in a strongly (respectively, uniformly strongly) ergodic class on $A \times B$.

Also, in these definitions we can simplify the language when referring to A and B (and Σ). These are left to the reader.

PROPOSITION 1.18. *Let* $\Sigma_1, \Sigma_2 \in \operatorname{Par}(A)$ *with* $\Sigma_1 \preceq \Sigma_2$. *If the chain* $(P_n)_{n \geq 1}$ *is strongly* (*respectively, uniformly strongly*) $[\Delta]$- *or* Δ-*ergodic on* $(A \times B, \Sigma_1)$, *then it is strongly* (*respectively, uniformly strongly*) $[\Delta]$-*ergodic on* $(A \times B, \Sigma_2)$.

Proof. Obvious. □

Complete Δ-ergodic problem. It has a 'weak-strong' part and one 'uniform weak-uniform strong'. The 'weak-strong' part refers to the determination of all distinct partitions $\Delta = \Delta(A \times B, \Sigma)$ ($\emptyset \neq A \subseteq S$, $\Sigma \in \operatorname{Par}(A)$, and $\emptyset \neq B \subseteq \mathbf{N}$) for which the chain is weakly Δ-ergodic on $A \times B$ (with respect to Σ) and the determination, for any C belonging to these partitions, of the largest, if any, $A_0 = A_0(C) \subseteq A$ with $A_0 = \bigcup_{u=1}^{p} K_u$, where $K_1, K_2, \ldots, K_p \in \Sigma$, and $B_0 = B_0(C) \subseteq B$ for which it is strongly ergodic on $A_0 \times B_0$ with respect to $A \times B$ (and Σ). The 'uniform weak-uniform strong' part refers to the determination of all distinct partitions $\Delta = \Delta(A \times B, \Sigma)$ for which the chain is uniformly weakly Δ-ergodic on $A \times B$ (with respect to Σ) and the determination, for any U belonging to these partitions, of the largest, if any, $A_0 = A_0(U) \subseteq A$ with $A_0 = \bigcup_{u=1}^{p} K_u$, where $K_1, K_2, \ldots, K_p \in \Sigma$, and $B_0 = B_0(U) \subseteq B$ for which it is uniformly strongly ergodic on $A_0 \times B_0$ with respect to $A \times B$ (and Σ).

In connection with the above problem we mention the result below (it is a generalization of the result from Remark 1.15 in [6]).

Proposition 1.19. *Let $\emptyset \neq A_1, A_2 \subseteq S$ and $\emptyset \neq B_1, B_2 \subseteq \mathbf{N}$. Let $\Sigma_1 \in \operatorname{Par}(A_1)$ and $\Sigma_2 \in \operatorname{Par}(A_2)$. If $A_1 \subseteq A_2, \Sigma_1 \subseteq \Sigma_2, B_1 \subseteq B_2$, and the chain is weakly (respectively, uniformly weakly) Δ_1-ergodic on $(A_1 \times B_1, \Sigma_1)$ and weakly (respectively, uniformly weakly) $[\Delta_2]$- or Δ_2-ergodic on $(A_2 \times B_2, \Sigma_2)$, then $\Delta_2 \preceq \Delta_1$.*

Proof. Obvious. □

2. LIMIT Δ-ERGODIC THEORY

In this section we deal with a generalization of the limit Δ-ergodic theory from [6]. This generalization will be also called *the limit Δ-ergodic theory.* Moreover, we shall indicate some connections between this and Δ-ergodic theory.

We shall agree that when writing

$$\lim_{u \to \infty} \lim_{v \to \infty} a_{u,v},$$

where $a_{u,v} \in \mathbf{R}$, $\forall u, v \in \mathbf{N}$ with $u \geq u_1$, $v \geq v_1(u)$, we assume that $\exists u_0 \geq u_1$ such that

$$\exists \lim_{v \to \infty} a_{u,v}, \quad \forall u \geq u_0.$$

As in Section 1, we consider $\emptyset \neq A \subseteq S$ and $\Sigma \in \operatorname{Par}(A)$. The three definitions below generalize Definitions 2.1, 2.2, and 2.3 from [6], respectively.

Definition 2.1. Let $i, j \in S$. We say that i and j are in the same *limit weakly ergodic class on* A (or *on* A *with respect to* Σ, or *on* (A, Σ) when confusion can arise) if $\forall K \in \Sigma$ we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{k\in K}\left[(P_{m,n})_{ik}-(P_{m,n})_{jk}\right]=0.$$

Write $i \overset{l,A}{\sim} j$ (with respect to Σ) (or $i \overset{l,(A,\Sigma)}{\sim} j$) when i and j are in the same limit weakly ergodic class on A. Then $\overset{l,A}{\sim}$ is an equivalence relation and determines a partition $\bar{\Delta} = \bar{\Delta}(A, \Sigma) = (L_1, L_2, \ldots, L_u)$ of S. The sets $L_1, L_2, \ldots, L_u$ are called *limit weakly ergodic classes on* A.

Definition 2.2. Let $\bar{\Delta} = (L_1, L_2, \ldots, L_u)$ be the partition of limit weakly ergodic classes on A. We say that the chain is *limit weakly* $\bar{\Delta}$*-ergodic on* A. In particular, a limit weakly (S)-ergodic chain on A is called *limit weakly ergodic on* A for short.

Definition 2.3. Let $(L_1, L_2, \ldots, L_u)$ be the partition of limit weakly ergodic classes on A of a Markov chain with state space S and $\bar{\Delta} \in \mathrm{Par}(S)$. We say that the chain is *limit weakly* $[\bar{\Delta}]$*-ergodic on* A if $\bar{\Delta} \preceq (L_1, L_2, \ldots, L_u)$.

In the above definitions we have used $\bar{\Delta}$ only for differing from Section 1, where we have used Δ. This section is called 'Limit Δ-ergodic theory', but not 'Limit $\bar{\Delta}$-ergodic theory' since the former is simply a generic name.

If $A = S$ then in the above definitions we can omit '*on* S' and can write $\overset{l}{\sim}$ instead of $\overset{l,S}{\sim}$ (or $\overset{l,\Sigma}{\sim}$ instead of $\overset{l,(S,\Sigma)}{\sim}$).

PROPOSITION 2.4. *Let* $\Sigma_1, \Sigma_2 \in \mathrm{Par}(A)$ *with* $\Sigma_1 \preceq \Sigma_2$.

(i) *If* $i \overset{l,(A,\Sigma_1)}{\sim} j$, *then* $i \overset{l,(A,\Sigma_2)}{\sim} j$.

(ii) *If the chain* $(P_n)_{n\geq 1}$ *is limit weakly* $[\Delta]$*- or* Δ*-ergodic on* (A, Σ_1), *then it is limit weakly* $[\Delta]$*-ergodic on* (A, Σ_2).

Proof. Obvious. $\square$

The definition below generalizes Definition 2.5 from [6].

Definition 2.5. Let $\emptyset \neq A_0 \subseteq A$ for which $\exists K_1, K_2, \ldots, K_p \in \Sigma$ such that $A_0 = \bigcup_{k=1}^{p} K_u$. Let L be a limit weakly ergodic class on A. We say that L is a *limit strongly ergodic class on* A_0 *with respect to* A (*and* Σ) if $\forall i \in L$, $\forall K \in \Sigma$ with $K \subseteq A_0$ the limit

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{j\in K}(P_{m,n})_{ij} := \sigma_K = \sigma_K(L)$$

exists and does not depend on i.

For simplification, in the above definition we say that L is a *limit strongly ergodic class on* A (*with respect to* Σ) when $A = A_0$ and L is a *limit strongly ergodic class* when $A = A_0 = S$.

The two definitions below generalize Definitions 2.6 and 2.7 from [6], respectively.

Definition 2.6. Let $(P_n)_{n\geq 1}$ be a limit weakly $\bar{\Delta}$-ergodic Markov chain on A. We say that the chain is *limit strongly* $\bar{\Delta}$*-ergodic on* A if any $L \in \bar{\Delta}$ is a limit strongly ergodic class on A.

Definition 2.7. Let $(P_n)_{n\geq 1}$ be a limit weakly $\left[\bar{\Delta}\right]$-ergodic Markov chain on A. We say that the chain is *limit strongly* $\left[\bar{\Delta}\right]$*-ergodic on* A if any $L \in \bar{\Delta}$ is included in a limit strongly ergodic class on A.

In the last two definitions we can omit '*on* A' if $A = S$.

Proposition 2.8. *Let* $\Sigma_1, \Sigma_2 \in \operatorname{Par}(A)$ *with* $\Sigma_1 \preceq \Sigma_2$. *If the chain* $(P_n)_{n\geq 1}$ *is limit strongly* $[\Delta]$*- or* Δ*-ergodic on* (A, Σ_1), *then it is limit strongly* $[\Delta]$*-ergodic on* (A, Σ_2).

Proof. Obvious. $\square$

Complete limit Δ*-ergodic problem.* This consists in the determination of all distinct partitions $\bar{\Delta} = \bar{\Delta}(A, \Sigma)$ ($\emptyset \neq A \subseteq S$, $\Sigma \in \operatorname{Par}(A)$) for which the chain is limit weakly $\bar{\Delta}$-ergodic on A and the determination, for any L belonging to these partitions, of the largest, if any, $A_0 = A_0(L) \subseteq A$ with $A_0 = \bigcup_{k=1}^{p} K_u$, where $K_1, K_2, \ldots, K_p \in \Sigma$, for which it is limit strongly ergodic on A_0 with respect to A. (We say 'Complete limit Δ-ergodic problem', but not 'Complete limit $\bar{\Delta}$-ergodic problem' since the former is simply a generic name.)

In connection with the above problem, we have the following result (it is a generalization of the result from Remark 2.8 in [6]).

Proposition 2.9. *Let* $\emptyset \neq A_1$, $A_2 \subseteq S$. *Let* $\Sigma_1 \in \operatorname{Par}(A_1)$ *and* $\Sigma_2 \in \operatorname{Par}(A_2)$. *If* $A_1 \subseteq A_2$, $\Sigma_1 \subseteq \Sigma_2$, *and the chain is limit weakly* $\bar{\Delta}_1$*-ergodic on* (A_1, Σ_1) *and limit weakly* $\left[\bar{\Delta}_2\right]$*- or* $\bar{\Delta}_2$*-ergodic on* (A_2, Σ_2), *then* $\bar{\Delta}_2 \preceq \bar{\Delta}_1$.

Proof. Obvious. $\square$

We shall now indicate connections between Δ-ergodic theory and limit Δ-ergodic theory. We begin with the basic result (it is a generalization of Theorem 2.9 from [6]).

Theorem 2.10. *The following statements hold* (*we only use a partition* $\Sigma \in \operatorname{Par}(A)$).

(i) *If* $i \overset{A}{\sim} j$, *then* $i \overset{l,A}{\sim} j$.

(ii) *If* $i \overset{A\times B}{\sim} j$, *then* $i \overset{l,A}{\sim} j$, *if* $\exists m \geq 0$ *such that* $B \supseteq \{m, m+1, \ldots\}$ (*in particular, if* $B = \mathbf{N}$ *we obtain* (i)).

(iii) *If the chain is weakly* $[\Delta]$*- or* Δ*-ergodic on* A, *then it is limit weakly* $[\Delta]$*-ergodic on* A.

(iv) *If the chain is weakly* $[\Delta]$*- or* Δ*-ergodic on* $A \times B$, *then it is limit weakly* $[\Delta]$*-ergodic on* A, *if* $\exists m \geq 0$ *such that* $B \supseteq \{m, m+1, \ldots\}$ (*in particular, if* $B = \mathbf{N}$ *we obtain* (iii)).

Proof. Obviously, (ii)⇒(i) and (iv)⇒(iii).

(ii) Obvious.

(iv) This follows from (ii). □

Remark 2.11. Theorem 2.10 does not provide a result similar to (iii) and (iv) in the 'strong' case. For this, see Remark 2.10 from [6].

The following result (it is a generalization of Theorem 2.12 from [6]) is a criterion of strong $[\Delta]$-ergodicity (respectively, Δ-ergodicity) when we know that $\exists \lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij}$, $\forall i \in S$, $\forall K \in \Sigma$, $\forall m \in B$.

THEOREM 2.12. *Consider a Markov chain* $(P_n)_{n\geq 1}$. *Then the chain is strongly* $[\Delta]$*-ergodic* (*respectively,* Δ*-ergodic*) *on* $A \times B$ *if and only if*

(i) *it is weakly* $[\Delta]$*-ergodic* (*respectively,* Δ*-ergodic*) *on* $A \times B$,

and

(ii) $\exists \lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij}$, $\forall i \in S$, $\forall K \in \Sigma$, $\forall m \in B$.

Proof. Obvious. □

In the theorem below (it is a generalization of Theorem 2.13 from [6]) we give a converse result related to Remark 2.11.

THEOREM 2.13. *Consider a Markov chain* $(P_n)_{n\geq 1}$. *If the chain is limit strongly* $\left[\bar{\Delta}\right]$*-ergodic* (*respectively,* $\bar{\Delta}$*-ergodic*) *on* A (*with respect to* Σ), *then* $\forall B \subseteq \mathbf{N}$ *with* $B \supseteq \{m, m+1, \ldots\}$ *for some* $m \geq 0$, $\exists \Delta = \Delta(A \times B, \Sigma) \in \mathrm{Par}(S)$ *with* $\Delta \preceq \bar{\Delta}$ *such that it is strongly* $[\Delta]$*-ergodic* (*respectively,* Δ*-ergodic*) *on* $A \times B$.

Proof. Obviously, for any $A \times B$ and $\Sigma \in \mathrm{Par}(A)$, $\exists \Delta \in \mathrm{Par}(S)$ such that the chain is weakly Δ-ergodic on $A \times B$. As the chain is limit strongly $\left[\bar{\Delta}\right]$-ergodic (respectively, $\bar{\Delta}$-ergodic) on A, it is limit weakly $\left[\bar{\Delta}\right]$-ergodic (respectively, $\bar{\Delta}$-ergodic) on A and $\exists \lim_{m\to\infty} \lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij}$, $\forall i \in S$, $\forall K \in \Sigma$. In the limit weakly $\left[\bar{\Delta}\right]$-ergodic case, if $\Delta \not\preceq \bar{\Delta}$, it is easy to modify Δ such that $\Delta \preceq \bar{\Delta}$ while in the limit weakly $\bar{\Delta}$-ergodic case, by Theorem 2.10, we have $\Delta \preceq \bar{\Delta}$. From $\exists \lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij}$, $\forall i \in S$, $\forall K \in \Sigma$, $\forall m \geq m_0$ ($\exists m_0$ ($m_0 \geq 0$) because the chain is finite (see Definitions 2.5, 2.6, and 2.7)), we have $\exists \lim_{n\to\infty} \sum_{j\in K} (P_{m,n})_{ij}$, $\forall i \in S$, $\forall K \in \Sigma$, $\forall m \geq 0$. Now, the result follows from Theorem 2.12. $\square$

The three results below generalize Theorems 2.14, 2.15, and 2.16 from [6], respectively.

The first one can be used to show that a chain is weakly Δ-ergodic on $A \times B$ when we know that it is weakly $[\Delta]$-ergodic on $A \times B$.

THEOREM 2.14. *Consider a Markov chain* $(P_n)_{n\geq 1}$, $m \geq 0$, *and* $B \supseteq \{m, m+1, \ldots\}$. *If the chain is*

(i) *weakly* $[\Delta]$*-ergodic on* $A \times B$,

and

(ii) *limit weakly* Δ*-ergodic on* A,

then it is weakly Δ*-ergodic on* $A \times B$.

Proof. Let $\Delta' \in \mathrm{Par}(S)$ such that the chain is weakly Δ'-ergodic on $A \times B$. Then, from (i) and (ii) we have $\Delta \preceq \Delta'$ and $\Delta' \preceq \Delta$, respectively. Consequently, $\Delta' = \Delta$. $\square$

The second one is a criterion of strong Δ-ergodicity.

THEOREM 2.15. *Consider a Markov chain* $(P_n)_{n\geq 1}$, $m \geq 0$, *and* $B \supseteq \{m, m+1, \ldots\}$. *If the chain is*

(i) *weakly* $[\Delta]$*-ergodic on* $A \times B$,

and

(ii) *limit strongly* Δ*-ergodic on* A,

then it is strongly Δ*-ergodic on* $A \times B$.

Proof. From (ii), by Theorem 2.13, $\exists \Delta' \in \mathrm{Par}(S)$ with $\Delta' \preceq \Delta$ such that the chain is strongly Δ'-ergodic on $A \times B$. It follows that it is weakly Δ'-ergodic on $A \times B$. By (i), we have $\Delta \preceq \Delta'$. Further, by $\Delta' \preceq \Delta$ and $\Delta \preceq \Delta'$, we have $\Delta' = \Delta$, i.e., the chain is strongly Δ-ergodic on $A \times B$. $\square$

The third one is a criterion of strong Δ-ergodicity when we know that a chain is strongly Δ'-ergodic on $A \times B$, where $B \supseteq \{m, m+1, \ldots\}$ for some $m \geq 0$, but we do not know Δ'.

THEOREM 2.16. *Consider a Markov chain* $(P_n)_{n\geq 1}$, $m \geq 0$, *and* $B \supseteq \{m, m+1, \ldots\}$. *If the chain is*

(i) *weakly* $[\Delta]$*-ergodic on* $A \times B$,

(ii) *limit weakly* Δ*-ergodic on* A,

and

(iii) *strongly* Δ'*-ergodic on* $A \times B$,
then it is strongly Δ*-ergodic on* $A \times B$.

Proof. We have $\Delta \preceq \Delta'$ from (i) and (iii) and $\Delta' \preceq \Delta$ from (ii) and (iii). It follows that $\Delta' = \Delta$. $\square$

For the generalization of other results from [6], a way is set forth below. We call it *the reduced matrix method*.

Let $E = (E_{ij})$ be a real $m \times n$ matrix. Let $\emptyset \neq U \subseteq \{1, 2, \ldots, m\}$, $\emptyset \neq V \subseteq \{1, 2, \ldots, n\}$, and $\Sigma = (K_1, K_2, \ldots, K_p) \in \operatorname{Par}(V)$. Define

$$E_U = (E_{ij})_{i \in U, j \in \{1,2,\ldots,n\}}, \quad E^V = (E_{ij})_{i \in \{1,2,\ldots,m\}, j \in V}, \quad E_U^V = (E_{ij})_{i \in U, j \in V},$$

$$Z = (Z_{ij})_{i \in \{1,2,\ldots,|V|\}, j \in \{1,2,\ldots,p\}}, \quad Z_{K_x}^{\{y\}} = \begin{cases} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} & \text{if } y = x, \\ \\ 0 & \text{if } y \neq x, \end{cases}$$

$\forall x, y \in \{1, 2, \ldots, p\}$,

$$|||E|||_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |E_{ij}|$$

(*the* ∞*-norm of* E), and

$$E^+ = \left(E_{ij}^+\right), \quad E_{ij}^+ = \sum_{k \in K_j} E_{ik}, \quad \forall i \in \{1, 2, \ldots, m\}, \ \forall j \in \{1, 2, \ldots, p\}.$$

We call $E^+ = \left(E_{ij}^+\right)$ *the reduced matrix of* E (on (V, Σ); $E^+ = E^+(V, \Sigma)$, i.e., it depends of (V, Σ) (if confusion can arise we write E^{+V} or $E^{+(V,\Sigma)}$ instead of E^+)).

The operators $(\cdot)_U$, $(\cdot)^V$, $(\cdot)_U^V$, $(\cdot)^+$, and $|||\cdot|||_\infty$ have the following basic properties.

Proposition 2.17. *Let* E *be a real* $m \times n$ *matrix and* F *and* G *two real* $n \times p$ *matrices. Let* $\emptyset \neq U \subseteq \{1, 2, \ldots, m\}$, $\emptyset \neq V \subseteq \{1, 2, \ldots, p\}$, *and* $\Sigma = (K_1, K_2, \ldots, K_q) \in \operatorname{Par}(V)$. *Then the following statements hold.*

(i) $(EF)_U = E_U F$, $(EF)^V = EF^V$, *and* $(EF)_U^V = E_U F^V$.

(ii) *If* $E, F \geq 0$, U *and* V *are as above, and* $\emptyset \neq W \subseteq \{1, 2, \ldots, n\}$, *then*

$$(EF)_U \geq E_U^W F_W, \ (EF)^V \geq E^W F_W^V, \ \text{and} \ (EF)_U^V \geq E_U^W F_W^V.$$

(iii) $F^+ = F^V Z$.

(iv) $(-F)^+ = -F^+$ *and* $(F + G)^+ = F^+ + G^+$.

(v) $(EF)^+ = EF^+$.
(vi) $|||F^+|||_\infty \le \left|\left|\left|F^V\right|\right|\right|_\infty \le |||F|||_\infty$ *and* $|||F^+|||_\infty = \left|\left|\left|F^V\right|\right|\right|_\infty$ *if* $F \ge 0$.

Proof. (i) Obvious.
(ii) By (i) we have

$$(EF)_U = E_U F \ge E_U^W F_W, \quad (EF)^V = EF^V \ge E^W F_W^V,$$

and

$$(EF)_U^V = E_U F^V \ge E_U^W F_W^V.$$

(iii) Obvious.
(iv) By (iii) we have

$$(-F)^+ = (-F)^V Z = -F^V Z = -F^+$$

and

$$(F+G)^+ = (F+G)^V Z = \left(F^V + G^V\right) Z = F^V Z + G^V Z = F^+ + G^+.$$

(v) By (i) and (iii) we have

$$(EF)^+ = (EF)^V Z = EF^V Z = EF^+.$$

(vi) By (iii) we have

$$\left|\left|\left|F^+\right|\right|\right|_\infty = \left|\left|\left|F^V Z\right|\right|\right|_\infty \le \left|\left|\left|F^V\right|\right|\right|_\infty |||Z|||_\infty = \left|\left|\left|F^V\right|\right|\right|_\infty \le |||F|||_\infty \,.$$

Now, if $F \ge 0$ then we have

$$|||F^+|||_\infty = \max_{1\le i\le n} \sum_{j=1}^{q} (F^+)_{ij} = \max_{1\le i\le n} \sum_{j=1}^{q} (F^V Z)_{ij} = \max_{1\le i\le n} \sum_{j=1}^{q} \sum_{k=1}^{|V|} (F^V)_{ik} Z_{kj} =$$

$$= \max_{1\le i\le n} \sum_{k=1}^{|V|} \left(F^V\right)_{ik} \sum_{j=1}^{q} Z_{kj} = \max_{1\le i\le n} \sum_{k=1}^{|V|} \left(F^V\right)_{ik} = \left|\left|\left|F^V\right|\right|\right|_\infty . \quad \square$$

THEOREM 2.18. *Consider a Markov chain* $(P_n)_{n\ge1}$.
(i) $\exists \Delta \in \mathrm{Par}(S)$ *such that the chain is strongly* $[\Delta]$- *or* Δ-*ergodic on* $A \times B$ *if and only if* $\exists \lim_{n\to\infty} (P_{m,n})^+ := \Pi_m$, $\forall m \in B$.
(ii) $\exists \Delta \in \mathrm{Par}(S)$ *such that the chain is limit strongly* $[\Delta]$- *or* Δ-*ergodic on* A *if and only if* $\exists \lim_{m\to\infty} \lim_{n\to\infty} (P_{m,n})^+ := \Pi$.

Proof. Obvious. $\square$

Definition 2.19. Under the conditions of Theorem 2.18 we say that a strongly $[\Delta]$- or Δ-ergodic Markov chain on $A \times B$ has *limit* Λ if $\Pi_m = \Lambda$, $\forall m \in B$. We say that Π from Theorem 2.18 is the (*iterated*) *limit* of limit strongly $[\Delta]$- or Δ-ergodic Markov chain on A.

The result below generalizes Theorem 2.6 from [8] (see also Theorem 2.27 from [6]).

THEOREM 2.20. *Consider a Markov chain* $(P_n)_{n\geq 1}$. *Then the chain is strongly ergodic on* A *with limit* Π *if and only if it is limit strongly ergodic on* A *with limit* Π.

Proof. "$\Rightarrow$" If the chain is strongly ergodic on A with limit Π then, by Theorem 2.18, $\lim_{n\to\infty}(P_{m,n})^+ = \Pi$, $\forall m \geq 0$ $(\Sigma \in \operatorname{Par}(A))$. It follows that $\lim_{m\to\infty}\lim_{n\to\infty}(P_{m,n})^+ = \Pi$, i.e., the chain is limit strongly ergodic on A with limit Π.

"$\Leftarrow$" If the chain is limit strongly ergodic on A with limit Π then, by Theorem 2.18, $\exists m_0 \geq 0$ such that $\exists \lim_{n\to\infty}(P_{m,n})^+$, $\forall m \geq m_0$, and

$$\lim_{m\to\infty}\lim_{n\to\infty}(P_{m,n})^+ = \Pi.$$

Further, by Proposition 2.17(v), $\exists \lim_{n\to\infty}(P_{m,n})^+$, $\forall m \geq 0$. Now, we show that $\lim_{n\to\infty}(P_{m,n})^+ = \Pi$, $\forall m \geq 0$. Setting $Q_m = \lim_{n\to\infty}(P_{m,n})^+$, $\forall m \geq 0$, we have

$$\left|\left|\left|(P_{m,n})^+ - \Pi\right|\right|\right|_\infty = \left|\left|\left|P_{m,k}\,(P_{k,n})^+ - P_{m,k}\Pi\right|\right|\right|_\infty \leq$$

$$\leq |||P_{m,k}|||_\infty \left|\left|\left|(P_{k,n})^+ - \Pi\right|\right|\right|_\infty = \left|\left|\left|(P_{k,n})^+ - \Pi\right|\right|\right|_\infty \leq$$

$$\leq \left|\left|\left|(P_{k,n})^+ - Q_k\right|\right|\right|_\infty + |||Q_k - \Pi|||_\infty, \quad \forall k,m,n,\ 0 \leq m < k < n,$$

which implies

$$\limsup_{n\to\infty}\left|\left|\left|(P_{m,n})^+ - \Pi\right|\right|\right|_\infty \leq |||Q_k - \Pi|||_\infty, \quad \forall k,m,\ 0 \leq m < k.$$

Since $\lim_{m\to\infty} Q_m = \Pi$, we have

$$\limsup_{n\to\infty}\left|\left|\left|(P_{m,n})^+ - \Pi\right|\right|\right|_\infty \leq \inf_{k>m}|||Q_k - \Pi|||_\infty = 0, \quad \forall m \geq 0.$$

Therefore, $\lim_{n\to\infty}(P_{m,n})^+ = \Pi$, $\forall m \geq 0$, i.e., the chain is strongly ergodic on A. $\square$

PROPOSITION 2.21. *Let* $(P_n)_{n\geq 1}$ *and* $(P'_n)_{n\geq 1}$ *be two Markov chains. Then*

$$\left|\left|\left|(P_{m,n})^+ - (P'_{m,n})^+\right|\right|\right|_\infty \leq$$

$$\leq \left|\left|\left|P_{m,n} - P'_{m,n}\right|\right|\right|_\infty \leq \sum_{u=1}^{n-m}\left|\left|\left|P_{m+u} - P'_{m+u}\right|\right|\right|_\infty, \quad \forall m,n,\ 0 \leq m < n.$$

Proof. The first inequality follows from Proposition 2.17 (iv) and (vi) while the second one follows directly by induction (see also Proposition 3.11 from [6]). $\square$

Definition 2.22 ([7]). Let $(P_n)_{n\geq 1}$ and $(P'_n)_{n\geq 1}$ be two Markov chains. We say that $(P'_n)_{n\geq 1}$ is a *perturbation of the first type of* $(P_n)_{n\geq 1}$ if

$$\sum_{n\geq 1} \left|\left|\left|P_n - P'_n\right|\right|\right|_\infty < \infty.$$

The result below generalizes Theorem 1.43 from [7]. (In fact, Theorem 1.43 (with a different proof) is due to J. Hajnal (see [1]).)

THEOREM 2.23. *Let $(P_n)_{n\geq 1}$ be a Markov chain and $(P'_n)_{n\geq 1}$ a perturbation of the first type of it. Then $\exists\Delta\in$Par(S) such that $(P_n)_{n\geq 1}$ is strongly Δ-ergodic on A if and only if $\exists\Delta'\in$ Par(S) such that $(P'_n)_{n\geq 1}$ is strongly Δ'-ergodic on A.*

Proof. By symmetry, it is sufficient to suppose that $(P_n)_{n\geq 1}$ is strongly Δ-ergodic on A and prove that $\exists\Delta'\in$ Par(S) such that $(P'_n)_{n\geq 1}$ is strongly Δ'-ergodic on A.

First, we show that $\left((P'_{m,n})^+\right)_{n>m}$ is a Cauchy sequence, $\forall m\geq 0$. Let $m\geq 0$. We have

$$\begin{aligned}\left|\left|\left|(P'_{m,n})^+ - (P'_{m,n+p})^+\right|\right|\right|_\infty &= \left|\left|\left|P'_{m,t}(P'_{t,n})^+ - P'_{m,t}(P'_{t,n+p})^+\right|\right|\right|_\infty \leq\\ \leq \left|\left|\left|P'_{m,t}\right|\right|\right|_\infty \left|\left|\left|(P'_{t,n})^+ - (P'_{t,n+p})^+\right|\right|\right|_\infty &= \left|\left|\left|(P'_{t,n})^+ - (P'_{t,n+p})^+\right|\right|\right|_\infty \leq\\ \leq \left|\left|\left|(P'_{t,n})^+ - (P_{t,n})^+\right|\right|\right|_\infty &+ \left|\left|\left|(P_{t,n})^+ - (P_{t,n+p})^+\right|\right|\right|_\infty +\\ &+ \left|\left|\left|(P_{t,n+p})^+ - (P'_{t,n+p})^+\right|\right|\right|_\infty \leq\end{aligned}$$

$$\leq 2\sum_{k\geq t+1} \left|\left|\left|P_k - P'_k\right|\right|\right|_\infty + \left|\left|\left|(P_{t,n})^+ - (P_{t,n+p})^+\right|\right|\right|_\infty, \quad \forall n,t,\ m<t<n,\ \forall p\geq 0.$$

Let $\varepsilon>0$. Then $\exists t_\varepsilon > m$ such that

$$2\sum_{k\geq t+1} \left|\left|\left|P_k - P'_k\right|\right|\right|_\infty < \frac{\varepsilon}{2}, \quad \forall t\geq t_\varepsilon.$$

Because $\left((P_{u,v})^+\right)_{v>u}$ is convergent, $\forall u\geq 0$ (see Theorem 2.18(i)), it is a Cauchy sequence, $\forall u\geq 0$. Hence $\exists n_\varepsilon > t_\varepsilon$ such that

$$\left|\left|\left|(P_{t_\varepsilon,n})^+ - (P_{t_\varepsilon,n+p})^+\right|\right|\right|_\infty < \frac{\varepsilon}{2}, \quad \forall n\geq n_\varepsilon,\ \forall p\geq 0.$$

Further, it follows that $\exists n_\varepsilon > m$ such that

$$\left|\left|\left|(P'_{m,n})^+ - (P'_{m,n+p})^+\right|\right|\right|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n\geq n_\varepsilon,\ \forall p\geq 0$$

(this is equivalent to $\lim_{n\to\infty}\left((P'_{m,n})^+ - (P'_{m,n+p})^+\right) = 0$ uniformly with respect to $p\geq 0$), i.e., $\left((P'_{m,n})^+\right)_{n>m}$ is a Cauchy sequence, therefore is convergent.

Now, by Theorem 2.18(i), $\exists\Delta'\in$ Par(S) such that the chain $(P'_n)_{n\geq 1}$ is strongly Δ'-ergodic on A. □

The result below generalizes Theorem 1.28 from [7].

THEOREM 2.24. *Let* $(P_n)_{n\geq 1}$ *be a strongly* Δ*-ergodic Markov chain on* A *and* $(P'_n)_{n\geq 1}$ *a perturbation of the first type of it.*

(i) $(P_n)_{n\geq 1}$ *is limit weakly* $\bar{\Delta}$*-ergodic on* A *if and only if* $(P'_n)_{n\geq 1}$ *is limit weakly* $\bar{\Delta}$*-ergodic on* A.

(ii) $\lim_{m\to\infty}\lim_{n\to\infty}(P_{m,n})^+ = \Pi$ *if and only if* $\lim_{m\to\infty}\lim_{n\to\infty}\left(P'_{m,n}\right)^+ = \Pi$.

(iii) $(P_n)_{n\geq 1}$ *is limit strongly* $\bar{\Delta}$*-ergodic on* A *with* (*iterated*) *limit* Π *if and only if* $(P'_n)_{n\geq 1}$ *is limit strongly* $\bar{\Delta}$*-ergodic on* A *with* (*iterated*) *limit* Π.

Proof. (i) Let $i, j \in S$. Then the conclusion is equivalent to $i \overset{l,A}{\sim} j$ for $(P_n)_{n\geq 1}$ if and only if $i \overset{l,A}{\sim} j$ for $(P'_n)_{n\geq 1}$. By symmetry, it is sufficient to prove that $i \overset{l,A}{\sim} j$ for $(P'_n)_{n\geq 1}$ when $i \overset{l,A}{\sim} j$ for $(P_n)_{n\geq 1}$. By Proposition 2.21 we have

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left|\left\|(P_{m,n})^+ - (P'_{m,n})^+\right\|\right|_\infty = 0.$$

(In [7] we proved that $\exists \lim_{n\to\infty}\left|\left\|P_{m,n} - P'_{m,n}\right\|\right|_\infty$, $\forall m \geq 0$; the problem whether $\exists \lim_{n\to\infty}\left|\left\|(P_{m,n})^+ - (P'_{m,n})^+\right\|\right|_\infty$, $\forall m \geq 0$, is left to the reader.)

Now, from

$$\left|\sum_{k\in K}\left[(P'_{m,n})_{ik} - (P'_{m,n})_{jk}\right]\right| \leq \left|\sum_{k\in K}\left[(P'_{m,n})_{ik} - (P_{m,n})_{ik}\right]\right| +$$

$$+\left|\sum_{k\in K}\left[(P_{m,n})_{ik} - (P_{m,n})_{jk}\right]\right| + \left|\sum_{k\in K}\left[(P_{m,n})_{jk} - (P'_{m,n})_{jk}\right]\right| \leq$$

$$\leq \left|\sum_{k\in K}\left[(P_{m,n})_{ik} - (P_{m,n})_{jk}\right]\right| +$$

$$+2\left|\left\|(P_{m,n})^+ - (P'_{m,n})^+\right\|\right|_\infty, \quad \forall m,n,\ 0\leq m<n,\ \forall K\in\Sigma,$$

we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{k\in K}\left[(P'_{m,n})_{ik} - (P'_{m,n})_{jk}\right] = 0, \quad \forall K\in\Sigma$$

($\exists \lim_{n\to\infty}\sum_{k\in K}\left[(P'_{m,n})_{ik} - (P'_{m,n})_{jk}\right]$, $\forall m \geq m_0$ ($m_0 \geq 0$), $\forall K \in \Sigma$, because of the hypothesis and Theorem 2.23), i.e., $i \overset{l,A}{\sim} j$ for $(P'_n)_{n\geq 1}$.

(ii) By symmetry, it is sufficient to prove that $\lim_{m\to\infty}\lim_{n\to\infty}(P'_{m,n})^+ = \Pi$ when $\lim_{m\to\infty}\lim_{n\to\infty}(P_{m,n})^+ = \Pi$ ($\exists \lim_{n\to\infty}(P'_{m,n})^+$, $\forall m \geq m_0$ ($m_0 \geq 0$), because of the hypothesis and Theorem 2.23). Obviously, this follows from

$$\left|\left|\left|(P'_{m,n})^+ - \Pi\right|\right|\right|_\infty \le \left|\left|\left|(P'_{m,n})^+ - (P_{m,n})^+\right|\right|\right|_\infty + \\ + \left|\left|\left|(P_{m,n})^+ - \Pi\right|\right|\right|_\infty, \quad \forall m, n,\ 0 \le m < n.$$

(iii) This follows from (i) and (ii) (see also Theorem 2.18 and Definition 2.19). □

Finally, we give an example.

Example 2.25. Let

$$P_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 \end{pmatrix} := P, \quad \forall n \ge 1.$$

Let $\Sigma_1 = (\{i\})_{i\in\{1,2,3,4\}}$ and $\Sigma_2 = (\{1,2\}, \{3\}, \{4\})$ $(\Sigma_1, \Sigma_2 \in \mathrm{Par}(\{1,2,3,4\}))$. Because

$$P^n = \begin{cases} P & \text{if } n \text{ is odd} \\ P^2 & \text{if } n \text{ is even,} \end{cases}$$

where

$$P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix},$$

the chain $(P_n)_{n\ge1}$ is weakly $(\{i\})_{i\in\{1,2,3,4\}}$-ergodic and is not strongly $(\{i\})_{i\in\{1,2,3,4\}}$-ergodic with respect to Σ_1 ($A\times B = \{1,2,3,4\}\times\mathbf{N}$) and strongly $(\{1,2\}, \{3\}, \{4\})$-ergodic with respect to Σ_2.

Now, consider the chain

$$P'_n = \begin{pmatrix} \frac{1}{n^2} & 1-\frac{1}{n^2} & 0 & 0 \\ 1-\frac{1}{n^2} & 0 & 0 & \frac{1}{n^2} \\ 0 & \frac{1}{2n^2} & 1-\frac{1}{n^2} & \frac{1}{2n^2} \\ \frac{1}{4}-\frac{1}{4n^2} & \frac{2}{4}-\frac{1}{4n^2} & \frac{1}{4}-\frac{1}{4n^2} & \frac{3}{4n^2} \end{pmatrix}, \quad \forall n \ge 1.$$

The chain $(P'_n)_{n\ge1}$ is a perturbation of the first type of $(P_n)_{n\ge1}$. It follows from Theorem 2.23 that $\exists\Delta' \in \mathrm{Par}(\{1,2,3,4\})$ such that $(P'_n)_{n\ge1}$ is strongly Δ'-ergodic with respect to Σ_2 because $(P_n)_{n\ge1}$ is strongly $(\{1,2\}, \{3\}, \{4\})$-ergodic with respect to Σ_2. In particular, this implies that $\exists \lim\limits_{n\to\infty} \sum\limits_{k\in K} \left(P'_{m,n}\right)_{ik}$,

$\forall i \in \{1,2,3,4\}$, $\forall K \in \Sigma_2$. Hence $\exists \lim_{n\to\infty} P(X'_n \in K)$, $\forall K \in \Sigma_2$, where $(X'_n)_{n\geq 0}$ is a chain with state space $\{1,2,3,4\}$ and transition matrices $(P'_n)_{n\geq 1}$. Further, by Theorem 2.24, $(P'_n)_{n\geq 1}$ is limit strongly $(\{1,2\},\{3\},\{4\})$-ergodic with respect to Σ_2 and has (iterated) limit

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix},$$

because $(P_n)_{n\geq 1}$ is limit strongly $(\{1,2\},\{3\},\{4\})$-ergodic with respect to Σ_2 and has limit $\overline{\Pi}$ above.

REFERENCES

[1] J. Hajnal, *Weak ergodicity in non-homogeneous Markov chains.* Proc. Cambridge Philos. Soc. **54** (1958), 233–246.

[2] M. Iosifescu, *Finite Markov Processes and Their Applications.* Wiley, Chichester & Editura Tehnică, Bucharest, 1980; corrected republication by Dover, Mineola, N.Y., 2007.

[3] M.V. Koutras, *On a Markov chain approach for the study of reliability structures.* J. Appl. Probab. **33** (1996), 357–367.

[4] P.J.M. van Laarhoven and E.H.L. Aarts, *Simulated Annealing*: *Theory and Applications.* D. Reidel, Dordrecht, 1987.

[5] W. Niemiro, *Limit distributions of simulated annealing Markov chains.* Discuss. Math. Algebra Stoch. Methods **15** (1995), 241–269.

[6] U. Păun, *General* Δ*-ergodic theory of finite Markov chains.* Math. Rep. (Bucur.) **8**(**58**) (2006), 83–117.

[7] U. Păun, *Perturbed finite Markov chains.* Math. Rep. (Bucur.) **9**(**59**) (2007), 183–210.

[8] U. Păun, Δ*-ergodic theory and simulated annealing.* Math. Rep. (Bucur.) **9**(**59**) (2007), 279–303.

[9] G. Winkler, *Image Analysis, Random Fields and Dynamic Monte Carlo Methods*: *A Mathematical Introduction*, 2nd Edition. Springer-Verlag, Berlin, 2003.

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