We study a class of Lindley processes whose distribution can be computed. We call them computable Lindley processes and investigate their applications in queueing theory and ruin theory.

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1. THE PROBLEM

I. A G/G/1 queue is defined by two independent sequences \((X_n)_{n \geq 1}\) and \((Y_n)_{n \geq 1}\) of i.i.d. random variables; \(X_n\) is the service time of the \(n\)th customer.

For \(n \geq 2\), \(Y_n\) is the time between the arrival moment of the \((n-1)\)th customer and the arrival moment of the \(n\)th customer while \(Y_1\) is the arrival moment of the first customer. Thus the \(n\)th customer arrives at \(t_n = Y_1 + \cdots + Y_n\). Let \(F_X, F_Y\) be the distributions of \(X_n\) and \(Y_n\). These two distributions are the parameters of the queue; we codify it as the queue \((F_X, F_Y)\).

It is important to keep in mind that we shall denote by the same letter both the distribution and its distribution function: if \(F\) is a distribution – a probability measure on the real line – then \(F(x)\) means actually \(F((−\infty, x])\). The right tail \(F((x, \infty))\) of \(F\), will be denoted by \(\mathcal{L}(x)\).

There are many characteristics of a queue worth to be studied. We shall focus on the waiting time of the \(n\)th customer: the time elapsed between joining the line and entering the service. This is a sequence of random variables \((W_n)_{n \geq 0}\) constructed by the recurrence

\[
W_1 = 0, \quad W_{n+1} = (W_n + X_n - Y_{n+1})_+,
\]

where \(x_+\) is the positive part of \(x\): \(x_+ = x\) for \(x > 0\) and \(x_+ = 0\) for \(x \leq 0\).

If we denote by \(Z_n\) the random variables \(X_n - Y_{n+1}\), then \((Z_n)_{n}\) will be a sequence of i.i.d. random variables having the distribution \(F = F_X * F_Y\). Replacing \(n\) by \(n-1\), (to start the count from 0) recurrence (1.1) becomes

\[
W_0 = 0, \quad W_{n+1} = (W_n + Z_n)_+.
\]

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Obviously, $W = (W_n)_{n \geq 0}$ is a Markov chain with the transition kernel
(1.3) $U_x(B) := P(W_{n+1} \in B \mid W_n = x) = (F \ast \delta_x)(B), \forall B \in \mathcal{R}$ Borel set.
If $G$ is a distribution on the real line $\mathcal{R}$, we denote by $G(\cdot)$ the distribution
(1.4) $G(\cdot)(B) = G(B \cap (0, \infty)) + \delta_0(B)G((-\infty, 0]).$

Sometimes it is easier to understand $G(\cdot)$ by means of its right tail: we clearly have
(1.5) $G(\cdot)(x) = G(x) \forall x \geq 0, \quad G(\cdot)(x) = 1$ if $x < 0.$

Let $G_n$ be the distribution of $W_n.$ Then we can write (1.2) as
(1.6) $G_0 = \delta_0, \quad G_{n+1} = (G_n \ast F)(\cdot).$

**Definition.** A Markov chain $(W_n)_{n \geq 0}$ given by recurrence (1.2) is called a Lindley process of parameter $F.$ Thus, its distributions are $G_1 = F(\cdot), \quad G_2 = (F(\cdot) \ast F)(\cdot), \quad G_3 = ((F(\cdot) \ast F)(\cdot) \ast F)(\cdot), \ldots.$

**II. A classic renewal risk model (the Sparre Andersen model)** is a stochastic process of the form
(1.7) $V(t) = S_{N(t)} - ct,$
where $S_0 = 0$ and $n \geq 1 \Rightarrow S_n = X_1 + \cdots + X_n, \quad N(t) = \sup \{k \mid t_k \leq t\}, \quad t_0 = 0$
and $k \geq 1 \Rightarrow t_k = \sigma_1 + \cdots + \sigma_k.$

Here, $(X_n)_{n \geq 1}$ and $(\sigma_n)_{n \geq 1}$ are two independent sequences of i.i.d. positive random variables which have the following meaning: the $X_n$ are the values of the claims coming from insured persons to an insurer and the $\sigma_n$ are the interarrival claim times. The $n$th claim has the size $X_n$ and comes at moment $t_n.$ The constant $c$ is the intensity of the cash flow coming from the insured persons. Then $V(t)$ is the loss of the insurer at time $t$: if $V(t) \leq 0,$ it is OK, but if the initial capital of the insurer is $u \geq 0$ and it happens that $V(t) > u,$ it is bad. The first moment $t$ when $V(t) > u$ stops the business: this is the ruin moment $\tau(u).$

Of course, the ruin may only occur at claim arrival moments $t_n.$ But
(1.8) $V(t_n) = (X_1 - c\sigma_1) + \cdots + (X_n - c\sigma_n) = Z_1 + \cdots + Z_n,$
where $Z_n = X_n - Y_n$ and $Y_n = c\sigma_n.$

Let $L_n = \max(0, S_1, \ldots, S_n),$ with $S_0 = 0$ and $n \geq 1 \Rightarrow S_n = Z_1 + \cdots + Z_n.$ Then $(L_n)_{n \geq 1}$ is not a Markov chain but, since
(1.9) $L_{n+1} = \max(0, Z_1 + \max(0, Z_2, Z_2 + Z_3, \ldots, Z_2 + \cdots + Z_{n+1}))$
$= (L_n^+ + Z_1)^+,$
it has the same distribution as a Markov chain given by the recurrence
(1.10) $L_0 = 0, \quad L_{n+1} = (L_n + Z_n)^+.$
This is exactly recurrence (1.2). Thus, $L_n$ has the same distribution as a Lindley process of parameter $F$, where $F$ is the distribution of $Z_n$. The right tail $\psi_n(u) = P(L_n > u)$ of $L_n$ is the probability that the ruin occurs before $n$ transactions.

It is well known (see, for instance [1]) that the necessary and sufficient condition that $L_n$ has a limit in distribution $L_\infty$ is that $EZ_n < 0$. Or, in terms of $X_n$ and $Y_n$, that $EX_n < EY_n$. The ratio $\frac{EX_n}{EY_n}$ is denoted in queuing theory by $\rho$ and it is called the traffic intensity. In the ruin theory the same ratio is denoted by $\frac{1}{1+\theta}$, where $\theta$ is the loading factor.

In other words, the sequence $(G_n)_n$ of distributions has a proper limit $G_\infty$ iff $\rho < 1$. We will always assume this condition in the sequel.

A last remark: the case where the distribution $F_Y$ is exponential is special. In ruin theory this is the Cramér-Lundberg model while in queuing theory is the “$M/G/1$ queue”. Most of the interesting results are proved under this hypothesis.

In short: if we are interested only in the waiting times $W_n$ in a queue or in the ruin probabilities in a risk model, then we deal with the same mathematical object: a Lindley process of parameter $F$, where $F = F_X * F_{-Y}$. We will also denote this Lindley process (maybe it is not very usual) by $\langle F_X, F_Y \rangle$ and its trajectories by $(L_n)_n$. The distribution of $L_n$ will be denoted by $G_n$; hence

$$G_0 = \delta_0, \quad G_{n+1} = (G_n * F)_{(\cdot)^{+}}.$$ 

Now, we state the problem. Suppose that we want to compare two Lindley processes: the first one has parameter $F$ (and distributions $G_n$) and the second one has parameter $F'$ (and distributions $G'_n$). Write $F \prec_{++} F'$ iff $G_n \prec_{st} G'_n \forall n$ and $F \prec_{st} F'$ iff $G_\infty \prec_{st} G'_\infty$. Or, in terms of random variables, $Z \prec_{++} Z' \iff L_n \prec_{st} L'_n \forall n$ and $Z \prec_{st} Z' \iff L_\infty \prec_{st} L'_\infty$. Do we have computational criteria to decide if $F \prec_{++} F'$ or $F \prec_{st} F'$?

We shall deal with particular types of Lindley process of the form $\langle F_X, F_Y \rangle$ and $\langle F_X', F_Y' \rangle$. In that case we write $\langle F_X, F_Y \rangle \prec \langle F_X', F_Y' \rangle$ instead of $F_X * F_{-Y} \prec_{++} F_X' * F_{-Y}'$ and $\langle F_X, F_Y \rangle \prec \langle F_X', F_Y' \rangle$ instead of $F_X * F_{-Y} \prec_{++} F_X' * F_{-Y}'$.

In [4] was considered the particular case $F_X' = \text{Exp}(a)$, $F_Y = \text{Exp}(b)$, $F_X = \text{Exp}(a')$, $F_Y' = \text{Exp}(b')$. The parameters of the two Lindley processes are $F_{a,b} = F_X * F_{-Y}$ and $F_{a',b'} = F_X' * F_{-Y}'$. In that situation, the result was $F_{a,b} \prec_{++} F_{a',b'} \iff a \geq a'$, $\rho \leq \rho'$, where $\rho = \frac{EX}{EY} = \frac{b}{a}$, $\rho' = \frac{EX'}{EY'} = \frac{b'}{a'}$. In terms of queues, the first queue is better if the service time is smaller and the traffic intensity is smaller. In the same paper a counterexample was given of two Lindley processes $\langle F_X, F_Y \rangle$ and $\langle F_X', F_Y' \rangle$ for which this is not true: $\rho = \rho'$, $X' \prec_{st} X$ and, on the contrary, $\langle F_X, F_Y \rangle \prec \langle F_X', F_Y' \rangle$. The domination goes in the opposite way.
The result was generalized a bit in [5]: suppose that the pairs \( \langle F_X, F_Y \rangle \) and \( \langle F_{X'}, F_{Y'} \rangle \) are conjugated, to mean that

\[
X, X' > 0 \text{ (a.s.)}, \quad F_X * F_{-Y} = pF_X + qF_{-Y}, \quad F_{X'} * F_{-Y'} = p'F_{X'} + q'F_{-Y'}.
\]

(1.11) \( X, X' > 0 \) (a.s.), \( F_X * F_{-Y} = pF_X + qF_{-Y}, \) \( F_{X'} * F_{-Y'} = p'F_{X'} + q'F_{-Y'} \)

In this case a weaker result than that from [4] holds, namely:

\[
X \prec_X X' \text{ and } \rho \leq P \Rightarrow \langle F_X, F_{-Y} \rangle \prec_X \langle F_{X'}, F_{-Y'} \rangle \Rightarrow \]

\[
F_X \leq \frac{P}{P'} F_{X'} \text{ and } \rho \leq P',
\]

where \( \rho = \frac{E_X}{E_Y} \) and \( \rho' = \frac{E_{X'}}{E_{Y'}} \).

Remark 1.1. Property (1.11) means that the convolution of \( F_X \) and \( F_{-Y} \) is a mixture of these distributions, and the same property hold for the convolution of \( F_{X'} \) and \( F_{-Y'} \). Notice that if this is indeed the case, then \( p \) and \( p' \) are uniquely determined by the expectations. It is easy to see that

\[
p = \frac{\rho}{\rho + 1}, \quad p' = \frac{\rho'}{\rho' + 1}; \quad q = \frac{1}{\rho + 1}, \quad q' = \frac{1}{\rho' + 1},
\]

where \( \rho = \frac{E_X}{E_Y} \) and \( \rho' = \frac{E_{X'}}{E_{Y'}} \).

In this paper we generalize this result and give more examples of Lindley processes for which the first implication from (1.12) holds. In terms of ruin theory, we want to give a partial answer to the question: for what kind of risk processes it is still true that a process with smaller claims and greater loading is safer?

Remark 1.2. We shall use the following notation:

- Negbin\((k, p)\) is the negative binomial distribution of parameters \( k \) and \( p \), that is the \( k \)th convolution of Negbin\((1, p)\) := \( 0 1 2 3 \cdots \)
  \[
  p \begin{array}{cccc}
    0 & 1 & 2 & 3 \\
    p & pq & pq^2 & pq^3 \\
  \end{array}
  \cdots \] ;

- Negbin\((k, -p)\) is the distribution of \(-X\) if \( X \sim \text{Negbin}(k, p)\); it is the \( k \)th convolution of Negbin\((1, -p)\) := \( 0 -1 -2 -3 \cdots \)
  \[
  p \begin{array}{cccc}
    0 & -1 & -2 & -3 \\
    p & pq & pq^2 & pq^3 \\
  \end{array}
  \cdots \] ;

- Gamma\((m, a)\) = Exp\((a)^m\);

- Geo\((m, p)\) = Geo\((1, p)^m = \text{Negbin}(m, p) \ast \delta_m\);

- Exp\((-b)\) is the distribution with the distribution function \( be^{bx}, x \leq 0 \).

It is the distribution of \(-X\) for \( X \sim \text{Exp}(b)\):

- Geo\((1, p)\) = \( \begin{array}{cccc}
    1 & 2 & 3 & 4 \\
    p & pq & pq^2 & pq^3 \\
  \end{array} \cdots \) = Negbin\((1, p) \ast \delta_1\)

- Geo\((m, p)\) = Geo\((1, p)^m = \text{Negbin}(m, p) \ast \delta_m\);

- Poisson\((-\lambda)\) is the distribution of \(-X\) for \( X \sim \text{Geo}(m, p)\);

- Uniform\((a, b)\) is the distribution with density \( 1/(b - a) \).
2. COMPUTABLE LINDLEY PROCESSES

The crucial fact in the approach in [4] and [5] was the possibility to derive an algebraic formula for $G_n$ of the form

\[(2.1) \quad G_n = \Gamma Q^n e,\]

where

- $\Gamma$ is a infinitely dimensional row vector $(\Gamma_0, \Gamma_1, \Gamma_2, \ldots)$ with components the distributions $\Gamma_n = F_X * F_X * \cdots * F_X$ (the $n$th convolution of $F_X$); by convention, $\Gamma_0 = \delta_0$;
- $Q$ is an infinite dimensional column stochastic matrix; and
- $e$ is the column vector $(1, 0, 0, \ldots)$.

In the case considered in [4] and [5], the components of the matrix $Q$ were

\[q(i, j) = \begin{cases} 
q_{i+1} & \text{if } i = 0 \\
 pq_{j-i+1} & \text{if } i \geq 1, \ j \geq i - 1 \\
0 & \text{if } j < i - 1.
\end{cases}\]

Here, $p = \frac{\rho}{\rho + 1} = \frac{E_X}{E_X + E_Y}$ and $q = \frac{1}{\rho + 1} = \frac{E_Y}{E_X + E_Y}$. Or, explicitly,

\[Q = \begin{pmatrix} 
r_0 & r_{-1} & r_{-2} & r_{-3} & \cdots \\
p_1 & p_0 & p_{-1} & p_{-2} & \cdots \\
0 & 0 & p_1 & p_0 & \cdots \\
0 & 0 & 0 & p_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},\]

If $\pi = \begin{pmatrix} \cdots & -2 & -1 & 0 & 1 \\
\cdots & pq^3 & pq^2 & pq & p \end{pmatrix}$ and we write $p_i$ instead of $\pi(\{i\})$, then we have

\[Q = \begin{pmatrix} 
r_0 & r_{-1} & r_{-2} & r_{-3} & \cdots \\
p_1 & p_0 & p_{-1} & p_{-2} & \cdots \\
0 & 0 & p_1 & p_0 & \cdots \\
0 & 0 & 0 & p_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},\]

where $r_k = \pi((-\infty, k])$. Now, we see that the transposed of $Q$ is the transition matrix of a new Lindley process of parameter $\pi$. Indeed, if we consider a sequence $(\xi_n)_n$ of i.i.d. random variables with distribution $\pi$ and the Markov chain given by the recurrence $N_{t+1} = (N_t + \xi_{t+1})^+$, starting from some $N_0$
independent of the $\xi_n$, then

\begin{align*}
P(N_{t+1} = n \mid N_t = k) &= P((N_t + \xi_{t+1})_+ = n \mid N_t = k) \\
&= P((k + \xi_{t+1})_+ = n \mid N_t = k) = P((k + \xi_{t+1})_+ = n)
\end{align*}

(because of independence!). Hence

\begin{align*}
P(N_{t+1} = 0 \mid N_t = k) &= P(\xi_{t+1} \leq -k) = r_{-k} \quad \text{if } n = 0 \quad \text{and} \\
P(N_{t+1} = n \mid N_t = k) &= P(\xi_{t+1} = n - k) = \pi(\{n - k\}) = p_{n-k} \quad \text{if } n \geq 1.
\end{align*}

Therefore,

\begin{align*}
P(N_{t+1} = n \mid N_t = k) &= q_{n,k}.
\end{align*}

We can restate this remark in terms of random variables as follows.

Let $\mathbf{N} = (N_n)_{n \geq 0}$ be a Lindley process of parameter $\pi$ starting from 0. Let $\mathbf{X} = (X_n)_n$ be a sequence of i.i.d. random variables with distribution $F_X$. Assume that $\mathbf{X}$ is independent of $\mathbf{N}$. Let $S(0) = 0$ and for $n \geq 1$ let $S(n) = X_1 + \cdots + X_n$. Then $L_n$ has the same distribution as $S(N_n)$.

(It seems that what really matters is the distribution $F_X$ while $F_{-Y}$ plays a secondary role.). This remark motivated us to give the following

**Definition.** Let $\mathbf{L} = (L_n)_n$ be a Lindley process of parameter $F$. If the distribution $G_n$ of $L_n$ can be written as

\begin{equation}
G_n = \Gamma Q^n e,
\end{equation}

where $\Gamma_n = \mu^{*m} = \mu * \mu * \cdots * \mu$, (the $m$th convolution of $\mu$), $\mu$ is a distribution on $(0, \infty)$ and $Q$ is the transposed of a transition matrix of a Lindley process on the set of nonnegative integers of parameter

\begin{equation}
\pi = \begin{pmatrix}
\ldots & k-3 & k-2 & k-1 & k \\
\ldots & p_{k-3} & p_{k-2} & p_{k-1} & p_k
\end{pmatrix}
\end{equation}

for some $k \geq 1$, then we say that $L$ is a **computable Lindley process with characteristics** $\mu$ and $\pi$.

**Remark 2.1.** It is easy to see that the above definition is equivalent to the existence (possibly on another suitable probability space) of a sequence $\mathbf{U} = (U_n)_{n \geq 1}$ of i.i.d. positive random variables with distribution $\mu$ and of a discrete Lindley process $\mathbf{N} = (N_n)_{n \geq 0}$ of parameter $\pi$, which is independent of $\mathbf{U}$ and starts from 0 such that

\begin{equation}
L_n \overset{D}{=} \sum_{i \leq N_n} U_i.
\end{equation}

The Lindley process $N_n$ is given by the recurrence

\begin{equation}
N_0 = 0, \quad N_{n+1} = (N_n + \xi_{n+1})_+
\end{equation}
and the distribution of $\xi_n$ is $\pi$. Next, the $m$th column of $Q$ is the distribution of $(m + \xi_{n+1})_+$, namely,

$$q_{i,m} = P((m + \xi_{n+1})_+ = i) = \begin{cases} P(\xi_{n+1} \leq -m) = r_{-m} & \text{if } i = 0 \\ P(\xi_{n+1} = i - m) = p_{i-m} & \text{if } i \geq 0 \end{cases}$$

Remark 2.2. It is very important that the support of $\pi$ is included in $(-\infty, k]$. We call such a Lindley process computable because if we can compute the convolutions $\Gamma_m$ for $m \leq kn$, then we can also compute $G_n$. This happens because $Q^n e$ is the first column of $Q^n$ and has at most $kn + 1$ nonzero entries. We do not need to keep in memory all the entries of $Q$ – which is, of course, impossible. The reader can check that the product $Q^n e$ is the same as $Q_n Q_{n-1} \cdots Q_2 Q_1 e$, where the matrices $Q_m$ are obtained from $Q$ as follows:

- $Q_1$ is the $(k+1) \times 1$ matrix obtained from $Q$ keeping the first column and the first $k+1$ rows;
- $Q_2$ is the $(2k+1) \times (k+1)$ matrix obtained from $Q$ keeping the first $k+1$ columns and the first $2k+1$ rows;
- $\cdots$
- $Q_m$ is the $(mk+1) \times (mk - k + 1)$ matrix obtained from $Q$ keeping only the first $mk - k + 1$ columns and the first $mk + 1$ rows.

Moreover, as $q_{i,m} = p_{i-m}$ we do not need to keep in memory the matrices $Q_m$ but only the last $nk + 1$ entries of $\pi$. For $k = 9$, we were able to compute $(G_n)_n$ for $n \leq 500$.

Remark 2.3. It is obvious that any Lindley process of parameter $F = \begin{pmatrix} \cdots & (m-3)h & (m-2)h & (m-1)h & mh \\ \cdots & p_{m-3} & p_{m-2} & p_{m-1} & p_m \end{pmatrix}$ is computable. Here, $\Gamma_n = \delta_{nh}$ and $Q$ is the same matrix with $h = 1$.

We shall give some examples of computable Lindley processes. May be, there are smarter methods of proof, ours are computational. All the examples we know fit into the pattern described below.

**Theorem.** Let $\mu$ be a probability distribution on $(0, \infty)$ and let $\Gamma_n = \mu^*n$, $\Gamma_0 = \delta_0$. Suppose that

(i) $F_X = \Gamma_m$ for some $m \geq 1$;
(ii) $F_{Y}$ is a distribution on $(-\infty, 0]$ such that

$$\Gamma_n + F_{Y})_{(+} = q_{0,n} \Gamma_0 + q_{1,n} \Gamma_1 + \cdots + q_{n,n} \Gamma_n.$$

Let $Q_{n,m}$ be the column vector $(q_{0,n}, q_{1,n}, \ldots, q_{n,n}, 0, 0, 0, \ldots)$ and $Q$ the column stochastic matrix obtained by putting together the columns starting with
Q_{.,m}, \, i.e.,
\begin{equation}
Q = (Q_{.,m}; Q_{.,m+1}; Q_{.,m+2}; \ldots).
\end{equation}

Then the distribution of $L_n$ is
\begin{equation}
G_n = \Gamma Q^n e,
\end{equation}
where $\Gamma$ is the row vector $(\Gamma_0, \Gamma_1, \Gamma_2, \ldots)$ and $e$ the column vector $(1, 0, 0, \ldots)$.

Proof. For $n = 1$ the claim is true since $G_1 = \Gamma Q e$. Suppose it holds for $n$, and let us check it for $n + 1$. We then have
\begin{align*}
G_n &= a_{n,0} \Gamma_0 + a_{n,1} \Gamma_1 + \cdots + a_{n,nm} \Gamma_{nm} \Rightarrow G_{n+1} = (G_n * \Gamma_m * F_{-Y})(+) = \\
&= ((a_{n,0} \Gamma_m + a_{n,1} \Gamma_{m+1} + \cdots + a_{n,nm} \Gamma_{nm+m}) * F_{-Y})(+) = \\
&= a_{n,0} \Gamma_m * F_{-Y} + a_{n,1} \Gamma_{m+1} * F_{-Y} + \cdots + a_{n,nm} \Gamma_{nm+m} * F_{-Y})(+) = \\
&= a_{n,0} (\Gamma_m * F_{-Y})(+) + a_{n,1} (\Gamma_{m+1} * F_{-Y})(+) + \cdots + a_{n,nm} (\Gamma_{nm+m} * F_{-Y})(+)
\end{align*}
(provided $\mu + \nu$)
\begin{align*}
&= a_{n,0} \Gamma_0 Q_{.,m} + a_{n,1} \Gamma_1 Q_{.,m+1} + \cdots + a_{n,nm} \Gamma_{nm+m} \\
&= \Gamma (a_{n,0} Q_{.,m} + a_{n,1} Q_{.,m+1} + \cdots + a_{n,nm} Q_{.,nm+m}) = \\
&= \Gamma (Q_{.,m}; Q_{.,m+1}; Q_{.,m+2}; \ldots)(a_{n,0}, a_{n,1}, \ldots, a_{n,nm}, 0, 0, \ldots)' \tag{here $\nu'$ is the transposed of the vector $\nu$}
\end{align*}
(by induction assumption) = $\Gamma Q^{n+1} e$. □

To use this result, we give a combinatorial lemma.

Lemma 2.1. Let $(A, +, \Delta)$ be an algebra over a commutative field $K$. Let $a, b \in A$. Suppose that
\begin{equation}
ab = ba = pa + qb,
\end{equation}
where $p, q \in K$. Then
\begin{equation}
a^m b^k = \sum_{m=1}^{n} p_{k,n-m} a^m + \sum_{i=1}^{k} q_{n,k-i} b^i,
\end{equation}
where $p_{k,m} = \binom{k+m-1}{k-1} p^k q^m$ and $q_{n,i} = \binom{n+i-1}{n-1} p^i q^n$.

Proof. Induction. Notice that $p_{1,m} = pq^m$, $q_{1,i} = qp^i$, $p_{k,0} = p^k$ and $q_{n,0} = q^n$. For $n = k = 1$, (2.8) becomes $ab = p_{1,0} a + q_{1,0} b = pa + qb$ and this is our very assumption (2.7). Next, $a^2 b = (pa + qb)a = pa^2 + qab = pa^2 + qpa + q^2 b$ $\Rightarrow a^3 b = (pa^2 + qpa + q^2 b)a = pa^3 + pqa^2 + q^2 (pa + qb)$; by recurrence, it is clear that (2.8) holds for $k = 1$ and for $n = 1$, that is,
\begin{equation}
a^n b = \sum_{m=1}^{n} pq^{n-m} a^m + q^n b, \quad ab^k = p^k a + \sum_{i=1}^{k} qp^{k-i} b^i.
\end{equation}
It follows that (2.8) holds for \( n = 1 \) or for \( k = 1 \). Suppose it holds for a pair \((n, k)\). We shall check it for \((n + 1, k)\) and \((n, k + 1)\), and this will end the proof. We have

\[
a^{n+1}b^k = a(a^n b^k) = a \left( \sum_{m=1}^{n} p_{k,n-m} a^m + \sum_{i=1}^{k} q_{n,k-i} b^i \right) =
\]

\[
= \sum_{m=1}^{n} p_{k,n-m} a^{m+1} + \sum_{i=1}^{k} q_{n,k-i} a b^i =
\]

\[
= \sum_{m=2}^{n+1} p_{k,n-m+1} a^m + \sum_{i=1}^{k} q_{n,k-i}(p^i a + \sum_{s=1}^{i} q p^{i-s} b^s) =
\]

\[
= \sum_{m=2}^{n+1} p_{k,n-m+1} a^m + a \sum_{i=1}^{k} q_{n,k-i} p^i + \sum_{i=1}^{k} \sum_{s=1}^{i} q_{n,k-i} q p^{i-s} b^s.
\]

Now,

\[
q_{n,k-i} = \left( \frac{n + k - i - 1}{n - 1} \right) \Rightarrow
\]

\[
\Rightarrow \sum_{i=1}^{k} q_{n,k-i} p^i = \sum_{i=1}^{k} \left( \frac{n + k - i - 1}{n - 1} \right) p^i q^n = \left( \frac{n + k - 1}{n} \right) p^k q^n = p_{k,n}
\]

hence

\[
\sum_{m=2}^{n+1} p_{k,n-m+1} a^m + a \sum_{i=1}^{k} q_{n,k-i} p^i = \sum_{m=1}^{n+1} p_{k,n-m+1} a^m.
\]

Next, interchanging the summation order we get

\[
\sum_{i=1}^{k} \sum_{s=1}^{i} q_{n,k-i} q p^{i-s} b^s = \sum_{s=1}^{k} \sum_{i=s}^{k} q_{n,k-i} q p^{i-s} b^s =
\]

\[
= \sum_{s=1}^{k} \sum_{i=s}^{k} \left( \frac{n + k - i - 1}{n - 1} \right) p^i q^n q p^{i-s} b^s =
\]

\[
= \sum_{s=1}^{k} \left( \sum_{i=s}^{k} \left( \frac{n + k - i - 1}{n - 1} \right) \right) q^{n+1} p^{k-s} b^s = \sum_{s=1}^{k} \left( \frac{n + k - s}{n} \right) q_{n+1,k-s} b^s,
\]

hence (2.8) holds in this case, too.

The fact that (2.8) holds for the pair \((n, k + 1)\) can be proved in the same manner. □

**Corollary 2.2.** Let \( \mu \) and \( \nu \) be two conjugated distributions, to mean that

\[
(2.10) \quad \mu * \nu = p\mu + q\nu \quad \text{for some } p, q \geq 0, \ p + q = 1.
\]
Then
\begin{equation}
\mu^{sn} \ast \nu^{rk} = \sum_{m=1}^{n} p_{k,n-m} \mu^{sm} + \sum_{i=1}^{k} q_{n,k-i} \nu^{si},
\end{equation}
where
\[ p_{k,m} = \binom{k + m - 1}{k - 1} p^k q^m = \text{Negbin}(k,p)(\{m\}) \]
and
\[ q_{n,i} = \binom{n + i - 1}{n - 1} p^i q^n = \text{Negbin}(n,q)(\{i\}). \]

Proof. This is an obvious particular case of Lemma 2.1. □

Remark 2.4. Notice that since \(\mu^{sn} \ast \nu^{rk}\) is a distribution, the equation below should also hold
\begin{equation}
\sum_{m=1}^{n} p_{k,n-m} \mu^{sm} + \sum_{i=1}^{k} q_{n,k-i} \nu^{si} = \text{Negbin}(k,p)(\{0, 1, \ldots, n - 1\}) + \text{Negbin}(n,q)(\{0, 1, \ldots, k - 1\}) = 1.
\end{equation}
This result gives us a formula for the tail of the negative binomial distribution which, maybe, is well known. Not to us. Anyway, as a consequence we have

Corollary 2.3. Let \(\mu\) and \(\nu\) be two conjugated distributions, to mean
\begin{equation}
(0, \infty) = \nu((0, \infty)) = \nu((-\infty, 0]) = 1. \text{ Then}
\end{equation}
\begin{equation}
(\mu^{sn} \ast \nu^{rk})(+) = r_{k,n} \Gamma_0 + \sum_{m=1}^{n} p_{k,n-m} \Gamma_m,
\end{equation}
where \(\Gamma_m = \mu^{sm}\) (with the convention \(\Gamma_0 = \delta_0\)) and \(r_{k,n}\) is the tail of \(\text{Negbin}(k,p)\) given by
\begin{equation}
 r_{k,n} = \text{Negbin}(k,p)([n, \infty)) = \text{Negbin}(n,q)(\{0, 1, \ldots, k - 1\})
\end{equation}
(the last equality follows from (2.12)!)\)

Proof. We have
\begin{align*}
(\mu^{sn} \ast \nu^{rk})(+) &= \left( \sum_{m=1}^{n} p_{k,n-m} \mu^{sm} + \sum_{i=1}^{k} q_{n,k-i} \nu^{si}\right)(+) = \\
&= \sum_{m=1}^{n} p_{k,n-m}(\mu^{sm})(+) + \sum_{i=1}^{k} q_{n,k-i}(\nu^{si})(+).
\end{align*}
(since \((\mu + \nu)(+) = \mu(+) + \nu(+)\)) for any measures \(\mu, \nu\)
\[
= \sum_{m=1}^{n} p_{k,n-m}\mu^m + \sum_{i=1}^{k} q_{n,k-i}\delta_0 = 
\]
(because \(\mu((0, \infty)) = 1 \Rightarrow \mu = \mu_+\) and \(\nu((-\infty, 0]) = 1 \Rightarrow \nu(+) = \delta_0\))
\[
= r_{k,n}\Gamma_0 + \sum_{m=1}^{n} p_{k,n-m}\Gamma_m.\]

Now, we can give our first example of a computable Lindley process.

**Proposition 2.4.** Let \(\mu\) and \(\nu\) be two distributions. Suppose that

(i) \(\mu((0, \infty)) = \nu((-\infty, 0]) = 1\);

(ii) \(F_X = \mu^*\), \(F_Y = \nu^k\), \(m, k \geq 1\);

(iii) \(\mu\) and \(\nu\) are conjugated, to mean (2.10).

Consider the Lindley process \(L = (L_n)_{n \geq 0}\) of parameter \(F = F_X \ast F_Y\). Then \(L\) is computable. Precisely, \(L_n \overset{D}{=} \sum_{i \leq N_n} U_i\) where \(U = (U_i)_{i \geq 1}\) is a sequence of i.i.d. random variables with distribution \(\mu\) and \((N_n)\), a Lindley process with parameter
\[
(2.15)\quad \pi = \begin{pmatrix}
\cdots & m-3 & m-2 & m-1 & m \\
\cdots & p_{k,3} & p_{k,2} & p_{k,1} & p_{k,0}
\end{pmatrix},
\]
where \(p_{k,n} = \text{Negbin}(k, p)(\{n\}) = \binom{k+n-1}{k-1} p^k q^n\). In terms of (2.2), \(\Gamma_n = \mu^m\) and
\[
(2.16)\quad Q = \begin{pmatrix}
\cdots & r_{k,m} & r_{k,m+1} & r_{k,m+2} & r_{k,m+3} & r_{k,m+4} & \cdots \\
\cdots & p_{k,m-1} & p_{k,m} & p_{k,m+1} & p_{k,m+2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
p_{k,1} & p_{k,2} & p_{k,3} & p_{k,4} & p_{k,5} & \cdots \\
p_{k,0} & p_{k,1} & p_{k,2} & p_{k,3} & p_{k,4} & \cdots \\
0 & 0 & p_{k,0} & p_{k,1} & p_{k,2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix},
\]
where \(r_{k,n} = p_{k,n} + p_{k,n+1} + p_{k,n+2} + \cdots\).

**Proof.** This is an immediate consequence of our theorem above. \(\square\)

**Corollary 2.5.** A Lindley process \(L = (L_n)_{n \geq 0}\) of parameter \(F = F_X \ast F_Y\) is computable if

(i) \(F_X = \text{Gamma}(m, a)\) and \(F_Y = \text{Gamma}(k, b)\). Here, \(\Gamma\) from (2.2) has the components \(\Gamma_n = \text{Gamma}(n, a)\) and \(Q\) is the matrix (2.16) with \(p = \frac{b}{a+b}\).
\[ q = \frac{a}{a+b}. \] The limit distribution \( G_\infty = G_\infty(m, a; k, b) \) does exist iff \( \frac{m}{a} < k < \frac{b}{a} \).

(ii) \( F_X = \text{Geom}(m, \alpha) \) and \( F_Y = \text{Negbin}(k, \beta). \) Here, \( \Gamma \) from (2.2) has the components \( \Gamma_n = \text{Geom}(n, \alpha) \) and \( Q \) is the matrix (2.16) with \( p = \frac{\beta}{\alpha+\beta-\alpha \beta} \).

The limit distribution \( G_\infty = G_\infty(m, \alpha; k, \beta) \) does exist iff \( m/\beta < k\alpha(1-\beta). \)

**Proof.** (i). We know [4] that if \( \mu = \text{Exp}(a) \), \( \nu = \text{Exp}(-b) \), then \( \mu * \nu = \frac{b}{a+b} \mu + \frac{a}{a+b} \nu. \) As \( F_X = \mu^{*m} \) and \( F_Y = \nu^{*k} \), we can apply Proposition 2.4. It is well known that the limit distribution does exist iff \( EX_n < \infty \) for some \( n \).

(ii). It was proved in [5] that if \( \mu = \text{Geom}(1, \alpha) \), \( \nu = \text{Negbin}(1, \beta) \), then \( \mu * \nu = \frac{\beta}{\alpha+\beta-\alpha \beta} \mu + \frac{\alpha-\alpha \beta}{\alpha+\beta-\alpha \beta} \nu. \) The same proof as before: \( F_X = \mu^{*m} \) and \( F_Y = \nu^{*k} \). As \( EX_n = \frac{m}{\alpha} \) and \( EY_n = \frac{k(1-\beta)}{\beta} \), \( G_\infty \) does exist iff \( \frac{m}{\alpha} < \frac{k(1-\beta)}{\beta}. \)

In other words, we arrived at the conclusion that if \( X_n \) are i.i.d. distributed as Gamma\((m, a)\), \( Y_n \) are i.i.d. distributed as Gamma\((k, b)\), \( Z_n = X_n - Y_n, S_0 = 0, S_n = Z_1 + Z_2 + \cdots + Z_n \) for \( n \geq 1 \), then we can compute the distribution of \( L_n = \max(S_0, S_1, \ldots, S_n) \). It can be represented as \( L_n = \sum_{j=1}^{N_n} \xi_j \), where \( \xi_j \sim \text{Exp}(a) \) and \( (N_n) \) is another Lindley process of parameter \( \text{Negbin}(k, -p) * \delta_n \) with \( p = \frac{b}{a+b} \). What does happen if \( Y_n \) is constant? This is our next example.

**Proposition 2.6.** Suppose that \( X_n \sim \text{Gamma}(m, \lambda) \) and \( Y_n = b = \text{constant}. \) Then \( L \) is a computable Lindley process. Precisely, \( L_n = \sum_{i \leq N_n} U_i \), where \( U = (U_i)_{i \geq 1} \) is a sequence of i.i.d. random variables distributed as \( \text{Exp}(\lambda) \) and \( (N_n) \) is a Lindley process independent of \( U \) with parameter

\[
(2.17) \quad \pi = \begin{pmatrix}
\cdots & m-3 & m-2 & m-1 & m & \cdots \\
\cdots & p_{b, \lambda, 3} & p_{b, \lambda, 2} & p_{b, \lambda, 1} & p_{b, \lambda, 0} & \cdots 
\end{pmatrix} = \delta_m * \text{Poisson}(-b\lambda),
\]

where \( p_{a,n} = \text{Poisson}(a)(\{n\}) = \frac{a^n}{n!}e^{-a}. \) The limit distribution \( G_\infty \) does exist iff \( b\lambda > m. \)

**Proof.** In order to apply our theorem, we notice that \( F_X = \Gamma_m \), where \( \Gamma_n = \text{Gamma}(n, \lambda) = \text{Exp}(\lambda)^n. \) We have to compute \( (\Gamma_n * F_{-Y})_{(1)} = (\Gamma_n * \delta_{-b})_{(1)} \) in order to check (2.5). Let \( U \sim \text{Exp}(\lambda) \) and \( \xi = U-b \). The distribution of \( \xi \) is \( F := \Gamma_1 * \delta_{-b} \). The density of \( \xi \) is \( f(x) = \lambda e^{-\lambda(b+x)}1_{(-b, \infty)}(x) \). We write \( F \) as a sum of two measures, the first concentrated on \((0, \infty)\) and the second on \((-\infty, 0)\). Precisely, write \( F = \mu + \nu \) with \( \mu, \nu \) absolutely continuous such that \( d\mu(x) = f(x)1_{(0, \infty)}(x)dx \) and \( d\nu(x) = f(x)1_{(-b, 0]}(x)dx \). Thus, \( \mu = e^{-\lambda} \Gamma_1 \)
and supp(\(\nu\)) = [\(-b, 0\)]. We claim that

\[
(2.18) \quad \Gamma_n * \nu = p_{b\lambda,n} \Gamma_1 + p_{b\lambda,n-1} \Gamma_2 + \cdots + p_{b\lambda,1} \Gamma_n + \nu_n,
\]

where \(\nu_n\) is a measure concentrated on \([-b, 0]\). If this holds, then we can write

\[
\Gamma_n * \delta_{-b} = \Gamma_{n-1} * (\Gamma_1 * \delta_{-b}) = \Gamma_{n-1} * (\mu + \nu) = \Gamma_{n-1} * (e^{-\lambda b} \Gamma_1 + \nu) = p_{b\lambda,0} \Gamma_n + \Gamma_{n-1} * \nu = p_{b\lambda,0} \Gamma_n + p_{b\lambda,1} \Gamma_{n-1} + \cdots + p_{b\lambda,n-1} \Gamma_1 + \nu_n,
\]

hence

\[
(2.19) \quad (\Gamma_n * F_Y)(+1) = r_{b\lambda,n} \Gamma_0 + p_{b\lambda,n-1} \Gamma_1 + \cdots + p_{b\lambda,1} \Gamma_{n-1} + p_{b\lambda,0} \Gamma_n,
\]

where \(r_{\lambda b,n} = \text{Poisson}(\lambda b)\([n, \infty)\). By (2.6) the matrix \(Q\) from (2.2) would have the columns \((Q_m; Q_{m+1}; \ldots)\), where the transposed of \(Q_m\) is \((r_{\lambda b,n}, p_{\lambda b,n-1}, \ldots, p_{\lambda b,1}, p_{\lambda b,0}, 0, 0, \ldots)\). So, in order to complete the proof we have to check (2.18). We compute the density of \(\Gamma_n * \nu\):

\[
f_{\Gamma_n * \nu}(x) = \int f_{\nu}(x - y) f_{\Gamma_n}(y) dy
\]

\((f_{\nu}\) is the density of \(\nu\) and \(f_{\Gamma_n}(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} 1_{[0, \infty)}(x)\) is the density of \(\Gamma_n = \text{Gamma}(n, \lambda)\))

\[
= \int e^{-\lambda(x-y+b)} 1_{(-b,0]}(x-y) \frac{\lambda^n y^{n-1}}{(n-1)!} e^{-\lambda y} 1_{[0, \infty)}(y) dy = \int \frac{\lambda^n}{(n-1)!} \frac{y^{n-1}}{\lambda^n} e^{-\lambda x + b} dy = \frac{(\lambda^n)}{n!} e^{-\lambda x + b} - \frac{(\max(x,0))^n}{n!}
\]

Thus, if \(x \in [-b, 0]\) then \(f_{\Gamma_n * \nu}(x) = \frac{\lambda^n + e^{-\lambda(x+b)} (b+x)^n}{n!}\). This is the density of \(\nu_n\).

If \(x > 0\) then

\[
f_{\Gamma_n * \nu}(x) = \lambda^n e^{-\lambda(x+b)} (b+x)^n e^{-\lambda x} = \frac{\lambda^n}{n!} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} e^{-\lambda x} = \sum_{k=0}^{n-1} \frac{\lambda^k}{k!} \frac{x^{n-k}}{(n-k)!} e^{-\lambda x} = \sum_{k=0}^{n-1} f_{\Gamma_{k+1}}(x) p_{\lambda,b,n-k} \Rightarrow \Gamma_n * \nu = \sum_{k=0}^{n-1} p_{\lambda,b,n-k} \Gamma_{k+1} + \nu_n.
\]

A slight generalization is
Corollary 2.7. Suppose that $X_n \sim \text{Gamma}(m, \lambda)$ and 

$$Y_n \sim \left( \begin{array}{cccc} b_1 & b_2 & \cdots & b_k \\ \beta_1 & \beta_2 & \cdots & \beta_k \end{array} \right)$$

with $0 < b_1 < \cdots < b_k$. Then $L$ is a computable Lindley process. Precisely, $L_n \overset{D}{=} \sum_{i \leq N_n} U_i$ where $U = (U_i)_{i \geq 1}$ is a sequence of i.i.d. random variables distributed as $\text{Exp}(\lambda)$ and $(N_n)_n$ is a Lindley process independent of $U$ with parameter

$$\pi = \sum_{i=1}^{k} \beta_i \pi_i \text{ with } \pi_i = \left( \begin{array}{cccc} \cdots & m-3 & m-2 & m-1 & m \\ \cdots & p_{b_i \lambda,3} & p_{b_i \lambda,2} & p_{b_i \lambda,1} & p_{b_i \lambda,0} \end{array} \right),$$

where $p_{a,n} = \text{Poisson}(a)(\{n\}) = \frac{a^n}{n!} e^{-a}$.

The limit distribution $G_\infty$ does exist iff $\lambda (\beta_1 b_1 + \cdots + \beta_k b_k) > m$.

Proof. We know from (2.19) that

$$(\Gamma_n * \delta_{-b})_{(+)} = r_{b \lambda, n} \Gamma_0 + p_{b \lambda, n-1} \Gamma_1 + \cdots + p_{b \lambda, 1} \Gamma_{n-1} + p_{b \lambda, 0} \Gamma_n.$$  

This means that

$$(\Gamma_n * \mathcal{F}_{-Y})_{(+)} = \sum_{i=1}^{k} \beta_i \Gamma_n * \delta_{-b_i} = \sum_{i=1}^{k} \beta_i (r_{b \lambda, n} \Gamma_0 + p_{b \lambda, n-1} \Gamma_1 + \cdots + p_{b \lambda, 0} \Gamma_n).$$

Applying our theorem completes the proof. $\square$

Remark 2.5. What does happen if we interchange $X$ and $Y$? Precisely, what if $X_n = a$ is constant while the $Y_n$ are distributed as $\text{Gamma}(m, \lambda)$? Well, in this case $(F_X, F_Y)$ is not a computable Lindley process. Our theorem cannot be applied since $F_X = \mu^* m$ for some distribution $\mu$ and positive integer $m$. Then $\mu$ can only be a Dirac measure $\mu = \delta_h$ and in this case (2.5) cannot hold. But the reader can verify that an equation of the form (2.2) is manifestly excluded. However, in the particular case where $m = 1$ (thus $Y_n \sim \text{Exp}(\lambda)$) we are in the framework of the classical Cramèr-Lundberg model (see, for instance, [8] or [1]). Even if we still are not able to compute the distributions $G_n$, we know their limit distribution $G_\infty$: the Hiičin-Pollaczek formula (see, for instance [2] or [1]) yields

$$(2.21) \quad G_\infty = \sum_{n=0}^{\infty} (1 - \rho) \rho^n (F_X)_I^n,$$

where $\rho = \frac{EX_n}{EY_n} = \lambda a < 1$, and $(F_X)_I$ is the integrated tail distribution defined by

$$(2.22) \quad (F_X)_I(x) = \frac{\int_{0}^{x} F_X(y)dy}{\int_{0}^{\infty} F_X(y)dy}.$$
Since in our case it happens that \((F_X)_1 = \text{Uniform}(0,a)\), we can compute \(G_\infty(x)\) with reasonable precision (using the well-known formula for convolutions of uniform distributions:

\[
\text{Uniform}(0,a)^*n(x) = \frac{x^n - C_1^n(x-a)_+^n + C_2^n(x-2a)_+^n + C_3^n(x-3a)_+^n + \cdots}{n!a^n}.
\]

3. COMPARISON OF LINDLEY PROCESSES

Let \(L = (L_n)_{n \geq 0}\) and \(L' = (L'_n)_{n \geq 0}\) be two Lindley processes. Assume that the first one has parameter \(F\) while the second one has parameter \(F'\) and, moreover, that the expectations of \(F\) and \(F'\) are negative. In this case the limits \(L_\infty\) and \(L'_\infty\) exist in distribution. We denote by \(\prec_{++} F''\) the fact that \(L_n \prec_{st} L'_n \ \forall n \geq 1\) and by \(\prec_{+} F''\) the fact that \(L_\infty \prec_{st} L'_\infty\). We may call the first domination the strong one and the second the weak one. Thus \(L\) is strongly dominated by \(L'\) iff \(L_n \prec_{st} L'_n \ \forall n \geq 1\) and weakly dominated by \(L'\) iff \(L_\infty \prec_{st} L'_\infty\).

In this very general situation, the only assertion that we can made is the trivial remark below.

**Proposition 3.1.** \(F \prec_{st} F' \Rightarrow F \prec_{++} F' \Rightarrow F \prec_{+} F'\).

**Proof.** Obvious. If \(G_n\) and \(G'_n\) are the distributions of \(L_n\) and \(L'_n\), \(G_\infty\) and \(G'_\infty\) are the distributions of \(L_\infty\) and \(L'_\infty\), then we see that \(F \prec_{st} F' \Rightarrow F'(+) \prec_{st} F'_n \Rightarrow F' \prec_{st} G'_1 \Rightarrow G_1 = F \prec_{st} G'_1 \Rightarrow G'_1 \prec F' \Rightarrow (G'_1 \ast F')_{(+)} \prec_{st} (G_1 \ast F')_{(+)} \Rightarrow G_2 \prec_{st} G'_2 \Rightarrow \cdots\).

As \(G_n \Rightarrow G_\infty\), and \(G'_n \Rightarrow G'\infty\), the second implication is obvious, too. \(\Box\)

Suppose that, moreover, both \(L\) and \(L'\) are computable and \(G_n\) and \(G'_n\) are the distributions of \(L_n\) and \(L'_n\). According to (2.2), we have \(G_n = \Gamma Q^n e\) and \(G'_n = \Gamma' Q'^n e\), where \(\Gamma, \Gamma' = \mu^{e} m, m = \mu^{e} m, \mu\) and \(\mu'\) are distributions on \((0, \infty)\) and \(Q, Q'\) are column stochastic matrices given by the distributions \(\pi\) and \(\pi'\). Now, we can say more, even in a slightly more general context.

**Definitions.** 1. Let \(\Delta = (\Delta_n)_{n \geq 0}\) be a sequence of distributions on the real line. We say that \(\Delta\) is monotone if \(\Delta_n \prec_{st} \Delta_{n+1} \ \forall n \geq 0\).

2. Let \(Q\) be a column-stochastic infinitely dimensional matrix. We say that \(Q\) is monotone if \(Q_{n,n} \prec_{st} Q_{n+1,n}\), where \(Q_{n,n}\) is the \(n\)th column of \(Q\), considered as a probability distribution on the set of non-negative integers.

**Lemma 3.2.** Let \(\Delta, \Delta'\) be sequence of distributions on the real line and \(Q, Q'\) infinitely dimensional column-stochastic matrices. Denote by \(\Delta \prec_{st} \Delta'\) the fact that \(\Delta_n \prec_{st} \Delta'_n \ \forall n \geq 0\) and by \(Q \prec_{st} Q'\) the fact that \(Q_{n,n} \prec_{st} Q'_{n,n}\) \(\forall n \geq 0\). Then
(i) if $\Delta$ and $Q$ are monotone then $\Delta Q$ is monotone, too;
(ii) if $\Delta, \Delta', Q, Q'$ are monotone and $\Delta \prec_{\text{st}} \Delta', Q \prec_{\text{st}} Q'$ then $\Delta Q \prec_{\text{st}} \Delta' Q'$.

Proof. (i) We have $(\Delta Q)_n = \Delta_0 Q_{0,n} + \Delta_1 Q_{1,n} + \Delta_2 Q_{2,n} + \cdots$. The tail of this distribution is

\begin{equation}
(\Delta Q)_n(x) = \Delta_0(x)Q_{0,n} + \Delta_1(x)Q_{1,n} + \Delta_2(x)Q_{2,n} + \cdots,
\end{equation}

where the $\Delta_k(x)$ are the tails of the distributions $\Delta_k$. As we supposed that $\Delta$ is monotone, the sequences $(\Delta_k(x))_k$ is non-decreasing, by the very definition of stochastic domination. On the other hand, the tail of the distribution $(\Delta Q)_n+1$ is

\begin{equation}
(\Delta Q)_{n+1}(x) = \Delta_0(x)Q_{0,n+1} + \Delta_1(x)Q_{1,n+1} + \Delta_2(x)Q_{2,n+1} + \cdots.
\end{equation}

But we know that $Q_{.,n} \prec_{\text{st}} Q_{.,n+1}$. This means that $\sum_{k=0}^{\infty} a_k Q_{k,n} \leq \sum_{k=0}^{\infty} a_k Q_{k,n+1}$ for any non-decreasing bounded sequence $(a_k)_{k \geq 0}$ while $(\Delta_k(x))_{k \geq 0}$ is exactly such kind of sequence.

ii) The tail of $(\Delta Q)_n$ computed at some real $x$ is $\sum_{k=0}^{\infty} \Delta_k(x)Q_{k,n}$ while the tail of $(\Delta' Q')_n$ computed at some real $x$ is $\sum_{k=0}^{\infty} \Delta'_k(x)Q'_{k,n}$. As $\Delta \prec_{\text{st}} \Delta'$, we have

$$\sum_{k=0}^{\infty} \Delta_k(x)Q_{k,n} \leq \sum_{k=0}^{\infty} \Delta'_k(x)Q'_{k,n} \leq \sum_{k=0}^{\infty} \Delta'_k(x)Q'_{k,n}.$$ 

The last inequality holds since $Q \prec_{\text{st}} Q'$. □

Here is a somewhat more general result that we need for computable Lindley processes.

**Proposition 3.3.** Suppose that $(L_n)_{n \geq 0}$ and $(L'_n)_{n \geq 0}$ are two sequences of random variables. Let $G_n$ and $G'_n$ be their distributions. Suppose that $G_n = \Gamma Q^n e$ and $G'_n = \Gamma' Q'^n e$ and

(i) $\Gamma$ and $\Gamma'$ are monotone sequences of distributions on the real line;
(ii) $Q$ and $Q'$ are monotone column-stochastic infinite dimensional matrices;
(iii) $\Gamma \prec_{\text{st}} \Gamma'$;
(iv) $Q \prec_{\text{st}} Q'$.

Then $G_n \prec_{\text{st}} G'_n \forall n$.

Proof. Let $\Delta_n = \Gamma Q^n$ and $\Delta'_n = \Gamma'(Q')^n$. According to Lemma 3.2 (i), $\Delta_n$ and $\Delta'_n$ are monotone for every $n$. We can check by induction that $\Delta_n \prec_{\text{st}} \Delta'_n \forall n \geq 0$. For $n = 0$ this is true due to our assumption (iii). Suppose now
that $\Delta_n \sim_{st} \Delta'_n$, and let us prove that $\Delta_{n+1} \sim_{st} \Delta'_{n+1}$. But $\Delta_{n+1} = \Delta_n Q$ and $\Delta'_{n+1} = \Delta'_n Q'$. The claim follows now using Lemma 3.2 (ii). \qed

**Corollary 3.4.** Let $L$ and $L'$ be computable Lindley processes with characteristics $\mu, \pi$ and $\mu', \pi'$. Suppose that $\mu \sim_{st} \mu'$ and $\pi \sim_{st} \pi'$. Then $L$ is strongly dominated by $L'$.

**Proof.** Let $G_n, G'_n$ be the distributions of $L_n$ and $L'_n$. Then $G_n = \Gamma Q^n e$ and $G'_n = \Gamma' Q'^n e$, $\Gamma_n = \mu^n$, $\Gamma'_n = (\mu')^n$ while the column-stochastic matrices $Q$ and $Q'$ are linked to the distributions $\pi$ and $\pi'$ as follows: if $\xi$ and $\xi'$ are random variables such that $\xi \sim \pi$ and $\xi' \sim \pi'$, then $Q_{n,n}$ is the distribution of $(\xi + n)_+$ and $Q'_{n,n}$ is the distribution of $(\xi' + n)_+$. We are under the conditions of the previous proposition: $\mu \sim_{st} \mu' \Rightarrow \mu^n \sim_{st} (\mu')^n \Rightarrow \Gamma_n \sim_{st} \Gamma'_n \forall n$ and $\pi \sim_{st} \pi'$ can be written in terms of random variables as $\xi \sim_{st} \xi'$, thus making obvious the relations $(\xi + n)_+ \sim_{st} (\xi' + n)_+ \forall n \geq 0$ (i.e., $Q \sim_{st} Q'$) and $(\xi + n)_+ \sim_{st} (\xi + n + 1)_+$. $(\xi' + n)_+ \sim_{st} (\xi' + n + 1)_+$ (meaning that $Q$ and $Q'$ are monotone). The fact that $\Gamma$ and $\Gamma'$ are monotone is a simple consequence of the fact that $\mu$ and $\mu'$ are distributions on $(0, \infty)$. So, we can apply Proposition 3.3. to conclude that $L$ is strongly dominated by $L'$. \qed

Now, we want to apply this result to our examples from Corollary 2.5 and Proposition 2.6.

Let $L$ and $L'$ be two computable Lindley processes with characteristics $\mu, \pi$ and $\mu', \pi'$. We shall investigate the various implications between the following assertions:

A. $\mu \sim_{st} \mu', \pi \sim_{st} \pi'$;

B. $G_n \sim_{st} G'_n \forall n$ (i.e., $F \sim_{+++} F'$);

C. $F_X \sim_{st} F_{X'}$ and $\rho \leq \rho'$ ($\rho$ and $\rho'$ are the traffic intensities);

D. $G_1 \sim_{st} G'_1$;

E. $G_\infty \sim_{st} G'_\infty$ (i.e., $F \sim F'$).

We begin with processes from Corollary 2.5. The assumption from (i) are:

$F_X = \text{Gamma}(m, a), F_Y = \text{Gamma}(k, b)$,

$F_{X'} = \text{Gamma}(m', a'), F_{Y'} = \text{Gamma}(k', b')$

while the ones from (ii) are

$F_X = \text{Geom}(m, a), F_Y = \text{Negbin}(k, \beta)$,

$F_{X'} = \text{Geom}(m', a'), F_{Y'} = \text{Negbin}(k', \beta')$.

For short, in the first case $L$ is of type $(m, k, \text{Gamma})$ and $L'$ is of type $(m', k', \text{Gamma})$ while in the second case $L$ is of type $(m, k, \text{Geom})$ and $L'$ is of type $(m', k', \text{Geom})$.

In the first case

\begin{equation}
\rho = \frac{EX}{EY} = \frac{mb}{ka}, \quad \rho' = \frac{EX'}{EY'} = \frac{m'b'}{k'a'}, \quad p = \frac{b}{a + b}, \quad p' = \frac{b'}{a' + b'}
\end{equation}
while in the second one

$$\rho = \frac{m\beta}{k\alpha(1-\beta)}, \quad \rho' = \frac{m'\beta'}{k'\alpha'(1-\beta')}.$$ (3.4)

$$p = \frac{\beta}{\alpha + \beta - \alpha\beta}, \quad p' = \frac{\beta'}{\alpha' + \beta' - \alpha'\beta'}.$$

The limit distributions exist iff $\rho, \rho' < 1$. The characteristics of these two processes from Corollary 2.5 are

$$\mu = \text{Exp}(a), \quad \pi = \text{Negbin}(k, -p) * \delta_m,$$
$$\mu' = \text{Exp}(a'), \quad \pi' = \text{Negbin}(k', -p') * \delta_{m'}.$$ (3.5)

in case (i) and

$$\mu = \text{Geom}(\alpha), \quad \pi = \text{Negbin}(k, -p) * \delta_m,$$
$$\mu' = \text{Geom}(\alpha'), \quad \pi' = \text{Negbin}(k', -p') * \delta_{m'}.$$ (3.6)

in case (ii).

We know that $G_n = \Gamma Q^m E, G'_n = \Gamma' Q'^m E$. Concerning our assumptions A–E, we know that $A \Rightarrow B$ (Corollary 3.4) and $B \Rightarrow D$, $D \Rightarrow E$.

**Proposition 3.5.** Under the assumption from Corollary 2.5, condition A implies $F_X \preceq_{st} F_{X'}$, $p \leq p'$ and condition D implies $\mu \preceq_{st} \mu'$.

**Proof.** In case (i) of Corollary 2.5, condition A says that $\text{Exp}(a) \preceq_{st} \text{Exp}(a')$, which is the same as $a \geq a'$. In case (ii) we have the condition $\text{Geom}(\alpha) \preceq_{st} \text{Geom}(\alpha')$ which is the same as $\alpha \geq \alpha'$ (because the mappings $a \rightarrow \text{Exp}(a), \alpha \rightarrow \text{Geom}(\alpha)$ are stochastically decreasing). On the other hand, if $\text{Negbin}(k, -p) * \delta_m \preceq_{st} \text{Negbin}(k', -p') * \delta_{m'}$, then $m \leq m'$ (otherwise, if $m > m'$ then $\pi((m, \infty)) > 0$ and $\pi'(([m, \infty])) = 0$, thus $\pi$ cannot be dominated by $\pi'$). Hence $F_X = \mu^m$ is surely dominated by $F_{X'} = (\mu')^m$. Next, $\xi \sim \text{Negbin}(k, p), \xi' \sim \text{Negbin}(k', p')$. Then $\pi$ is the distribution of $m - \xi$ and $\pi'$ is the distribution of $m' - \xi'$. Thus, $\pi \preceq_{st} \pi' \iff m - \xi \preceq_{st} m' - \xi' \iff \xi + m - m' \preceq_{st} \xi \Rightarrow \text{Exp} \geq e^{t(m - m')} e^{t\xi}$ for $t \geq 0$ (because for $t \geq 0$ the mapping $x \rightarrow e^{tx}$ is increasing) $\iff \left(\frac{p'}{1 - q't}\right)^k \leq e^{t(m - m')} \left(\frac{p}{1 - q\alpha}\right)^k$ for $t > 0$. Let $t = \ln(1/q)$ and $t' = \ln(1/q')$. As $t \rightarrow 0$ the left hand becomes infinite, and that violates the above inequality. Therefore, $t \leq t' \iff q \geq q' \iff p \leq p'$.

Now, suppose that condition D holds. If we consider the moment generating functions $m$ and $m'$ of $G_1$ and $G'_1$, then in case (i) we have

$$m(t) = r_{k,m} + pk_{m-1} \left(\frac{a}{a-t}\right) + pk_{m-2} \left(\frac{a}{a-t}\right)^2 + \cdots + pk_0 \left(\frac{a}{a-t}\right)^m,$$ (3.7)
\[ m'(t) = r'_{k',m'} + p'_{k',m'-1} \left( \frac{a'}{a'-t} \right) + p'_{k',m'-2} \left( \frac{a'}{a'-t} \right)^2 + \cdots + p'_{k',0} \left( \frac{a'}{a'-t} \right)^m, \]

(3.8)

where \( p_{k,n} = \text{Negbin}(k,p)(\{n\}) \) and \( p'_{k',n'} = \text{Negbin}(k',p)(\{n\}) \) and \( r_{k,n}, r'_{k',n'} \) are the corresponding tails.

We want that \( m(t) \leq m'(t) \) \( \forall t > 0 \). This cannot happen if \( a < a' \) because \( m(a-0) = \infty \), hence the only possibility is \( a \geq a' \Leftrightarrow \mu \prec_{st} \mu' \). If we are in case (ii), thus dealing with processes of type \( (m,k,\text{Geom}) \), we have to replace the m.g.f. \( \frac{a}{a-t} \) and \( \frac{a'}{a'-t} \) of the exponential distributions with those of the geometric distribution, namely, \( \frac{\alpha}{1-(1-\alpha)e^t} \) and \( \frac{\alpha'}{1-(1-\alpha')e^t} \). If we want that \( m(t) \leq m'(t) \) \( \forall t > 0 \), then \( \frac{1}{1-\alpha} \) should be not smaller than \( 1/(1-\alpha') \) hence \( \alpha \geq \alpha' \Leftrightarrow \mu \prec_{st} \mu' \). □

If we deal with processes of the same type, we can say more.

**Proposition 3.6.** If we assume the conditions of Corollary 2.5 and \( \mathbf{L} \) and \( \mathbf{L}' \) are of the same type (to mean that \( m = m' \) and \( k = k' \)), then \( \mathbf{A} \iff \mathbf{B} \iff \mathbf{C} \iff \mathbf{D} \).

*Proof.* We shall check that \( \mathbf{D} \Rightarrow \mathbf{A} \) and this will mean that \( \mathbf{A} \iff \mathbf{B} \iff \mathbf{D} \).

From Proposition 3.5 we deduce that \( \mathbf{D} \Rightarrow \mu \prec_{st} \mu' \). As \( m = m' \), this is equivalent to \( F_X \prec_{st} F_{X'} \). In order to prove that \( \pi \prec_{st} \pi' \), we only have to check that \( p \leq p' \). Let

\[
\begin{align*}
p_j &= \text{Negbin}(k,p)(\{j\}), \quad p_j' = \text{Negbin}(k',p')(\{j\}), \\
\Gamma_n &= \text{Gamma}(n,a), \quad \Gamma'_n = \text{Gamma}(n,a'), \\
r_j &= \text{Negbin}(k,p)(\{j, \infty\}), \quad r_j' = \text{Negbin}(k',p')(\{j, \infty\}).
\end{align*}
\]

We know that

\[
\begin{align*}
G_1 &= p_0 \Gamma_m + p_1 \Gamma_{m-1} + \cdots + p_{m-1} \Gamma_1 + r_m \Gamma_0, \\
G_1' &= p_0' \Gamma'_m + p_1' \Gamma'_{m-1} + \cdots + p_{m-1}' \Gamma'_1 + r_m' \Gamma'_0.
\end{align*}
\]

Then

\[
\begin{align*}
G_1 \prec_{st} G_1' &\Leftrightarrow p_0 \Gamma_m(x) + p_1 \Gamma_{m-1}(x) + \cdots + p_{m-1} \Gamma_1(x) + r_m \Gamma_0(x) \\
&\leq p_0' \Gamma'_m(x) + p_1' \Gamma'_{m-1}(x) + \cdots + p_{m-1}' \Gamma'_1(x) + r_m' \Gamma'_0(x)
\end{align*}
\]
for every $x > 0$. Letting $x \to 0$ and taking into account that $\Gamma_m(0) = \Gamma_m'(0) = 1$, we get $p_0 + p_1 + \cdots + p_{m-1} \leq p'_0 + p'_1 + \cdots + p'_{m-1}$ or, equivalently $1 - r_m \leq 1 - r'_m \iff r_m \geq r'_m$. Therefore, the tails satisfy the inequality $\text{Negbin}(k, p)((m, \infty)) \geq \text{Negbin}(k, p')((m, \infty))$. But the mapping $p \to \text{Negbin}(k, p)$ is stochastically decreasing. It follows that $p \leq p'$ or, since $p = \frac{\rho}{\rho + 1}, p' = \frac{\rho'}{\rho' + 1}$, that $\rho \leq \rho'$.

So, $D$ implies both $A$ and $C$ and Corollary 3.4 says that $A$ implies $B$ and $D$. □

Remark 3.1. Both the theorem from [4] and example 2 from [5] are particular case of Proposition 3.6. for $m = m' = k = k' = 1$.

We investigate now the case where $L, L'$ are processes of the type described in Proposition 2.6. The parameters of the two processes are

\[(3.11) \quad F_X = \text{Gamma}(m, \lambda), \quad F_X' = \text{Gamma}(m', \lambda'), \quad F_Y = \delta_b, \quad F_Y' = \delta_{b'}\]

and the characteristics are

\[(3.12) \quad \mu = \text{Exp}(\lambda), \quad \pi = \delta_m \ast \text{Poisson}(-b\lambda), \]
\[\mu' = \text{Exp}(\lambda'), \quad \pi' = \delta_{m'} \ast \text{Poisson}(-b'\lambda').\]

The result is similar to that from Proposition 3.6.

Proposition 3.7. If $m = m'$ then conditions $A, B, C$ and $D$ are equivalent.

Proof. The weakest is condition $D$. Let $p_j = \text{Poisson}(b\lambda)((j))$, $p'_j = \text{Poisson}(b'\lambda')((j))$, $\Gamma_j = \text{Gamma}(j, \lambda)$ and $\Gamma'_j = \text{Gamma}(j, \lambda')$. According to Proposition 2.6 we have

\[(3.13) \quad G_1 = p_0\Gamma_m + p_1\Gamma_{m-1} + \cdots + p_{m-1}\Gamma_1 + r_m\Gamma_0, \]
\[(3.14) \quad G'_1 = p'_0\Gamma_{m'} + p'_1\Gamma'_{m'-1} + \cdots + p'_{m'-1}\Gamma'_1 + r'_m\Gamma'_0.\]

Assertion $D$ is that $G_1 \prec_{st} G'_1$. The method of moment generating functions which we used in the proof of Proposition 3.5 points out that $\lambda \geq \lambda'$, hence $\mu \prec_{st} \mu'$. The relation between the tails is $G_1(x) \leq G'_1(x) \forall x > 0$. Letting $x \to 0$, this inequality yields $1 - r_m \leq 1 - r'_m \iff r_m \geq r'_m$.

If $m \neq m'$ this is not very helpful, but if, as in our statement, $m = m'$, then the conclusion is that $\text{Poisson}(b\lambda)((m, \infty)) \geq \text{Poisson}(b'\lambda')((m, \infty))$. The mapping $a \to \text{Poisson}(a)$ is stochastically increasing. Thus, $b\lambda \geq b'\lambda'$. For short,
On the other hand, \( \rho = \frac{m}{b\lambda} \), \( \rho' = \frac{m'}{b'\lambda'} \). Obviously, if \( m = m' \) then \( b\lambda \geq b'\lambda' \iff \rho \leq \rho' \). So, in the case \( m = m' \) we have

\[
D \iff \lambda \geq \lambda', \quad \rho \leq \rho'.
\]

As \( \lambda \geq \lambda', m = m' \iff F_X \prec_{st} F_{X'} \), and we have proved the equivalence \( C \iff D \).

Assertion \( A \) is \( \mu \prec_{st} \mu', \pi \prec_{st} \pi' \iff \lambda \geq \lambda', \delta_m \ast \text{Poisson}(\lambda) \prec_{st} \delta_m \ast \text{Poisson}(\lambda') \). The second relation can also be written as \( \text{Poisson}(\lambda) \prec_{st} \text{Poisson}(\lambda') \ast \delta_{m' - m} \), which further implies \( m \leq m', b\lambda \leq b'\lambda \). If \( m = m' \) it is obvious that \( A \iff D \). Hence all the four claimed conditions are equivalent. \( \square \)

If \( L \) and \( L' \) are of different types, the problem is very difficult.

What about the weak domination, \( F \prec_{+} F' \)?

**We are not aware of a general formula for** \( G_{\infty} \). However, there are at least two cases where we can say something about this distribution: when \( m = 1 \) and when \( L \) is of type \((m, 1, \text{Gamma})\).

In the first case, the characteristics of \( L \) are \( \mu \) and

\[
\pi = \begin{pmatrix}
\cdots & -2 & -1 & 0 & 1 \\
\cdots & p_{-2} & p_{-1} & p_0 & p_1 \\
0 & p_1 & p_0 & p_{-1} & \cdots \\
0 & 0 & p_1 & p_0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
\]

hence the distribution of \( L_n \) is \( \Gamma Q^n \) with

\[
Q = \begin{pmatrix}
\begin{pmatrix}
r_0 & r_{-1} & r_{-2} & r_{-3} & \cdots \\
p_1 & p_0 & p_{-1} & p_{-2} & \cdots \\
0 & p_1 & p_0 & p_{-1} & \cdots \\
0 & 0 & p_1 & p_0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\end{pmatrix},
\]

where \( r_j = p_{j+1} + p_{j-1} + p_{j-2} + \cdots \). The transposed of \( Q^n \) is the distribution of \( N_n \) where \( N \) is the Lindley process of parameter \( \pi \), starting from 0. The limit \( Q^\infty \) does exist iff the expectation of \( \pi \) is negative and is equal to a matrix with one repeating column; this column is the distribution of \( N_{\infty} \). Let \( \psi(u) = P(N_{\infty} > u) = G_{\infty}(u) \) (the ruin probability). As the distribution \( \pi \) is skip-free, the severity of ruin, denoted by \( D_u \), is equal to 1. The Lundberg equation (see, for instance \([1], [3] \) or \([8] \)) gives us the possibility to compute

\[
\psi(u) = \frac{e^{-Ru}}{Ee^{Rd_{\pi}}} = e^{-R(u+1)},
\]

where \( R \) is the Lundberg constant, i.e., the unique positive solution of the equation

\[
m_{\pi}(R) = 1 \iff p_1 R + p_0 + p_{-1} R^{-1} + p_{-2} R^{-2} + \cdots = 1.
\]

In this way we proved

**Proposition 3.8. If** \( L \) **is a computable Lindley process with characteristics** \((\mu, \pi)\), **where** \( \mu \) **is any distribution on** \((0, \infty)\) **and** \( \pi \) **has the
\[
\begin{pmatrix}
\cdots & -2 & -1 & 0 & 1 \\
\cdots & p_2 & p_1 & p_0 & p_1 \\
\end{pmatrix},
\]
then
\begin{equation}
G_\infty = \left(1 - e^{-R}\right) \sum_{n=0}^{\infty} e^{-Rn} \Gamma_n,
\end{equation}
where \( R \) is the unique positive solution of (3.16).

Examples. 1. Processes from Corollary 2.5 with \( m = 1 \). Equation (3.16) becomes \( pe^{R} + qe^{-R} = 1 \) with the unique solution \( e^{-R} = \frac{p}{q} := \rho \) (notice that \( \rho < 1 \iff p < \frac{1}{2} \)). Using the moment generating function, we see that if \( \mu = \text{Exp}(a) \), then
\begin{equation}
G_\infty = (1 - \rho) \sum_{n=0}^{\infty} \rho^n \text{Gamma}(n,a) = (1 - \rho) \delta_0 + \rho \text{Exp}((1 - \rho)a)
\end{equation}
while if \( \mu = \text{Geom}(\alpha) \), then
\begin{equation}
G_\infty = (1 - \rho) \sum_{n=0}^{\infty} \rho^n \text{Geom}(n,\alpha) = (1 - \rho) \delta_0 + \rho \text{Geom}((1 - \rho)\alpha).
\end{equation}

2. Processes from Proposition 2.6 with \( m = 1 \). Here \( \pi = \delta_0 \ast \text{Poisson}(-b\lambda) \). Now the Lundberg equation is transcendent, namely, \( b\lambda(1 - e^{-R}) = R \). However, it can be solved with numerically any reasonable precision. We get
\begin{equation}
G_\infty = (1 - e^{-R}) \delta_0 + e^{-R} \text{Exp}((1 - e^{-R})\lambda).
\end{equation}
The other case is special: it concerns processes of type \((m,1,\text{Gamma})\). Or, explicitly,
\[
X \sim \text{Gamma}(m,a), \quad Y \sim \text{Exp}(b).
\]
Now \( \rho = \frac{mb}{a} < 1 \). We can apply the Hincin-Pollaczek formula (2.21). We only have to notice that the integrated tail of \( F_X \) is
\begin{equation}
(\text{Gamma}(m,a))_I = \frac{1}{m} \sum_{k=1}^{m} \text{Gamma}(k,a)
\end{equation}
and get
\begin{equation}
G_\infty = (1 - \rho) \sum_{n=0}^{\infty} \rho^n \left(\frac{1}{m} \sum_{k=1}^{m} \text{Gamma}(k,a)\right)^n.
\end{equation}
We doubt that one can find an analytical expression for the tail of \( G_\infty \) in the general case. Anyway, this situation allows us to characterize some of the conditions \( A-E \). We intend to compare two processes of type \((m,1,\text{Gamma})\).

Proposition 3.9. Suppose that \( F_X = \text{Gamma}(m,a) \), \( F_Y = \text{Exp}(b) \), \( F_{X'} = \text{Gamma}(m',a') \), \( F_{Y'} = \text{Exp}(b') \).
Let
- $L$ and $L'$ be the computable Lindley processes generated by them;
- $G_n, G_n'$ be the distributions of $L_n$ and $L'_n$;
- $G_\infty, G_\infty'$ be the distributions of $L_\infty$ and $L'_\infty$;
- $\rho = E[X] = \frac{mb}{a}$, $\rho' = E[X'] = \frac{mb'}{a'}$;
- $\mu = \text{Exp}(a)$, $\pi = \delta_m * \text{Negbin}(1,-p)$ be the characteristics of $L$ (as in the previous sections, $p = \frac{b}{a+b} = \frac{\rho}{\rho+m}$);
- $\mu' = \text{Exp}(a')$, $\pi' = \delta_{m'} * \text{Negbin}(1,-p')$ be the characteristics of $L'$ (again, $p' = \frac{b'}{a'+b'} = \frac{\rho'}{\rho'+m'}$).

Conditions A–E become

A. $\mu \prec_{st} \mu'$, $\pi \prec_{st} \pi'$ $\iff a \geq a'$, $m \leq m'$, $\rho \leq \frac{m}{m'}\rho'$.

B. $F_X \sim F_X'$, and $\rho \leq \rho'$ $\iff a \geq a'$, $m \leq m'$, $\rho \leq \rho'$.

C. $G_1 \prec_{st} G_1'$ $\Rightarrow a \geq a'$, $(1+\frac{m}{m'})^{m} \leq (1+\frac{m}{m'})^{m'}$.

D. $G_\infty \prec_{st} G_\infty'$ $\Rightarrow \rho \leq \rho'$.

As a consequence, conditions A and C cannot hold if $m \leq m'$.

Proof. A. As $\mu = \text{Exp}(a)$ and $\mu' = \text{Exp}(a')$, the conditions $\mu \prec_{st} \mu'$ and $a \geq a'$ are equivalent. Let $\xi \sim \text{Negbin}(1,p)$ and $\xi' \sim \text{Negbin}(1,p')$. So $m - \xi \sim \pi$ and $m' - \xi' \sim \pi'$. Then $\pi \prec_{st} \pi'$ $\iff m - \xi \prec_{st} m' - \xi' \iff \xi' \prec_{st} m' - m + \xi$.

Therefore $P(\xi' \geq n) = (q')^n \leq P(m' - m + \xi \geq n) = q^{m'-m}q^n \forall n$. For $n = 0$ we get $m - m' \leq 0$ while for $n \rightarrow \infty$ we get $q' \leq q \iff \frac{m'}{m+p} \leq \frac{m}{m+p} \iff \rho \leq \frac{m}{m'}\rho'$.

C. $F_X \prec_{st} F_X'$ means $\text{Gamma}(m,a) \prec_{st} \text{Gamma}(m',a')$. The moment generating functions must satisfy the inequalities $t \geq 0 \Rightarrow (1+\frac{t}{a})^{m} \geq (1+\frac{t}{a'})^{m'}$, $t < 0 \Rightarrow (1+\frac{t}{a})^{m} \leq (1+\frac{t}{a'})^{m'}$, hence $a \geq a'$ (we let $t \rightarrow a$ !) and $m \leq m'$ (let $t \rightarrow -\infty$ !). The converse is obvious.

D. We have
$$G_1 = q^{m}\delta_0 + pq^{m-1}G_1 + \cdots + pqG_{m-1} + pG_m,$$
$$G_1' = (q')^{m'}\delta_0 + p'(q')^{m'-1}G_1' + \cdots + p'qG_{m'-1} + p'G_{m'},$$
where $\Gamma_k = \text{Gamma}(k,a)$ and $\Gamma'_k = \text{Gamma}(k,a')$. Comparing the tails, we see that $G_1(0) = q^m$ and $G_1'(0) = (q')^{m'}$. So, we must have $q^m \leq (q')^{m'} \iff (1+\frac{m}{m'})^{m} \leq (1+\frac{m'}{m'})^{m'}$. If we let $t \rightarrow a$ in the moment generating functions $mG_1$ and $mG_1'$, we see that $a \geq a'$. The converse is not true, as we shall see in examples.

E. Comparing the tails of $G_\infty$ and $G_\infty'$ at $x = 0$ in the Hincin-Pollaczek formula (3.22), we see that $1 - \rho > 1 - \rho' \iff \rho \leq \rho'$.

We notice that in D and E we do not have equivalences, but only one implication. To find equivalent conditions for them, we shall investigate the case $m = 2$. 

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Proposition 3.10. If \( m = 2 \) then (3.22) can be written as

\[
G_\infty = (1 - \rho) \left( \delta_0 + \frac{(a - t_1)^2}{t_1(t_2 - t_1)} \text{Exp}(t_1) - \frac{(a - t_2)^2}{t_2(t_2 - t_1)} \text{Exp}(t_2) \right),
\]

where \( 0 < t_1 < t_2 \) are the solutions of the equation

\[
t^2a^2 - 2at \left(1 - \frac{\rho}{4}\right) + a^2(1 - \rho) = 0.
\]

Remark 3.2. We cannot say that \( G_\infty \) is a mixture of exponentials and \( \delta_0 \) because the coefficient of \( \text{Exp}(t_2) \) is negative!

Proof. Let \( m_\infty \) be the m.g.f. of \( G_\infty \) and \( m(t) = \frac{a}{a-t} \) the m.g.f. of \( \text{Exp}(a) \).

It follows from (3.22) that

\[
m_\infty(t) = (1 - \rho) \sum_{n=0}^{\infty} \rho^n \left( \frac{m(t) + m^2(t)}{2} \right)^n = \frac{1 - \rho}{1 - \rho m(t)(1+m(t))} = \frac{1 - \rho}{1 - \rho a(t - a) \rho \frac{(2a-t)}{2(a-t)^2}}.
\]

Here, as usual, \( \rho = \frac{\text{EX}}{\text{FY}} = \frac{2b}{a} < 1 \). The above expression is a linear combination of a constant and two simple fractions—which are moment generating functions of exponential distributions. Precisely, we write \( m_\infty \) as

\[
m_\infty(t) = A + B \frac{t_1}{t_1 - t} + C \frac{t_2}{t_2 - t},
\]

where \( t_{1,2} = \frac{a}{4}(4 - \rho \pm \sqrt{\rho^2 + 8\rho}) \) are the roots of the denominator. Notice that

\[
(t_{1,2} - a)^2 = \frac{a^2}{8} \left( \rho^2 + 4\rho \pm \sqrt{\rho^2 + 8\rho} \right),
\]

\[
t_1 + t_2 = 2a \left(1 - \frac{\rho}{4}\right), \quad t_1t_2 = a^2(1 - \rho), \quad \text{and} \quad t_2 - t_1 = \frac{a}{2} \sqrt{\rho^2 + 8\rho}.
\]

Therefore both these roots are positive. It follows that \( G_\infty \) can be written as

\[
G_\infty = A \delta_0 + B \text{ Exp}(t_1) + C \text{ Exp}(t_2).
\]

Letting \( t \to -\infty \), we see that \( A = 1 - \rho \). To find \( B \) and \( C \) write (3.23) as

\[
(1 - \rho)(t-a)^2 = (1 - \rho)(t_1-t)(t_2-t) + Bt_1(t_2-t) + Ct_2(t_1-t).
\]

For \( t = t_1 \) we get \( B = \frac{(1-\rho)(t_1-a)^2}{t_1(t_2-t_1)} \) while for \( t = t_2 \) we get \( C = \frac{(1-\rho)(t_2-a)^2}{t_2(t_2-t_1)} \).

Example. For \( a = 4, b = 1 \) (hence \( \rho = \frac{1}{2} \)) we get

\[
G_\infty = \frac{1}{2} \delta_0 + \frac{5 + \sqrt{17}}{4\sqrt{17}} \text{Exp} \left( \frac{7 - \sqrt{17}}{2} \right) - \frac{5 - \sqrt{17}}{4\sqrt{17}} \text{Exp} \left( \frac{7 + \sqrt{17}}{2} \right).
\]
In terms of ruin theory, we can write $G_\infty$ as

$$
\psi(u) = \frac{5 + \sqrt{17}}{4\sqrt{17}} e^{-\left(\frac{\sqrt{17} - \sqrt{7}}{2}\right)u} - \frac{5 - \sqrt{17}}{4\sqrt{17}} e^{-\left(\frac{\sqrt{17} + \sqrt{7}}{2}\right)u} \approx 0,5532e^{-1.4384u} - 0,0532e^{-55616u}.
$$

**Remark 3.3.** It is interesting to compare this precise value with Lundberg’s estimation $\psi(u) \leq e^{-Ru}$. Solving Lundberg’s we find $R = \frac{7 - \sqrt{17}}{2}$, hence his estimation is $\psi(u) \leq e^{-\left(\frac{7 - \sqrt{17}}{2}\right)u}$. It is known that the limit $C_3 = \lim_{u \to \infty} \psi(u)e^{Ru}$ does exist: it is called Cramér’s constant. In our case, $C_3$ is the constant $B$ from (3.27). In our particular case, $C_3 = \frac{5 + \sqrt{17}}{4\sqrt{17}} \approx 0,5532$.

**Example.** For $a = 4$, $b = \frac{2}{3}$ (hence $\rho = \frac{1}{3}$) we get $t_1 = 2$, $t_2 = \frac{16}{9}$ and

$$
G_\infty = \frac{4}{3}\delta_0 + \frac{2}{5} \text{Exp}(2) - \frac{1}{15} \text{Exp}(16/3).
$$

In terms of ruin theory we can write (3.30) as

$$
\psi(u) = \frac{2}{5} e^{-2u} - \frac{1}{15} e^{-\frac{16}{9}u} \approx 0,4e^{-2u} - 0,0667e^{-5,3333u}.
$$

**Remark 3.4.** We compare once again the precise value from the Lundberg equation with the estimated value in (3.31). Solving Lundberg’s equation, we find $R = 2$, hence his estimation is $\psi(u) \leq e^{-2u}$. In this case, $C_3 = 0,4$.

In order to see various difficulties that may occur, we shall compare a process of type $(2, 1, \text{Gamma})$ with one of type $(1, 1, \text{Gamma})$. Thus $F = \text{Gamma}(2, a) * \text{Exp}(-b)$ and $F' = \text{Gamma}(1, a') * \text{Exp}(-b')$, $\rho = \frac{2b}{a}$, $p = \frac{\rho}{\rho + 2}$, $q = \frac{2}{\rho + 2}$, $p' = \frac{\rho'}{\rho' + 1}$, $q' = \frac{1}{\rho' + 1}$. Next, $G_1 = q^2\delta_0 + pq \text{Gamma}(1, a) + p \text{Gamma}(2, a)$, hence

$$
G_1(x) = pe^{-ax}(q + 1 + ax), \quad G'_1(x) = p'e^{-a'x}.
$$

By Proposition 3.9, $G_1 \prec_{\text{st}} G'_1 \Rightarrow a \geq a'$, $(1 + \frac{q}{a})^2 \leq 1 + p' \Leftrightarrow \rho' \geq \rho + \frac{q^2}{a}$. So, it is clear that the condition $\mu \prec_{\text{st}} \mu', \rho \leq \rho'$ cannot ensure the domination of $L$ by $L'$. But we want to find an equivalent condition for $D$. In order to do that, we shall investigate the behaviour of the function $f(x) = e^{ax}(G'_1(x) - G_1(x)) = p'e^{(a - a')x} - apx - p(1 + q)$. Its derivative is the function $g(x) = (a - a')p'e^{(a - a')x} - ap$. Then

$$
f(0) = p' - p(1 + q) = \frac{4\left(p' - \frac{\rho + \rho^2}{4}\right)}{(\rho' + 1)(\rho + 2)(\rho + 3)}, \quad g(0) = (a - a')p' - ap.
$$
We want that \( \min f \geq 0 \). Anyway, \( f(0) \) must be nonnegative, hence a necessary condition is that \( p' \geq p(1 + q) \iff \rho' \geq \rho + \frac{p^2}{4} \). We already know that. But \( f(0) \geq 0 \Rightarrow g(0) \geq p((a - a')(1 + q) - a). \) It may happen that \( p((a - a')(1 + q) - a) \geq 0 \iff \frac{a'}{a} \leq \frac{3}{4 \rho}. \) In that case, \( f \) is non-decreasing, hence \( \min f = f(0) \).

We arrive at

\[
(3.34) \quad \text{if } \frac{a'}{a} \leq \frac{2}{4 + \rho} \text{ then } G_1 \prec_{st} G_1' \iff \rho' \geq \rho + \frac{p^2}{4}.
\]

If not, the minimum is attained at the unique solution \( x_0 \) of the equation

\[
g(x) = 0 \iff e^{(a-a')x} = \frac{ap}{p'(a-a')} \iff x_0 = \frac{1}{a-a'} \ln \frac{ap}{p'(a-a')}.
\]

Thus, in general, we have

\[
(3.35) \quad \min f = \min(f(x_0), f(0)) = \min \left( \frac{ap}{(a - a')} - \frac{ap}{(a - a')} \ln \frac{ap}{p'(a-a')} - p(1 + q), p' - p(1 + q) \right).
\]

If we denote by \( \alpha \) the ratio \( \alpha = \frac{a'}{a} \), we can write

\[
(3.36) \quad \min f = \min \left( \frac{p}{1 - \alpha} - \frac{p}{p'(1 - \alpha)} - (1 + q)(1 - \alpha)), p' - p(1 + q) \right).
\]

After these considerations, the result is

**Proposition 3.11.** *In the above situation the condition \( G_1 \prec_{st} G_1' \) is equivalent to*

- either \( p' \geq \frac{p}{1 - \alpha} \) and \( \rho' \geq \rho + \frac{p^2}{4} \);
- or \( p' < \frac{p}{1 - \alpha} \) and \( 1 - \ln \frac{p}{p'(1 - \alpha)} - (1 + q)(1 - \alpha) \geq 0 \).

**Example.** If \( \rho = \frac{1}{2} \), hence \( p = \frac{1}{9}, q = \frac{4}{5} \), then \( G_1 \prec_{st} G_1' \) iff

- either \( p' \geq \frac{2}{5} \) and \( \rho' \geq \frac{9}{16} \) (hence if \( \rho \geq \frac{2}{3} \));
- or \( p' < \frac{2}{5} \) but \( 5p'(1 - \alpha) \geq \frac{9}{16}(1 - \alpha)^{-1} \).

For \( a = 4, a' = 2 \) we get \( \alpha = \frac{1}{2} \). Now, \( G_1 \prec_{st} G_1' \) iff \( \rho \geq \frac{2}{3} \) or \( 5p' \geq \frac{1}{2} \leq \frac{1}{e - 1} \iff \rho' \geq \frac{2}{5e + 2} \geq 0, 567 \ldots \)

Finally, we investigate the condition \( G_\infty \prec_{st} G_\infty' \) in the same situation when we compare processes \( (2, 1, \text{Gamma}) \) with one of type \( (1, 1, \text{Gamma}) \).

According to Proposition 3.10, for any \( x \geq 0 \) we have

\[
G_\infty(x) = (1 - \rho) \left( \frac{(a - t_1)^2}{t_1(t_2 - t_1)} e^{-t_1 x} - \frac{(a - t_2)^2}{t_2(t_2 - t_1)} e^{-t_2 x} \right)
\]

with \( t_1 + t_2 = 2a(1 - \frac{p}{4}), t_1t_2 = a^2(1 - \rho) \) and

\[
G_\infty'(x) = (1 - \rho) \left( \frac{(a - t_1)^2}{t_1(t_2 - t_1)} e^{-t_1 x} - \frac{(a - t_2)^2}{t_2(t_2 - t_1)} e^{-t_2 x} \right)
\]

with \( t_1 + t_2 = 2a \left( 1 - \frac{p}{4} \right), \ t_1t_2 = a^2(1 - \rho) \).
On the other hand, it is easy to see that \( G'_{\infty} = (1 - \rho')\delta_0 + \rho \exp((1 - \rho')a') \), hence
\[
G'_{\infty}(x) = \rho'e^{-(1 - \rho')a'x}.
\]
We want to find conditions for the parameters \( a, b, a', b' \) such that \( G_{\infty} \leq G'_{\infty} \).
Or, in our situation, such that \( (1 - \rho)\left( \frac{(a-t_1)^2}{t_1(t_2-t_1)}e^{-t_1x} - \frac{(a-t_2)^2}{t_2(t_2-t_1)}e^{-t_2x} \right) \leq \rho'e^{-(1 - \rho')a'x} \). Let \( A_1 = (1 - \rho)\frac{(a-t_1)^2}{t_1(t_2-t_1)} \) and \( A_2 = (1 - \rho)\frac{(a-t_2)^2}{t_2(t_2-t_1)} \) (so that \( A_2 - A_1 = \rho \)). The function \( x \rightarrow \rho'e^{-(1 - \rho')a'x} + A_2e^{-t_2x} - A_1e^{-t_1x} \) should be non-negative. If we multiply it by \( e^{tx} \), that means
\[
\rho'e^{(t_1-(1 - \rho')a')x} + A_2e^{-(t_2-t_1)x} \geq A_1 \quad \forall x \geq 0.
\]
Let \( f(x) = \rho'e^{(t_1-(1 - \rho')a')x} + A_2e^{-(t_2-t_1)x} \). As \( f(0) = \rho' + A_2 \geq A_1 \), we re-discover the necessary condition \( \rho' \geq \rho \), which we know already from Proposition 3.9. Another necessary condition is that \( t_1 \geq (1 - \rho')a' \) (otherwise, as \( t_1 < t_2 \), \( \lim_{x \to \infty} f(x) = 0 \)). As \( f(x) > \rho'e^{(t_1-(1 - \rho')a')x} \), a sufficient condition is that \( \rho' \geq A_1 \).

Sometimes, these two conditions \( \rho \leq \rho' \), \( t_1 \geq (1 - \rho')a' \) are also sufficient: if the minimum of \( f \) is \( f(0) \). The derivative of \( f \) is the function
\[
g(x) = \rho'(t_1 - (1 - \rho')a')e^{(t_1-(1 - \rho')a')x} - A_2(t_2-t_1)e^{-(t_2-t_1)x},
\]
that is obviously increasing. Thus, \( \min f = f(0) \Leftrightarrow g(0) \geq 0 \Leftrightarrow \rho'(t_1 - (1 - \rho')a') \geq A_2(t_2-t_1) \).

**Example.** If \( a = 4, \rho = \frac{1}{3} \), we have seen in the example following Proposition 3.11 that the inequality becomes \( \frac{2}{3}e^{-2u} - \frac{1}{15}e^{-\frac{16}{3}u} \leq \rho'e^{-(1 - \rho')a'x} \), or
\[
15\rho'e^{(2-(1 - \rho')a')x} + e^{-\frac{16x}{3}} \geq 6 \quad \forall x \geq 0.
\]

The necessary conditions are \( \rho' \geq \frac{1}{15} \) and \( (1 - \rho')a' \leq 2 \). The sufficient one comes by neglecting the second term and is \( 15\rho' \geq 6 \Leftrightarrow \rho' \geq \frac{2}{5} \). In other words, any \( \rho' \geq 0, 4 \) is good, provided that \( a' \leq \frac{2}{1 - \rho} \). For instance, \( \rho' = 0, 4 \) is good for any \( a' \leq \frac{10}{3} \). For \( \rho' = 0, 9 \) any \( a \leq 20 \) is good. There is no necessity that \( a \geq a' \).

**Final remark.** Unfortunately, we were unable to find a more reasonable condition to imply what we really were searching for: condition B. We know that \( A \Rightarrow B \) and that \( C \) does not imply \( A \). The examples we tried make us believe in the following

**Conjecture.** If \( G_1 \approx_{st} G_1' \) and \( G_\infty \approx_{st} G_\infty' \), then \( G_n \approx_{st} G_n' \).

This open problem was formulated in [5] in a somewhat simpler context.
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