We deal with asymptotic properties of the $[\alpha m]$th order statistic of a type-II progressively censored sample of size $m$ (quantile process) and with the counting process $m^{-1} \sum_{i=1}^{m} 1(X_{i,m,n} \leq t)$, where the $X_{i,m,n}$ are the observed $m$-sample obtained by progressively censoring from the sample $X_1, \ldots, X_n$.

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1. INTRODUCTION

Progressive censoring schemes are very useful in life-test experiments and in clinical studies. Montanari and Cacciari [13] reported results of progressively censored data aging tests on XLPE-insulated cable models under combined thermal-electrical stresses. Bhattacharya [5] gives an example, in a clinical trial study. The monograph of Balakrishnan and Aggarwala [2] (see also [5]) gives an interesting review of the background and of developments in this field. Progressive censoring is a particular model based on order statistics and record values, see for instance [12].

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random lifetimes of $n$ items. A progressive type-II right censored sample may be obtained in the following way: at the time of the first failure, noted $X_{1,m:n}$, $r_1$ surviving items are removed at random from the $n - 1$ remaining surviving items, at the time of the next failure, noted $X_{2,m:n}$, $r_2$ surviving items are removed at random from the $n - r_1 - 2$ remaining items, and so on. At the time of the $m$th failure, all the remaining $r_m = n - m - r_1 - \cdots - r_{m-1}$ surviving items are censored.

Our first result concerns the asymptotic behaviour of the $[\alpha m]$th order statistic of a type-II progressively censored sample of size $m$ (quantile process), defined by $(X_{[\alpha m]:m:n})_{\alpha \in [0,1]}$, where $[x]$ is the greatest integer satisfying $[x] \leq x < [x] + 1$. Our aim is to establish a result of the type

$$\sqrt{m} \left( X_{[\alpha m]:m:n} - u_\alpha \right) = C_\alpha W'(\alpha) + O \left( \delta'_m \right)$$

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in probability. Here $W'$ is a standard Brownian motion, $u_\alpha$ is the $\alpha$-quantile of the distribution function, $C_\alpha$ is an expression depending on $\alpha$ while $\delta'_m$ gives the rate of convergence (see our Theorem 2). In order to establish our rate of convergence result (Theorem 2) in Section 2, we begin by establishing in Theorem 1 a result about the associated triangular array resulting from the almost sure (a.s.) representation (see (1) below). Theorem 3 is related to the counting process $m^{-1} \sum_{i=1}^{m} 1(X_{i:n:m} \leq t)$, for which a Poisson approximation is given. Finally, in Theorem 4, we give a result on induced order statistics under progressively type-II censoring scheme applied to the $X$ sample. In Section 2, we make precise the regularity conditions and state our results. Section 3 is devoted to the proof of our results.

2. ASSUMPTIONS AND RESULTS

From now on, we assume that the $X_i$ have a common distribution function $F$ with density $f$. We denote by $\lambda$ the hazard rate function, i.e. $\lambda(t) = f(t)/R(t)$, where $R(\cdot) = 1 - F(\cdot)$ is the survival function (reliability function) and $\Lambda(t) = \int_0^t f(x)/R(x)\,dx$ the cumulative hazard rate function. Define $\log n$ as the two-iterated logarithm, with $\log n = \log(n \lor e)$.

Let us first introduce the assumptions under which these asymptotic properties are obtained. All asymptotic results are given for $m \to +\infty$.

A1. $\alpha \in [0, a]$ and $0 < a < 1$.

A2. $(r_i)_{i \geq 1}$ is a bounded sequence of nonnegative integers, $r_i \leq K$ for all $i \geq 1$, such that $\bar{r} = m^{-1} \sum_{i=1}^{m} r_i = r + O\left(\sqrt{\log m}\right)$, where $r$ is a nonnegative number.

A3. $\tau_1$ is a real number such that $F(\tau) < 1$.

A4. Let $G = 1 - R_{\tau+1}$. There exists a real number $\varepsilon \in [0, a)$ such that $G^{-1}([\varepsilon, a]) \subset (c, b) \subseteq \mathbb{R}^+$ and $\lambda$ is continuous and strictly positive on $(c, b)$.

Now, we introduce some useful notation. Let $(\alpha_{jm}^m)_{1 \leq j \leq m}$ be a triangular array of non-negative integers defined by $\alpha_{1m}^m = m$, $\alpha_{jm}^m = r_j + \cdots + r_m + m - j + 1$ for $2 \leq j \leq m - 1$, and $\alpha_{mm}^m = r_m + 1$. Denote by $u_\alpha$ the $\alpha$-quantile of the distribution function $G$, $u_\alpha = G^{-1}(\alpha)$, where $G^{-1}$ is taken in the generalized inverse sense ($G^{-1}(x) = \inf\{y : G(y) > x\}$) when $G$ is not invertible.

Finally, we recall the almost sure representation given in [2]: there exists a triangular array $(Z_{jm}^m)_{1 \leq j \leq m}$ of i.i.d. exponentially distributed random variables with mean 1 such that

$$\Lambda(X_{i:m:n}) = \sum_{j=1}^{i} \frac{Z_{jm}^m}{\alpha_{jm}^m} \quad \text{a.s.}$$
Theorem 1. Under assumptions A1–A4 and in a probability space $(\Omega_0, \mathcal{F}_0, P_0)$ rich enough containing all random variables and processes that we need, we have

$$\left| \sum_{j=1}^{[am]} \tilde{Z}_j^{(m)} - 1 - \frac{W'\left(\frac{a(1-a)}{a(1-a)}\right)}{\theta_m(a)} \right| = O \left( \frac{\delta_m}{\theta_m(a)} + \frac{1}{m} \right)$$

in probability, where $W'$ is a standard Brownian motion, $\theta_m(a) = \sqrt{1/(\sum_{k=1}^{[am]} (\alpha_k^n)^{-2})}$ and $\delta_m$ is a non-increasing positive sequence such that $\delta_m = o(\sqrt{m})$ as $m \to \infty$.

Remark 2.1. On account of the a.s. representation (1), we have to deal with the triangular array $(Z_j^{(m)})_{1 \leq j \leq m}$ of i.i.d. exponentially distributed random variables with mean 1. In Theorem 1, we write $\tilde{Z}_j^{(m)}$ in place of $Z_j^{(m)}$ because the probability space $(\Omega_0, \mathcal{F}_0, P_0)$ is a common probability space where $\tilde{Z}_j^{(m)}$ and $W'$ are well defined. In the proof of Theorem 1 (see Fact 4), we justify the existence of this space by standard arguments.

Theorem 2. Under the assumptions of Theorem 1, we have

$$\sqrt{m} \left( X_{[am]:m:n} - u_m \right) = \frac{\sqrt{a}}{(r+1)\lambda(u_m)} W'(\alpha) + O \left( \frac{\delta_m + 1}{m} \right)$$

in probability, where $\{W'(t), 0 \leq t \leq 1\}$ is a standard Brownian motion.

In the next theorem, we are concerned with the behaviour of the counting process

$$N_m(t) = \frac{1}{m} \sum_{i=1}^{m} 1(X_{i,m:n} \leq t), \quad t \geq 0.$$  

Our result is based on a Poisson approximation for the empirical distribution function. Roughly speaking, our result can be seen as a corollary of Lemma 3.1 of [11].

Theorem 3. If the assumptions of Theorem 1 are satisfied, then we can define Poisson processes $M_m(x)$ with $E[M_m(x)] = x$ such that

$$\sup_{0 \leq t < \infty} \left| N_m(t) - \frac{M_m(t)}{M_m(m)} \right| = O \left( \sqrt{\frac{\log_2 m}{m}} \right)$$

in probability.

Let $(X_i, Y_i), i = 1, 2 \ldots$, be independent copies of $(X, Y) \in \mathbb{R}^+ \times \mathbb{R}^+$. Suppose a progressively censored scheme is adopted for the $X$-sample and let


\[ Y_{1:n}, \ldots, Y_{m:n} \] the corresponding values of \( Y \). The random variables \( Y_{i:n} \), \( i = 1, \ldots, m \), are called induced order statistics (cf. [4] and [8] for instance).

Set \( m(x) = E[Y|X = x] \), \( \sigma^2(x) = \text{Var}(Y|X = x) \), \( \beta(x) = E[(Y - m(x))^4|X = x] \), and define

\[
A_m(t) = m^{-1/2} \sum_{i=1}^{m} (Y_{i:n} - m(X_{i:n})) 1(X_{i:n} \leq t), \quad t \geq 0.
\]

**Theorem 4.** Under the assumptions of Theorem 1, let \( \sigma^2(x) \) be of bounded variation and \( \beta(x) \) bounded. Then we can define Poisson processes \( M_m(x) \) with \( E[M_m(x)] = x \) and Brownian motions \( W_m \) such that

\[
\sup_{t \geq 0} |A_m(t) - W_m(G_m(t))| \to 0 \quad (6)
\]
in probability, where

\[
G_m(t) = (M_m(m))^{-1/2} \int_0^{1-R_{r+1}(t)} \sigma^2(s) dM_m(m s).
\]

3. **Proofs**

Proof of Theorem 1. Keeping in mind approximations results for triangular arrays given in [10], set

\[
\alpha_m \xi_j^{(m)} = \theta_m(a)(\tilde{Z}_j^{(m)} - 1) \quad \text{with} \quad \theta_m(a) = \sqrt{1/\sum_{k=1}^{[am]} (\alpha_k^{m})^{-2}}.
\]

Define the polygonal process \( \tilde{S}_{(m)}(t) \) by

\[
S_{(m)}(t) = \sum_{k=1}^{r} \xi_k^{(m)} + \frac{t - t_{m,r}}{t_{m,r+1}} \xi_{r+1}^{(m)},
\]

where \( 0 < r < [am] \) and \( t_{m,r} = \sum_{k=1}^{r} \sigma_{m,k}^{2} \) for \( 1 \leq r \leq [am] \), with \( t_{m,[am]} = \alpha_{[1-a]}^{m} \) for \( a \in [0, a] \) and \( m \) large enough (see Lemma 1 of [1]).

Under our assumptions, we have the following facts:

Fact 1. From \( m - k + 1 \leq \alpha_k^m \leq (K + 1)(m - k + 1) \), for \( k \leq am \) we have \( \alpha_k^m \geq (1 - a)m \) which yields

\[
\frac{am}{(1 - a)^2 m^2} \geq \sum_{j=1}^{[am]} \frac{1}{(\alpha_k^m)^2}.
\]

Fact 2. From Fact 1, for

\[
B_{m,j} = \alpha_j^m \sqrt{\sum_{k=1}^{[am]} \frac{1}{(\alpha_k^m)^2}}
\]
we have
\[(8) \quad c\sqrt{m} \leq B_{m,j} \leq \frac{(K + 1)\sqrt{am}}{(1 - \alpha)} ,\]
where \(c\) is a constant > 0.

Fact 3. For a random variable \(Z\) with exponential law of mean 1, we have
\[(9) \quad P\left(\frac{Z - 1}{B_{m,j}} \leq x\right) = \left(1 - e^{-(1+xB_{m,j})}\right)1(1 + xB_{m,j} \geq 0),\]
where \(1(\cdot)\) stands for the indicator function.

Fact 4. The result
\[(10) \quad L_m(\delta) = \sum_{j=1}^{k_m} E\left[\hat{\xi}_{j}^{(m)} 1(\hat{\xi}_{j}^{(m)} > \delta)\right] \rightarrow 0 \text{ as } m \rightarrow \infty \]
corresponds to Condition (C) in Corollary 4 of [10].

By using Fact 3, the moment part of (10) can be evaluated as
\[
E\left\{\left(\frac{Z - 1}{B_{m,j}}\right)^2 1(|Z - 1| > \delta B_{m,j})\right\} = \\
\int_{\mathbb{R}} B_{m,j}x^2 e^{-(1+xB_{m,j})}1(1 + xB_{m,j} \geq 0) 1(|x| > \delta) \, dx.
\]
Then we have to consider the following two cases:
\[
I = \int_{\mathbb{R}} \frac{B_{m,j}x^2}{e^{(1+xB_{m,j})}}1(1 + xB_{m,j} \geq 0) 1(x > \delta) \, dx = e^{-1} \int_{\delta B_{m,j}}^{+\infty} \frac{t^2 e^{-t}}{(B_{m,j})^2} \, dt,
\]
\[
II = \int_{\mathbb{R}} \frac{B_{m,j}x^2}{e^{(1+xB_{m,j})}}1(1 + xB_{m,j} \geq 0) 1(x < -\delta) \, dx \leq \\
\leq \int_{0}^{1-\delta B_{m,j}} \left(\frac{u - 1}{B_{m,j}}\right)^2 e^{-u} \, du.
\]
Using (8), we have \(1 - \delta B_{m,j} \leq 1 - \min_{1 \leq j \leq [am]} B_{m,j} \delta \leq 1 - c\delta \sqrt{m}\), hence \(II \rightarrow 0\).

Let us now evaluate \(I\). The sum term is bounded by \(\delta^2 \sum_{j=1}^{[am]} e^{-\delta \sqrt{m}}\), which goes to 0 by the integral test, so (10) is obtained by (8).

By (7) and Corollary 4 of [10], we have \(PS_{(m)}|B(C[0,1]) \rightarrow W|B(C[0,1])\). On account of this last argument, the probability space \((\Omega_0, \mathcal{F}_0, P_0)\) exists and is well defined (see Theorem 11.7.1 of [9]). In that space, we can define \(\hat{S}_m\) and a standard Wiener process \(W\) such that \(PS_m = P_0\hat{S}_m\) and \(W' = W\) in law, with \(d(\hat{S}_m, W') \rightarrow 0\) both in probability and a.s.
Fact 5. By [9], there exist positive numbers $\delta_m \to 0$ such that $ho(PS(m), W) \leq \delta_m$, i.e., where $\rho(\cdot, \cdot)$ stands for the Prohorov distance which associates a metric with weak convergence.

This last inequality yields
\[
\left| \sum_{j=1}^{[am]} \frac{\tilde{Z}_j^{(m)}}{\alpha_j^{m}} - W' \left( \frac{\alpha(1-a)}{a(1-a)} \right) \right| = O \left( \frac{\delta_m}{\theta_m(a)} + \frac{1}{m} \right),
\]
in probability, where the rate of convergence is obtained as in the proof of Theorem 11.7.1 of [9], which completes the proof.

\[\square\]

Proof of Theorem 2. By (1) and Assumption A4, $\Lambda^{-1}$ admits a first order Taylor expansion. Then we have
\[
X_{[am];m} \overset{a.s.}{=} \Lambda^{-1} \left( \Lambda(u_\alpha) + \sum_{j=1}^{[am]} \frac{\tilde{Z}_j^{(m)}}{\alpha_j^{m}} - \Lambda(u_\alpha) \right)
\]
\[
= \Lambda^{-1} (\Lambda(u_\alpha)) + \frac{\beta_{[am]}}{\Lambda^{-1}(\Lambda(\phi_m))},
\]
where $\beta_{[am]} = \sum_{j=1}^{[am]} \frac{Z_j^{(m)}}{\alpha_j^{m}} - \Lambda(u_\alpha)$ and $\phi_m$ is such that $\Lambda(\phi_m)$ belongs to the segment with extremities $\Lambda(u_\alpha)$ and $\Lambda(u_\alpha) + \beta_{[am]}$.

It is sufficient to study the asymptotic behaviour of $\beta_{[am]}$. We have
\[
\sqrt{m} \beta_{[am]} = I^{(m)} + II^{(m)} + III^{(m)}
\]
with
\[
I^{(m)} = \sqrt{m} \sum_{j=1}^{[am]} \frac{\tilde{Z}_j^{(m)} - 1}{\alpha_j^{m}},
\]
\[
II^{(m)} = \sqrt{m} \sum_{j=1}^{[am]} \frac{1}{\alpha_j^{m}} - \frac{1}{(r+1)(m-j+1)} = o(1)
\]
and
\[
III^{(m)} = \sqrt{m} \left( \sum_{j=1}^{[am]} \frac{1}{(r+1)(m-j+1)} - \Lambda(u_\alpha) \right) = O \left( \frac{1}{m} \right),
\]
where the $o(\cdot)$ and $O(\cdot)$ terms are obtained by Corollary 1 of [1] under Assumptions A1–A4. Moreover, $I^{(m)}/\sqrt{m}$ is given by (2). Then we can write
\[
\sqrt{m} \beta_{[am]} = \frac{\sqrt{m} W' \left( \frac{\alpha(1-a)}{a(1-a)} \right)}{\theta_m(a)} + O \left( \delta_m + \frac{1}{m} \right).
\]
in probability and, by (12),
\[ \sqrt{m} \left( X_{[m]} - u_\alpha \right) = \sqrt{m} W \left( \frac{\alpha(1-\alpha)}{\theta_m(a)\lambda(u_\alpha)} \right) + O \left( \delta_m + \frac{1}{m} \right) \]
in probability, with
\[ \frac{\sqrt{m}}{\theta_m(a)} \to \sqrt{\frac{a}{(1+r)^2(1-a)}}, \]
which completes the proof. □

Proof of Theorem 3. As in [6], let us define the counting process
\[ \hat{N}_m(t) = \frac{1}{m} \sum_{i=1}^{m} 1(\hat{Y}_{i:m} \leq \Lambda(t)), \quad t \geq 0, \]
where
\[ \hat{Y}_{i:m} = \sum_{j=1}^{i} Z_j^{(m)} \frac{1}{(r+1)(m-j+1)}, \quad 1 \leq i \leq m. \]

Remark 3.1. In [6] it was stated that the \( \hat{Y}_{i:m} \) have the same distribution as an order statistic of an \( m \)-sample of exponential random variables of mean \( 1/(r+1) \).

In order to prove our result we establish the following facts.

Fact 6. Having in mind Remark 3.1, we can rewrite (13) as
\[ \hat{N}_m(\tau) = \frac{1}{m} \sum_{i=1}^{m} 1(U_{i:m} \leq \tau), \]
where \( \tau = 1 - \exp \{ (r+1)\Lambda(t) \} \), \( t \geq 0 \), and the \( U_{i:m} \) are the order statistics of an \( m \)-sample of uniformly distributed on \([0,1]\) random variables (see [2] about monotone transformations in this context).

On account of the above transformation, we can also rewrite (4) as
\[ N_m(\tau) = \frac{1}{m} \sum_{i=1}^{m} 1(V_{i:m} \leq \tau), \]
where \( 1 - \exp \{ (r+1)Y_{i:m} \} = V_{i:m} \) with \( Y_{i:m} = \Lambda(X_{i:m}) \) \( i = 1, \ldots, m \).

Fact 7. By Lemma 3.1 of [11] and (14), we can define Poisson processes \( M_m(\tau) \) with \( E[M_m(\tau)] = \tau \) such that
\[ P \left( \sup_{0 \leq \tau \leq 1} \left| \frac{\hat{N}_m(\tau) - M_m(\tau)}{M_m(m)} \right| > \frac{1}{m} (x + c \log m) \right) \leq \exp (-cx) \]
for all \( x > 0 \).
Fact 8. On account of Assumption A2, we can mimic the proof of Proposition 3 of [6], but we use the law of the iterated logarithm instead of the strong law of large numbers, to get

$$\sup_{0 \leq t \leq \tau_1} \left| \hat{N}_m(t) - N_m(t) \right| = O \left( \sqrt{\frac{\log 2}{m}} \right)$$

in probability.

Facts 6, 7 and 8 suffice to establish (5).

Proof of Theorem 4. Consider the monotone transformations defined in (14) and for $0 \leq t \leq 1$ consider

$$S_m([mt]) = m^{-1/2} \sum_{i=1}^{[mt]} (Y_{i;m:n} - m(U_{i;m:n})), \quad k = 1, \ldots, m.$$ 

Sums of conditional independent random variables can be represented by Brownian motion at stopping times. This is proved in [3] by applying the Skorokhod embedding theorem. Then we obtain the representation of $S_m([mt])$ by $W_m(t)$ in some common probability space. Further, we have

$$\sup_{0 \leq t \leq 1} \left| W_m \left( \int_0^t \sigma^2(s) dN_m(s) \right) - M_m^{-1/2} W_m \left( \int_0^t \sigma^2(s) dM_m(s) \right) \right| \leq I + II,$$

where

$$I = \sup_{0 \leq t \leq 1} \left| W_m \left( \int_0^t \sigma^2(s) dN_m(s) \right) - W_m \left( \int_0^t \sigma^2(s) d\hat{N}_m(s) \right) \right|$$

and

$$II = \sup_{0 \leq t \leq 1} \left| W_m \left( \int_0^t \sigma^2(s) d\hat{N}_m(s) \right) - M_m^{-1/2} W_m \left( \int_0^t \sigma^2(s) dM_m(s) \right) \right|,$$

where without loss of generality we have considered $t = \tau$ (see (14)).

Remark by using (16) that

$$\sup_{0 \leq t \leq 1} \left| \int_0^t \sigma^2(s) dN_m(s) - \int_0^t \sigma^2(s) d\hat{N}_m(s) \right| \leq C \sup_{0 \leq t \leq 1} \left| N_m(t) - \hat{N}_m(t) \right| \leq 1,$$

where $C$ is a constant.

Since $\sigma^2$ is of bounded variation, we apply Lemma 1.1.1 of [7] to get $I \to 0$. In the same way as in Lemma 3.2 of [11], we obtain that $II \to 0$ in probability.
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