We consider a class of second order elliptic problems on perforated domains with small holes of size $\varepsilon \delta$, distributed periodically with the period $\varepsilon$. A non homogeneous Neumann condition is prescribed on the boundary of some holes; on the boundary of the other ones, it is a homogeneous Dirichlet condition that is considered. We are interested here to give the limit behaviour of the problems when $\varepsilon \to 0$ and $\delta = \delta(\varepsilon) \to 0$. To do so, we apply the periodic unfolding method introduced in [5], that allows us to consider general operators with highly oscillating (with $\varepsilon$) coefficients and rather general geometries.

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1. INTRODUCTION

The aim of this work is to apply the periodic unfolding method introduced by Cioranescu, Damlamian and Griso [3] to the homogenization of a class of elliptic second-order equations with highly oscillating coefficients, in perforated domains in $\mathbb{R}^n$ ($n > 2$), with mixed-type conditions on the boundary of holes. The holes we consider here are $\varepsilon$-periodically distributed and their size $r(\varepsilon)$ is such that $r(\varepsilon)/\varepsilon \to 0$. Throughout the paper, we call such holes small holes. We consider the case where in each period there are two different kinds of holes: some are of size of order of $\varepsilon \delta_1$, and the other ones of size of order of $\varepsilon \delta_2$ with $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$, $\delta_1 \to 0$ and $\delta_2 \to 0$ as $\varepsilon \to 0$. On the boundary of holes of size $\varepsilon \delta_1$ a non homogeneous Neumann condition is prescribed while on the boundary of holes of size $\varepsilon \delta_2$ and on the exterior boundary of the domain, a homogeneous Dirichlet condition is imposed.

The asymptotic behaviour of the homogeneous Dirichlet problem for the Poisson equation in perforated domains with holes of size $\varepsilon^k$, $k > 0$, was studied

REV. ROUMAINE MATH. PURES APPL., 53 (2008), 5-6, 389-406
by Cioranescu and Murat [11]. They showed that the size $\varepsilon^{n/n-2}$ is “critical” in the following sense: the limit problem not only contains the Laplacian but also an additional zero order term, called “strange term” in [11], depending on the capacity of the set of holes in the limit. Conca and Donato [12] studied the non homogeneous Neumann problem for the Laplacian in the same geometrical framework. Now, the critical size is of order $\varepsilon^{n/n-1}$, and the contribution of the holes in the limit is reflected by an additional right-hand side integral term.

The case of mixed-type boundary conditions on the holes has been studied by Cardone, D’Apice and De Maio [2]. They consider the following setting: the size of the holes is of order of $\varepsilon$ and a homogeneous Neumann boundary condition is assumed on their boundaries, except on a flat portion of size $\varepsilon^{n/n-2}$, where a homogeneous Dirichlet condition is prescribed. In the limit problem, as expected, the “strange term” appears again. A related problem was studied by Corbo Esposito, D’Apice and Gaudiello [13] where the holes are now of the critical size $\varepsilon^{n/n-1}$. A non homogeneous Neumann condition is imposed on the boundary of each hole, except a flat portion of size $\varepsilon^{n/n-2}$, where a homogeneous Dirichlet condition is given. In the limit problem the two additional terms appear, the one giving the contribution of the Neumann condition as in [12], and the strange term corresponding to the critical size for Dirichlet conditions from [11]. In all these papers, standard variational homogenization methods are used. In particular, they need to introduce extension operators (since the domains are changing with $\varepsilon$) and to construct test functions, specific for each situation.

In the present paper we take the advantage of the simplicity of the periodic unfolding when applied to perforated domains, as it can be seen in Cioranescu, Donato and Zaki [8]-[10]. Indeed, the periodic unfolding, being a fixed-domain method, no extension operator is needed. Moreover, it does not use any construction of special test functions and so, one can treat general second order operators with highly oscillating (in $\varepsilon$) coefficients, which was not the case in the papers quoted above.

The standard case of homogenization in perforated domains, i.e., with holes of size $\varepsilon$ was studied via the periodic unfolding method in [8]-[10] where Robin or nonlinear boundary conditions were treated. To do so, a boundary unfolding operator was introduced and studied. For small holes, applications of the same method to sieve type problems, can be found in Cioranescu, Damlamian, Griso and Onofrei [7] which also contains a complete list of the properties of the unfolding operator for fixed domains, for perforated domains with holes of size $\varepsilon\delta$ ($\delta$ is another small parameter), and of a boundary unfolding operator corresponding to these small holes. Let us mention that the unfolding operator for “small” holes appeared for the first time (as a change of variables) in Casado-Díaz [3] and a boundary layer unfolding operator for
this case in Onofrei [16]. The situation from [12], of small holes of size $\varepsilon^{n/n-1}$ with non homogeneous Neumann conditions, was treated by unfolding in Ould Hammouda [15]. Our results here rely extensively on this last work.

The paper is organized as follows. In Section 2, following [3], [14] and [7], we recall the definitions and properties of the unfolding operator $T_\varepsilon$ for fixed domains and of the unfolding operator $T_{\varepsilon,\delta}$ for domains with “small” holes. We also recall the properties of a boundary unfolding operator $T_{b,\varepsilon,\delta}$ that was introduced in [15]. Section 3 is devoted to the setting of the problem and to the proof of our main homogenization results, Theorems 3.2 and 3.3. We show that if $\delta_1$ and $\delta_2$ are chosen in order to get the critical size corresponding to Neumann, respectively, to Dirichlet small holes, the limit problem contains the two contributions of the holes, an additional right hand side term, and a “strange” term. Due to the oscillating character of the coefficients in the original problem, in the homogenized equation, the partial differential operator with constant coefficients, is the “standard” homogenized one (see, for instance, Bensoussan, Lions and Papanicolaou [1]).

2. UNFOLDING OPERATORS

2.1. General notation

We start by introducing some general notation, in particular the definition of perforated domains with small holes, the geometric setting in this paper.

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ such that $|\partial \Omega|=0$, and $Y=\left[-\frac{1}{2},\frac{1}{2}\right]^n$ the periodicity (or reference) cell. Let now introduce the notation

$$\hat{\Omega}_\varepsilon = \text{interior}\left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon (\xi + \bar{Y}) \right\},$$

where $\Xi_\varepsilon = \{ \xi \in \mathbb{Z}^n, \varepsilon (\xi + Y) \subset \Omega \}$,

and set $\Lambda_\varepsilon = \Omega \setminus \hat{\Omega}_\varepsilon$. The set $\hat{\Omega}_\varepsilon$ is the largest finite union of $\varepsilon Y$ cells contained in $\Omega$ while $\Lambda_\varepsilon$ is the subset of $\Omega$ containing the parts from $\varepsilon Y$ cells intersecting the boundary $\partial \Omega$ (see Figure 1).

Let $B$ be an open set such that $B \subset \subset Y$. Introduce now the set $Y_\delta = Y \setminus \delta B$ supposed to be connected; $Y_\delta$ correspond to the part occupied by the material in the cell $Y$.

The set $B_{\varepsilon,\delta}$ of $\varepsilon$-periodic holes of size $\varepsilon \delta$ is defined as

$$B_{\varepsilon,\delta} = \bigcup_{\xi \in \mathbb{Z}^n} \varepsilon (\xi + \delta B).$$

The perforated domain $\Omega_{\varepsilon,\delta}$, with holes of size $\varepsilon \delta$ is defined as

$$\Omega_{\varepsilon,\delta} = \left\{ x \in \Omega \left| \left\{ \frac{x}{\varepsilon} \right\} \in Y_\delta \right\} \right.$$. 

(2.2)
If \( \{e_1, \ldots, e_n\} \) is the canonical basis of \( \mathbb{R}^n \), for any \( z \in \mathbb{R}^n \) we denote by \([z]_Y\) the unique integer combination \( \sum_{j=1}^{n} \ell_j e_j \) such that \( z - [z]_Y \) belongs to \( Y \). Set \( \{z\}_Y = z - [z]_Y \in Y \) a.e. for \( z \in \mathbb{R}^n \) (see for more details, [5], [6] and [14]). Then for each \( x \in \mathbb{R}^n \), one has

\[
x = \varepsilon \left( \left\lfloor \frac{x}{\varepsilon} \right\rfloor_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \text{a.e. for } x \in \mathbb{R}^n
\]

(see Figure 2).
2.2. The unfolding operator $T_\varepsilon$ for fixed domains

In this section we recall the general properties of the periodic unfolding operator introduced in [5], for more details see [7], and [14].

**Definition 2.1.** For $\phi$ Lebesgue-measurable on $\Omega$, the unfolding operator $T_\varepsilon$ is defined as:

$$T_\varepsilon(\phi)(x, y) = \begin{cases} 
\phi\left(\frac{x}{\varepsilon}\right) + \varepsilon y & \text{a.e. for } (x, y) \in \hat{\Omega}_\varepsilon \times Y, \\
0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y.
\end{cases}$$

It is obvious from (2.1) that for $v$ and $w$ Lebesgue-measurable we have

(2.3) $$T_\varepsilon(vw) = T_\varepsilon(v) T_\varepsilon(w).$$

**Proposition 2.2.** Let $\phi$ be measurable on $Y$ and extend it by periodicity to the whole of $\mathbb{R}^n$. Set

$$\phi_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \text{ a.e. for } x \in \mathbb{R}^n.$$ 

Then

$$T_\varepsilon(\phi_\varepsilon|\Omega)(x, y) = \begin{cases} 
\phi(y) & \text{a.e. for } (x, y) \in \hat{\Omega}_\varepsilon \times Y, \\
0 & \text{a.e. for } (x, y) \in \Lambda_\varepsilon \times Y.
\end{cases}$$

If $\phi$ belongs to $L^p(Y)$, $p \in [1, +\infty]$, and if $\Omega$ is bounded, then

$$T_\varepsilon(\phi_\varepsilon|\Omega) \rightarrow \phi \text{ strongly in } L^p(\Omega \times Y).$$

Let $\{v_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$ such that $v_\varepsilon \rightarrow v$ strongly in $L^p(\Omega)$. Then

$$T_\varepsilon(v_\varepsilon) \rightarrow v \text{ strongly in } L^p(\Omega \times Y).$$

**Remark 2.3.** Observe that an equivalent way to define $\phi_\varepsilon$ on $\mathbb{R}^n$, is to take simply

$$\phi_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right).$$

**Proposition 2.4.** Let $p$ in $[1, +\infty]$ and $v \in L^p(\Omega)$. Then

$$\int_{\Omega \times Y} T_\varepsilon(v)(x, y) \, dx \, dy = \int_\Omega v(x) \, dx - \int_{\Lambda_\varepsilon} v(x) \, dx = \int_{\hat{\Omega}_\varepsilon} v(x) \, dx.$$

As a consequence of Proposition 2.4,

(2.4) $$\left| \int_\Omega v \, dx - \int_{\Omega \times Y} T_\varepsilon(v) \, dx \, dy \right| \leq \int_{\Lambda_\varepsilon} |v| \, dx,$$

and so, any integral of a function $v$ on $\Omega$, is “almost equivalent” to the integral of its unfolded on $\Omega \times Y$. The “integration defect” only comes from the cells intersecting the boundary $\partial \Omega$ and is controlled by the right hand side integral...
in (2.4). This remark led in [5] the introduction of the so-called “unfolding criterion for integrals” (u.c.i.):

**Proposition 2.5.** If \( \{ \phi_\epsilon \} \) is a sequence in \( L^1(\Omega) \) satisfying \( \int_{\Lambda_\epsilon} |\phi_\epsilon| \, dx \to 0 \), then

\[
\int_{\Omega} \phi_\epsilon \, dx - \int_{\Omega \times Y} T_\epsilon(\phi_\epsilon) \, dxdy \to 0,
\]

property that is denoted

\[
\int_{\Omega} \phi_\epsilon \, dx \overset{T_\epsilon}{\to} \int_{\Omega \times Y} T_\epsilon(\phi_\epsilon) \, dxdy.
\]

This immediately justifies the next result, essential when dealing with homogenization problems.

**Proposition 2.6.** Let \( \{ u_\epsilon \} \) be a bounded sequence in \( L^p(\Omega) \) with \( p \in ]1, +\infty[ \) and \( v \) in \( L^{p'}(\Omega) \) (\( 1/p + 1/p' = 1 \)). Then

\[
\int_{\Omega} u_\epsilon v \, dx \overset{T_\epsilon}{\to} \int_{\Omega \times Y} T_\epsilon(u_\epsilon)T_\epsilon(v) \, dxdy.
\]

Assume now that \( \partial \Omega \) is bounded. Let \( \{ u_\epsilon \} \) be a bounded sequence in \( L^p(\Omega) \) and \( \{ v_\epsilon \} \) a bounded sequence in \( L^q(\Omega) \) with \( 1/p + 1/q < 1 \). Then

\[
\int_{\Omega} u_\epsilon v_\epsilon \, dx \overset{T_\epsilon}{\to} \int_{\Omega \times Y} T_\epsilon(u_\epsilon)T_\epsilon(v_\epsilon) \, dxdy.
\]

The main results concerning the unfolding operator \( T_\epsilon \) is as follows

**Proposition 2.7.** Let \( \{ w_\epsilon \} \) be a sequence in \( H^1(\Omega) \) such that \( w_\epsilon \rightharpoonup w \) weakly in \( H^1(\Omega) \). Then, up to a subsequence, there exists \( \hat{w} \in L^2(\Omega; H^1_{\text{per}}(Y)) \) such that

\[
T_\epsilon(\nabla w_\epsilon) \rightharpoonup \nabla_x w + \nabla_y \hat{w} \quad \text{weakly in} \quad L^2(\Omega \times Y).
\]

2.3. **The unfolding operator** \( T_{\epsilon,\delta} \) **depending on two parameters** \( \epsilon \) **and** \( \delta \)

In the next section we will consider domains perforated by “small” holes of size \( \epsilon \delta \), periodically distributed with period \( \epsilon Y \). This geometry of domains requires the introduction of a new unfolding operator \( T_{\epsilon,\delta} \) depending on both parameters \( \epsilon \) and \( \delta \). As mentioned above, this operator was first introduced in [3]. It was used for the study of reticulated structures in [4] and for sieve problems in [7].
Definition 2.8. For \( \phi \in L^p(\Omega) \), \( p \in [1, \infty] \), the unfolding operator \( T_{\varepsilon, \delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^n) \) is defined as

\[
T_{\varepsilon, \delta}(\phi)(x, z) = \begin{cases} 
T_{\varepsilon}(\phi)(x, \delta z) & \text{a.e. for } (x, z) \in \tilde{\Omega}_\varepsilon \times \frac{1}{\delta} Y, \\
0 & \text{otherwise.}
\end{cases}
\]

For \( u \in L^2(\Omega) \), from Definition 2.8 the estimates

\[
(2.8) \quad \begin{align*}
(i) \quad & \| T_{\varepsilon, \delta}(u) \|_{L^2(\Omega \times \mathbb{R}^n)}^2 \leq \frac{1}{\delta^n} \| u \|_{L^2(\Omega)}^2, \\
(ii) \quad & \left| \int_{\Omega} u \, dx - \delta^n \int_{\Omega \times \mathbb{R}^n} T_{\varepsilon, \delta}(u) \, dx \, dz \right| \leq \int_{\Lambda_\varepsilon} |u| \, dx,
\end{align*}
\]

are straightforward.

The operator \( T_{\varepsilon, \delta} \) was studied in details in [7]. To recall its properties (that will be widely used in the present paper), we need to introduce the notion of local average of a function.

Definition 2.9. The local average \( M_{Y}^\varepsilon : L^p(\Omega) \to L^p(\Omega) \) is defined for any \( \phi \) in \( L^p(\Omega) \), \( 1 \leq p < \infty \), as

\[
M_{Y}^\varepsilon(\phi)(x) = \int_Y T_{\varepsilon}(\phi)(x, y) \, dy.
\]

It is classical that if \( \{ v_\varepsilon \} \) is a bounded sequence in \( L^p(\Omega) \) such that \( v_\varepsilon \to v \) strongly in \( L^p(\Omega) \), then

\[
(2.9) \quad M_{Y}^\varepsilon(v_\varepsilon) \to v \text{ strongly in } L^p(\Omega).
\]

Now, we can list some of the properties of \( T_{\varepsilon, \delta} \) from [7], needed in the next section.

Proposition 2.10. Suppose \( n \geq 3 \) and denote by \( 2^* \) the Sobolev exponent \( \frac{2n}{n-2} \) associated with 2. Let \( \omega \) be open and bounded in \( \mathbb{R}^n \). Then

\[
\begin{align*}
\| \nabla_z(T_{\varepsilon, \delta}(u)) \|_{L^2(\Omega \times \frac{1}{\delta} Y)}^2 & \leq \frac{\varepsilon^2}{\delta^{n-2}} \| \nabla u \|_{L^2(\Omega)}^2, \\
\| T_{\varepsilon, \delta}(u - M_{Y}^\varepsilon(u)) \|_{L^2(\Omega; L^{2^*}(\mathbb{R}^n))}^2 & \leq \frac{C \varepsilon^2}{\delta^{n-2}} \| \nabla u \|_{L^2(\Omega)}^2, \\
\| T_{\varepsilon, \delta}(u) \|_{L^2(\Omega \times \omega)}^2 & \leq \frac{2C \varepsilon^2}{\delta^{n-2}} |\omega|^{\frac{2}{n}} \| \nabla u \|_{L^2(\Omega)}^2 + 2 |\omega| \| u \|_{L^2(\Omega)}^2,
\end{align*}
\]

where \( C \) denotes the Sobolev-Poincaré-Wirtinger constant for \( H^1(Y) \).

An unfolding criterion for integrals also holds for \( T_{\varepsilon, \delta} \).
Proposition 2.11. If \( \{w_\varepsilon\} \) is a sequence in \( L^1(\Omega) \) such that \( \int_{A_\varepsilon} |w_\varepsilon| \, dx \to 0 \), then

\[
\int_\Omega w_\varepsilon \, dx \lesssim \delta^n \int_{\Omega \times \mathbb{R}^n} T_{\varepsilon,\delta}(w_\varepsilon) \, dxdz.
\]

If \( \{u_\varepsilon\} \) is bounded in \( L^2(\Omega) \) and \( \{v_\varepsilon\} \) is bounded in \( L^p(\Omega) \) with \( p > 2 \), then

\[
\int_\Omega u_\varepsilon v_\varepsilon \, dx \lesssim \delta^n \int_{\Omega \times \mathbb{R}^n} T_{\varepsilon,\delta}(u_\varepsilon)T_{\varepsilon,\delta}(v_\varepsilon) \, dxdz.
\]

2.4. The boundary unfolding operator \( T_{\varepsilon,\delta}^b \)

We again use the notation from Section 2.1. From now on, we suppose that the set \( B \) has a Lipschitz boundary. We define a linear unfolding operator on the boundary of the holes \( B_{\varepsilon,\delta} \), specific to the case of domains with volume-distributed very small holes.

Definition 2.12. Let \( \phi \in L^p(\partial B_{\varepsilon,\delta}) \), with \( p \in [1, +\infty[ \). The boundary unfolding operator \( T_{\varepsilon,\delta}^b \) is defined as

\[
T_{\varepsilon,\delta}^b(\phi)(x, z) = \phi\left( \frac{x}{\varepsilon} + \varepsilon \delta z \right) \quad \text{a.e. for } x \in \mathbb{R}^n, \ z \in \partial B.
\]

For holes of size of order of \( \varepsilon \) (i.e., with \( \delta = 1 \)), such an operator, denoted \( T_{\varepsilon}^b \), was introduced for the first time in [8], its definition is exactly (2.12) with \( \delta = 1 \). Most of the properties of \( T_{\varepsilon,\delta}^b \) are almost transcriptions of the corresponding ones of \( T_{\varepsilon}^b \) and are obtained by a simple change of variables. For more details we refer the reader to [15].

Let \( g \) belong to \( L^2(\partial B) \). Denote by \( \mathcal{M}_{\partial B}(g) \) its mean value on \( \partial B \), namely,

\[
\mathcal{M}_{\partial B}(g) = \frac{1}{|\partial B|} \int_{\partial B} g \, ds.
\]

We now recall the following two propositions from [15], needed later on.

Proposition 2.13. Let \( \phi \in L^2(\partial B_{\varepsilon,\delta}) \). Then

\[
\int_{\partial B_{\varepsilon,\delta}} \phi(x) \, ds = \frac{\delta^{n-1}}{\varepsilon} \int_{\mathbb{R}^n \times \partial B} T_{\varepsilon,\delta}^b(\phi^{\varepsilon,\delta})(x, z) \, dx \, ds.
\]

Proposition 2.14. Let \( g \in L^2(\partial B) \) and set

\[
g_\varepsilon(x) = g\left( \frac{1}{\delta} \left\{ \frac{x}{\varepsilon} \right\} \right), \quad \text{for all } x \in \partial B_{\varepsilon,\delta}.
\]

Then

\[
\left| \int_{\partial B_{\varepsilon,\delta}} g_\varepsilon(x)\phi \, ds \right| \leq C \frac{\delta^{n-1}}{\varepsilon} (|\mathcal{M}_{\partial B}(g)| + \varepsilon \delta) \|\nabla \phi\|_{L^2(\Omega)^N}.
\]
Moreover, for all $\phi \in H^1(\Omega)$, as $\varepsilon \to 0$, one has the convergence

\begin{equation}
\frac{\varepsilon}{\delta^{N-1}} \int_{\partial B_{\varepsilon, \delta}} g_\varepsilon(x) \phi(x) \, ds \to |\partial B| \mathcal{M}_{\partial B}(g) \int_{\Omega} \phi(x) \, dx.
\end{equation}

3. HOMOGENIZATION RESULTS

3.1. Setting of the problem

As in Section 1, $\Omega$ is a bounded open set in $\mathbb{R}^n$ such that $|\partial \Omega| = 0$. Let $B$ and $T$ be two open sets such that $B \subset Y$, $T \subset Y$ and $B \cap T = \emptyset$. The part occupied by the material in the cell $Y$ is now $Y_{\delta_1, \delta_2} = Y \setminus (\delta_1 T \cup \delta_2 B)$, supposed to be connected. Here, $\delta_1$ and $\delta_2$ are two small parameters going to zero independently. The perforated domain $\Omega_{\varepsilon, \delta_1, \delta_2}$ where we will set the problem is obtained by removing from $\Omega$ the set of holes $B_{\varepsilon, \delta_1}$ and $T_{\varepsilon, \delta_2}$, namely,

\begin{equation}
B_{\varepsilon, \delta_1} = \bigcup_{\xi \in \mathbb{Z}^n} \varepsilon (\xi + \delta_1 B), \quad T_{\varepsilon, \delta_2} = \bigcup_{\xi \in \mathbb{Z}^n} \varepsilon (\xi + \delta_2 T).
\end{equation}

An example of such a geometry is depicted in Figure 3 below.

Fig. 3. The perforated domain $\Omega_{\varepsilon, \delta_1, \delta_2}$. 
This means that $\Omega_{\varepsilon,\delta_1,\delta_2}$ has perforations of size of order of $\varepsilon\delta_1$ and of size of order $\varepsilon\delta_2$ at the same time. Actually (see definition (2.2) for comparison),

$$
\Omega_{\varepsilon,\delta_1,\delta_2} = \Omega \setminus (B_{\varepsilon,\delta_1} \cup T_{\varepsilon,\delta_2}) = \left\{ x \in \Omega \mid \left\{ \frac{x}{\varepsilon} \right\} \in \mathcal{Y}_{\delta_1,\delta_2} \right\}.
$$

Assume that the matrix field $A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1 \leq i,j \leq n}$ is such that there exist two real numbers $\alpha$ and $\beta$ satisfying

$$
\alpha |\lambda|^2 \leq (A^\varepsilon(x)\lambda, \lambda) \quad \text{and} \quad |A^\varepsilon(x)\lambda|^2 \leq \beta(A^\varepsilon(x)\lambda, \lambda)
$$

for any $\lambda \in \mathbb{R}^n$ and a.e. $x$ in $\Omega$.

Let $g \in L^2(\partial B)$ and (recalling Remark 2.3), set

$$
g_{\varepsilon,\delta_1} (x) = g \left( \frac{1}{\delta_1} \left\{ \frac{x}{\varepsilon} \right\} \right) \quad \text{for all} \quad x \in \partial B_{\varepsilon,\delta_1}.
$$

For $f \in L^2(\Omega)$ consider the problem

$$
\begin{cases}
-\text{div}(A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2}) = f & \text{in} \ \Omega_{\varepsilon,\delta_1,\delta_2}, \\
A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2} \nu_B = g_{\varepsilon,\delta_1} & \text{on} \ \partial B_{\varepsilon,\delta_1}, \\
u_{\varepsilon,\delta_1,\delta_2} = 0 & \text{on} \ \partial_{\text{ext}} \Omega_{\varepsilon,\delta_1,\delta_2} \cup \partial T_{\varepsilon,\delta_2},
\end{cases}
$$

where $\partial_{\text{ext}} \Omega_{\varepsilon,\delta_1,\delta_2}$ is the exterior part of the boundary $\partial \Omega_{\varepsilon,\delta_1,\delta_2}$ and $\nu_B$ is the unit exterior normal to the set $B$. Observe that, by construction, $\nu_B$ also is the unit exterior normal to the set $B_{\varepsilon,\delta_1}$ (see Figure 2).

We introduce the space

$$V_{\delta_1,\delta_2}^\varepsilon = \{ \varphi \in H^1(\Omega_{\varepsilon,\delta_1,\delta_2}) \mid \varphi = 0 \text{ on } \partial_{\text{ext}} \Omega_{\varepsilon,\delta_1,\delta_2} \cup \partial T_{\varepsilon,\delta_2} \},$$

and in the sequel will still denote by $\varphi$ in $V_{\delta_1,\delta_2}^\varepsilon$, its extension by zero in $T_{\varepsilon,\delta_2}$.

Then the variational formulation of problem (3.5) is

$$
\begin{cases}
\text{Find } u_{\varepsilon,\delta_1,\delta_2} \in V_{\delta_1,\delta_2}^\varepsilon \text{ satisfying } \\
\int_{\Omega_{\varepsilon,\delta_1,\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2} \nabla \varphi \, dx = \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} f \varphi \, dx + \int_{\partial B_{\varepsilon,\delta_1}} g_{\varepsilon,\delta_1} \varphi \, ds \\
\forall \varphi \in V_{\delta_1,\delta_2}^\varepsilon,
\end{cases}
$$

The existence and uniqueness of the solution $u_{\varepsilon,\delta_1,\delta_2}$ in the space $V_{\varepsilon,\delta_1,\delta_2}$ are given by the Lax-Milgram theorem (due, in particular, to properties (3.3) of the matrix $A^\varepsilon$).

From now on, we assume $n \geq 3$. Suppose that $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$ satisfy

$$
k_1 = \lim_{\varepsilon \to 0} \frac{\delta_1^{n-1}}{\varepsilon}, \quad 0 \leq k_1 < \infty \quad \text{and} \quad k_2 = \lim_{\varepsilon \to 0} \frac{\delta_2^{2-1}}{\varepsilon}, \quad 0 \leq k_2 < \infty.
$$

We will study the asymptotic behaviour of problem (3.6) as $\varepsilon \to 0$ under assumptions (3.7).
Remark 3.1. Observe that $k_1$ corresponds to the critical size of Neumann small holes from [12] while $k_2$ corresponds to that of Dirichlet small holes from [11].

3.2. Main results

Let $K_T$ be the functional space (see [7] for more details)

\[ K_T = \{ \varphi \in L^2_{\text{loc}}(\mathbb{R}^n); \nabla \varphi \in L^2_{\text{loc}}(\mathbb{R}^n), \varphi = \text{const. on } T \}. \]

We are now able to state the homogenization results concerning problem (3.6).

Theorem 3.2. (Unfolded limit problem). Suppose that (3.7) holds. Let $A^\varepsilon$ satisfy (3.3) and suppose that, as $\varepsilon \to 0$, there exists two matrix fields $A$ and $A_0$, such that

\[ \begin{cases} \mathcal{T}_\varepsilon(A^\varepsilon)(x,y) \to A(x,y) \quad \text{a.e. in } \Omega \times Y, \\ \mathcal{T}_{\varepsilon,\delta_2}(A^\varepsilon)(x,z) \to A_0(x,z) \quad \text{a.e. in } \Omega \times (\mathbb{R}^n \setminus T). \end{cases} \]

Let $u_{\varepsilon,\delta_1,\delta_2}$ be the solution of problem (3.6) with $g_{\delta_1}$ defined by (3.4). Then, up tp a subsequence, there exists a function $u \in H^1_0(\Omega)$ such that

\[ u_{\varepsilon,\delta_1,\delta_2} \rightharpoonup u \text{ weakly in } L^2(\Omega). \]

Also, there exist $\hat{u} \in L^2(\Omega; H^1_{\text{per}}(Y))$ and $U \in L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^n))$, with $U - k_2u$ in $L^2(\Omega; K_T)$, such that $(u, \hat{u}, U)$ solves the equation

\[ \int_Y A(x,y)(\nabla u(x) + \nabla_y \hat{u}(x,y)) \nabla \phi(y) \, dy = 0 \]

for a.e. $x$ in $\Omega$ and all $\phi \in H^1_{\text{per}}(Y)$. Next,

\[ \int_{\mathbb{R}^n \setminus T} A_0(x,z) \nabla_z U(x,z) \nabla v(z) \, dz = 0, \]

for a.e. $x$ in $\Omega$ and all $v \in K_T$ with $v(T) = 0$ and, finally,

\[ \int_{\Omega \times Y} A(\nabla_x u + \nabla_y \hat{u}) \nabla \psi \, dx \, dy - k_2 \int_{\Omega \times \partial T} A_0 \nabla_z U \nu_T \psi \, ds \]

\[ = \int_\Omega f \psi \, dx + k_1 |\partial B| \mathcal{M}_{\partial B}(g) \int_\Omega \psi(x) \, dx \]

for all $\psi \in H^1_0(\Omega)$, where $\nu_T$ is the unit exterior normal to the set $T$.

The next theorem gives the classical (standard) form of the homogenized system (3.11)–(3.13). To state it, we follow the procedure from [7], where more details can be found. Introduce first the classical correctors $\hat{\chi}_j$, $j = 1, \ldots, n$, for
the homogenization in fixed domains (see for instance, [1]). They are defined by the cell problems

\[
\begin{array}{l}
\hat{\chi}_j \in L^\infty(\Omega; H^1_{\text{per}}(Y)), \\
\int_{Y} (A(x,y) \nabla (\hat{\chi}_j - y_j) \nabla \phi = 0 \quad \text{a.e. } x \in \Omega, \\
\forall \phi \in H^1_{\text{per}}(Y).
\end{array}
\tag{3.14}
\]

Let also \( \chi \) be the solution of the cell problem corresponding to the small holes \( \delta_2 T \), namely,

\[
\begin{array}{l}
\chi \in L^\infty(\Omega; K_T), \quad \chi(x,T) \equiv 1, \\
\int_{\mathbb{R}^n \setminus T} tA_0(x,z) \nabla_z \chi(x,z) \nabla_z \Psi(z) \, dz = 0 \quad \text{a.e. for } x \in \Omega, \\
\forall \Psi \in K_T \quad \text{with } \Psi(T) = 0,
\end{array}
\tag{3.15}
\]

and set

\[
\Theta(x) = \int_{\partial T} tA_0(x,z) \nabla_z \chi(x,z) \nu_T \, d\sigma_z.
\tag{3.16}
\]

We then have

**Theorem 3.3.** The limit function \( u \in H^1_0(\Omega) \) given by Theorem 3.2 is the unique solution of the homogenized equation

\[
\begin{array}{l}
-\text{div} (A^\text{hom} \nabla u) + k_1^2 \Theta u = f + |\partial B| M_{\partial B}(g), \\
u = 0 \quad \text{on } \partial \Omega,
\end{array}
\tag{3.17}
\]

where \( A^\text{hom} \) is the classical homogenized matrix

\[
A^\text{hom}_{ij}(x) = \int_{Y} \left( a_{ij}(x,y) - \sum_{k=1}^n a_{ik}(x,y) \frac{\partial \hat{\chi}_j}{\partial y_k}(x,y) \right) \, dy.
\tag{3.18}
\]

**Remark 3.4.** The contribution in the limit of the oscillations of the matrix \( A^\epsilon \) in the original problem (3.5), is reflected by the appearance of the operator \( A^\text{hom} \) in the homogenized system (3.17). The contribution in the limit of the set of small “Neumann” holes \( B_{\varepsilon,\delta_1} \), is the constant \( |\partial B| M_{\partial B}(g) = \int_{\partial B} g \, ds \). The contribution of the set of small “Dirichlet” holes \( T_{\varepsilon,\delta_2} \) is the zero order “strange term” \( k_1^2 \Theta u \).

### 3.3. Proof of Theorem 3.2

In the proof of Theorem 3.2, we will use the following lemma from [7]. We adapted its statement to our situation.
Lemma 3.5. Let $v$ in $\mathcal{D}(\mathbb{R}^n) \cap K_T$ (with $K_T$ defined by (3.8)). Set
\[ w_{\varepsilon, \delta_2}(x) = v(T) - v\left(\frac{1}{\delta_2} \left\{ \frac{x}{\varepsilon} \right\} \right), \quad x \in \mathbb{R}^n. \]
Then
\[ w_{\varepsilon, \delta_2} \rightharpoonup v(T) \text{ weakly in } H^1(\Omega). \]

Proof of Theorem 3.2. We start by establishing a priori estimates for $u_{\varepsilon, \delta}^*$ in $\Omega_{\varepsilon, \delta_1, \delta_2}$. Considering $u_{\varepsilon, \delta_1, \delta_2}$ as a test function in (3.5), by (3.3) and (2.13) we obtain
\[ \alpha \| \nabla u_{\varepsilon, \delta_1, \delta_2} \|^2_{(L^2(\Omega_{\varepsilon, \delta_1, \delta_2}))^n} \leq C\|f\|_{L^2(\Omega)} \|u_{\varepsilon, \delta_1, \delta_2}\|_{L^2(\Omega_{\varepsilon, \delta_1, \delta_2})} + C\frac{n-1}{\varepsilon} \| \nabla u_{\varepsilon, \delta_1, \delta_2} \|_{(L^2(\Omega_{\varepsilon, \delta_1, \delta_2}))^n}, \]
whence, by the Poincaré inequality and assumption (3.7),
\[ \|u_{\varepsilon, \delta_1, \delta_2}\|_{H^1(\Omega_{\varepsilon, \delta_1, \delta_2})} \leq C, \]
so convergence (3.10) holds. Since $\varepsilon \to 0$ and $\delta_2 \to 0$, it follows from Proposition 2.7 (see for more details [15]) that, up to a subsequence, there exists a $Y$-periodic $\hat{u}$ in $L^2_{\text{loc}}(\Omega; H^1_{\text{loc}}(Y))$ such that
\[ \begin{cases} 
T_{\varepsilon}(u_{\varepsilon, \delta_1, \delta_2}) \rightharpoonup u \text{ weakly in } L^2(\Omega; H^1_{\text{loc}}(Y)), \\
T_{\varepsilon}(\nabla u_{\varepsilon, \delta_1, \delta_2}) \rightharpoonup \nabla u + \nabla_y \hat{u} \text{ weakly in } L^2(\Omega; H^1_{\text{loc}}(Y)).
\end{cases} \]
Moreover, by Proposition 2.10 (again up to a subsequence) there exist $U$ in $L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^n))$ and $W$ in $L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^n))$ with $\nabla_z W$ in $L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^n))$ such that
\[ \begin{cases} 
\frac{\delta_2^n}{\varepsilon} (T_{\varepsilon, \delta_2}(u_{\varepsilon, \delta_1, \delta_2}) - M_{T_{\varepsilon, \delta}} u_{\varepsilon, \delta_1, \delta_2}) \rightharpoonup W \text{ weakly in } L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^n)), \\
\frac{\delta_2^n}{\varepsilon} \nabla_z (T_{\varepsilon, \delta_2}(u_{\varepsilon, \delta_1, \delta_2})) 1_{\frac{1}{\delta_2} Y} \rightharpoonup \nabla_z W \text{ weakly in } L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^n)), \\
\frac{\delta_2^n}{\varepsilon} T_{\varepsilon, \delta_2}(u_{\varepsilon, \delta_1, \delta_2}) \rightharpoonup U \text{ weakly in } L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^n)).
\end{cases} \]
The argument used in the proof of Theorem 3.1 from [7] shows actually that $U = W + k_2 u_0$ and $\nabla_z U = \nabla_z W$, with $W$ belonging to $L^2(\Omega; K_T)$. What is essential in this argument, is the convergence
\[ \frac{\delta_2^n}{\varepsilon} M_{T_{\varepsilon, \delta}}^* u_{\varepsilon, \delta_1, \delta_2} 1_{\frac{1}{\delta_2} Y} \rightharpoonup k_2 u_0 \text{ strongly in } L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^n)), \]
an easy consequence of (2.9).
Let $\psi \in D(\Omega)$ and $\phi \in C^1_{per}(Y)$ vanishing in a neighborhood of the origin. For $\varepsilon$ and $\delta$ small enough, the function $\Phi(\cdot) = \varepsilon \psi(\cdot)\phi\left(\frac{x}{\varepsilon}\right)$ belongs to $V^1_{\delta_1,\delta_2}$, so from (3.6) we have

\[ (3.23) \quad \varepsilon \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2} \nabla \psi \phi\left(\frac{x}{\varepsilon}\right) \, dx + \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2} \psi \nabla \phi\left(\frac{x}{\varepsilon}\right) \, dx = \varepsilon \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} f \phi\left(\frac{x}{\varepsilon}\right) \, dx. \]

Letting $\varepsilon \to 0$, all the terms go to zero, except the second one. Unfolding it by $T_{\varepsilon}$ and recalling Proposition 2.4, yield

\[ 0 = \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2} \psi \nabla \phi\left(\frac{x}{\varepsilon}\right) \, dx = \int_{\Omega \times Y} T_{\varepsilon}(A^\varepsilon) \, T_{\varepsilon}(u_{\varepsilon,\delta_1,\delta_2}) \, T_{\varepsilon}(\psi) \nabla \phi(y) \, dxdy. \]

We can now let $\varepsilon \to 0$ in the last term, thanks to hypotheses (3.9) and convergences (3.21). We get immediately (3.11), since the limit is

\[ \lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2} \psi \nabla \phi\left(\frac{x}{\varepsilon}\right) \, dx = \int_{\Omega \times Y} A(x, y)(\nabla u + \nabla \tilde{u}) \nabla \phi(y) \psi(x) \, dxdy = 0. \]

In order to obtain the other equations from the statement of Theorem 3.1, multiply (3.5) by $w_{\varepsilon,\delta_2} \psi$, where $w_{\varepsilon,\delta_2}$ was defined in Lemma 3.5, and $\psi$ is in $D(\Omega)$. Since $w_{\varepsilon,\delta} = 0$ on the set of holes $T_{\varepsilon,\delta_2}$, integrating by parts over $\Omega_{\varepsilon,\delta_1,\delta_2}$ yields

\[ (3.24) \quad \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2} \nabla w_{\varepsilon,\delta_2} \psi \, dx + \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1,\delta_2} w_{\varepsilon,\delta_2} \nabla \psi \, dx = \int_{\Omega_{\varepsilon,\delta_1,\delta_2}} f \, w_{\varepsilon,\delta_2} \psi \, dx + \int_{\partial B_{\varepsilon,\delta_1}} g_{\varepsilon,\delta_1} w_{\varepsilon,\delta_2} \psi \, ds. \]
Unfolding with $T_{\varepsilon,\delta}$ the first integral above, the choice of the test function implies that u.c.i. is satisfied. So, by Proposition 2.11,

$$\int_{\Omega_{\varepsilon,\delta_1\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1\delta_2} \nabla w_{\varepsilon,\delta_2} \psi \, dx$$

$$\simeq \delta_2^{\alpha} \int_{\Omega \times \mathbb{R}^n} T_{\varepsilon,\delta_2}(A^\varepsilon) T_{\varepsilon,\delta_2}(\nabla u_{\varepsilon,\delta_1\delta_2}) T_{\varepsilon,\delta_2}(\nabla w_{\varepsilon,\delta_2}) T_{\varepsilon,\delta_2}(\psi) \, dxdy$$

$$= \frac{\delta_2^\gamma - 1}{\varepsilon} \int_{\Omega \times \mathbb{R}^n} T_{\varepsilon,\delta_2}(A^\varepsilon) \delta_2^{\beta} T_{\varepsilon,\delta_2}(\nabla u_{\varepsilon,\delta_1\delta_2})(-\nabla z v) T_{\varepsilon,\delta_2}(\psi),$$

where we used the fact that $T_{\varepsilon,\delta_2}(\nabla w_{\varepsilon,\delta_2}) = -\nabla z v$ (see Lemma 3.5).

It is obvious from Definition 2.8 that

$$T_{\varepsilon,\delta_2}(\psi) \nabla v \rightarrow \psi \nabla v \quad \text{strongly in} \quad L^2(\Omega) \times L^2_{\text{loc}}(\mathbb{R}^n).$$

We are now able to let $\varepsilon \to 0$ in (3.25) thanks to hypothesis (3.9) and convergences (3.21) and (3.26), to obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon,\delta_1\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1\delta_2} \nabla w_{\varepsilon,\delta_2} \psi \, dx$$

$$= -k_2 \int_{\Omega \times (\mathbb{R}^n \setminus T)} A_0(x, z) \nabla z U(x, z) \nabla v(z) \psi(x) \, dx dz,$$

which, by density, is true for any $v \in K_T$.

Unfolding the second integral in (3.24) by $T_{\varepsilon}$ yields

$$\int_{\Omega_{\varepsilon,\delta_1\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1\delta_2} w_{\varepsilon,\delta_2} \nabla \psi \, dx \simeq$$

$$\simeq \int_{\Omega \times Y} T_{\varepsilon}(A^\varepsilon) T_{\varepsilon}(\nabla u_{\varepsilon,\delta_1\delta_2}) T_{\varepsilon}(w_{\varepsilon,\delta_2}) T_{\varepsilon}(\nabla \psi) \, dxdy,$$

where we let $\varepsilon \to 0$, to get

$$\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon,\delta_1\delta_2}} A^\varepsilon \nabla u_{\varepsilon,\delta_1\delta_2} w_{\varepsilon,\delta_2} \nabla \psi \, dx$$

$$= v(T) \int_{\Omega \times Y} A \left( \nabla_x u + \nabla_y \hat{u} \right) \nabla \psi \, dxdy,$$

where we used again hypotheses (3.9) and convergences (3.21), as well as Theorem 2.4 and convergence (3.19). Analogously,

$$\lim \int_{\Omega_{\varepsilon,\delta_1\delta_2}} f w_{\varepsilon,\delta_2} \psi \, dx = v(T) \int_{\Omega} f \psi \, dx.$$
For the fourth term we use Proposition 2.14, to get
\[
\lim_{\epsilon \to 0} \int_{\partial B_{\epsilon,\delta_1}} g_{\epsilon,\delta_1} w_{\epsilon,\delta_2} \psi \, ds = k_1 v(T) |\partial B| \mathcal{M}_{\partial B}(g) \int_{\Omega} \psi(x) \, dx.
\]
This, together with (3.27), (3.28) and (3.29), leads to the limit equation of (3.24):
\[
v(T) \int_{\Omega \times Y} A(\nabla u + \nabla_g \tilde{u}) \nabla \psi \, dx dy - k_2 \int_{\Omega \times (\mathbb{R}^n \setminus T)} A_0 \nabla z U \nu_B \psi \, d\sigma
\]
\[
= v(T) \int_{\Omega} f \psi \, dx + k_1 v(T) |\partial B| \mathcal{M}_{\partial B}(g) \int_{\Omega} \psi(x) \, dx,
\]
for all \( \psi \in H^1_0(\Omega) \) and \( v \in K_T \). Equation (3.12) is then obtained by taking \( v(T) = 0 \) while (3.13) follows by integrating by parts. □

3.4. Proof of Theorem 3.3

The proof follows the reasoning from [7, Section 4.3]. We just emphasize the main points. The correctors defined by (3.14) enable us to express \( \tilde{u} \) in equation (3.11) in terms of \( u \) as
\[
\tilde{u}(x,y) = - \sum_{j=1}^n \frac{\partial u_0}{\partial x_j}(x) \tilde{\chi}_j(x,y).
\]
Replacing this expression in (3.11), it is easily seen that the limit function \( u \) is solution of
\[
\int_{\Omega} A_{\text{hom}} \nabla u_0 \nabla \psi \, dx - k_2 \int_{\Omega \times \partial B} A_0 \nabla z U \nu_B \psi \, d\sigma_z
\]
\[
= \int_{\Omega} f \psi \, dx + k_1 |\partial B| \mathcal{M}_{\partial B}(g) \int_{\Omega} \psi(x) \, dx,
\]
with \( A_{\text{hom}} \) given by (3.18). Now, by integrating by parts in (3.15), one easily gets
\[
\int_{\partial T} A_0 \nabla z U \nu_T \, ds = \int_{\partial T} A_0 \nabla z (U - k_2 u) \nu_T \, ds = -k_2 u \left( \int_{\partial T} A_0 \nabla z \chi \nu_T \, d\sigma \right),
\]
which, replaced into (3.30) gives (3.17) with \( \Theta \) defined by (3.16).

It remains to show that the existence and uniqueness of the homogenized problem. To do so, it is sufficient to notice that from (3.16) and using system (3.15) defining the corrector \( \chi \), one has
\[
\Theta(x) = \int_{\mathbb{R}^n \setminus B} A_0(x,z) \nabla_z \chi(x,z) \nabla_z \chi(x,z) \, dz \geq 0.
\]
We are thus enabled to apply Lax-Milgram theorem and so, to conclude the proof. □

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