Dedicated to Professor Philippe G. Ciarlet
on his 70th birthday

ON POINCARÉ AND DE RHAM'S THEOREMS

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We prove Poincaré's theorem under general assumptions on the data. Then we derive the regularity of the solution from a result of Borchers and Sohr [6]. Finally, we give an elementary proof of the de Rham's theorem in the case of 1-dimensional flows on the Euclidean space by applying the techniques introduced in the proof of Poincaré's theorem.

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1. INTRODUCTION

In differential geometry, the theorems of Poincaré and de Rham give a characterization of the de Rham cohomology groups. Poincaré's theorem states that de Rham's cohomology groups of a contractible manifold coincide with those of a single point. If one is only interested in 1-forms – i.e. in the de Rham cohomology group H^1_{dR} – the simple-connectedness of the manifold is enough. For arbitrary smooth manifolds, de Rham's theorem states that de Rham's cohomology groups are isomorphic with the singular cohomology groups.

In a partial differential equations setting, these two theorems solve an over-determined system of linear partial differential equations of order one. The proof of these theorems are by no means simple, especially if they are stated in their most general setting, i.e., when the data are only distributions. To this day, there is no proof, to the best knowledge of the author, of the Poincaré theorem in the general case where the data are distributions in simply-connected domains. However, we wish to emphasize that the Poincaré theorem was already proved by Schwartz [16] in the case where the domain is the whole Euclidean space $\mathbb{R}^n$ and by several authors (see Section 3) in the case where the data are sufficiently smooth. By contrast, de Rham's theorem was proved in its whole generality in [15]. However, an important prerequisite about chains and flows on differential manifolds is needed in order to understand the proof in [15].

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The most relevant case for the partial differential equations theory is when the data are 1-forms defined on Euclidean spaces. For instance, de Rham's theorem is used in fluid mechanics theory in order to find the pressure component of the unknown of the Stokes or Navier-Stokes equations, once the velocity field is found.

In this paper, we restrict the presentation to the case of 1-forms defined on Euclidean spaces. More specifically, the manifold will be an open subset of the Euclidean space $\mathbb{R}^n$. First, we give elementary proofs of Poincaré and de Rham's theorems in this setting (Theorems 2.1 and 4.1). Then we study the regularity of the solution to the Poincaré problem in the setting of Sobolev spaces. The regularity of the de Rham problem was already studied by Amrouche and Girault in [2]. We will use the same regularity result as in [2] (first proved by Borchers and Sohr in [6]) in conjunction with the "distributional" Poincaré theorem (Theorem 2.1) in order to derive the regularity of the solution to the Poincaré problem. Since we want to make our proof as elementary as possible, in all that follows we will use the terminology of the partial differential equations theory. For convenience of the readers, we also restate our results in the terminology of differential geometry in remarks following each theorem.

We end this section by recalling two well known theorems in the distribution theory that will be used in the next sections. Throughout the paper, $\Omega \subset \mathbb{R}^n$ denotes an open set, $\mathcal{D}(\Omega)$ the space of indefinitely differentiable functions with compact support contained in Ω and $\mathcal{D}'(\Omega)$ the space of all distributions over Ω. The gradient of a distribution u is denoted ∇u. If $u, v \in \mathcal{D}'(\Omega)$ and $\omega \subset \Omega$, we say that "$u = v$ in ω" if $\langle u, \varphi\rangle = \langle v, \varphi\rangle$ for all $\varphi \in \mathcal{D}(\omega)$ (the inclusion $\mathcal{D}(\omega) \subset \mathcal{D}(\Omega)$ is defined by extending the functions in $\mathcal{D}(\omega)$ by zero).

The first result below states that, in a connected open set, a distribution whose gradient vanishes is constant (see also Remark 2.2).

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $u \in \mathcal{D}'(\Omega)$ such that $\nabla u = 0$ in $(\mathcal{D}'(\Omega))^n$, i.e.,*

$$\Big\langle u, \frac{\partial \varphi}{\partial x_i}\Big\rangle = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega) \text{ and } i \in \{1, 2, \dots, n\}.$$

Then there exists a constant $C \in \mathbb{R}$ such that $u = C$.

The second result shows how to reconstruct a distribution over Ω from its restriction to smaller domains covering Ω. It has been first introduced by Schwartz [16] under the name "principe du recollement des morceaux".

THEOREM 1.2. *Let $\{\Omega_i\}_{i\in I}$ be a family of open sets in $\mathbb{R}^n$ and $\Omega := \bigcup_{i\in I} \Omega_i$. Let $\{T_i \in \mathcal{D}'(\Omega_i)\}_{i\in I}$ be a family of distributions such that $T_i = T_j$ in $\Omega_i \cap \Omega_j$ for all $i, j \in I$ satisfying $\Omega_i \cap \Omega_j \neq \emptyset$.*

Then there exists one and only one distribution $T \in \mathcal{D}'(\Omega)$ such that

$$T = T_i \text{ on } \Omega_i \quad \text{for all } i \in I. \tag{1.1}$$

The idea of the *proof* is as follows. For any $\varphi \in \mathcal{D}(\Omega)$ one defines

$$\langle T, \varphi \rangle := \sum_{j=1}^{m} \langle T_{i_j}, \theta_j \varphi \rangle, \tag{1.2}$$

where the functions $(\theta_j)_{j=1}^m$ form a partition of unity subordinated to a covering $(\Omega_{i_j})_{j=1}^m$ of $\operatorname{supp}\varphi$ ($\operatorname{supp}\varphi \subset \bigcup_{j=1}^m \Omega_{i_j}$ for some $m \in \mathbb{N}^*$ and $i_1, \dots, i_m \in I$ since $\operatorname{supp}\varphi$ is compact and $\operatorname{supp}\varphi \subset \bigcup_{i\in I} \Omega_i$). Then one proves that the definition (1.2) is independent of the choice of the couple $((\Omega_{i_j}), (\theta_j))$ satisfying the above properties. In turn, this independence allows to prove that T is a distribution over Ω and that this distribution satisfies (1.1).

2. POINCARÉ'S THEOREM

Poincaré's theorem, also known in the literature as Poincaré's lemma, states that on a contractible manifold of dimension n, any k-form, $1 \le k \le n$, is exact if and only if it is closed. We consider here the particular case of 1-forms on an open subset of the Euclidean space $\mathbb{R}^n$. The main point here is that the domain Ω can be any simply-connected open set of $\mathbb{R}^n$, regardless of the regularity of its boundary. The main result of this section is as follows.

THEOREM 2.1. *Let Ω be a simply-connected open subset of $\mathbb{R}^n$ and let $f_1, f_2, \dots, f_n \in \mathcal{D}'(\Omega)$ such that*

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \text{in } \mathcal{D}'(\Omega) \tag{2.1}$$

for all $i, j \in \{1, 2, \dots, n\}$. Then there exists $u \in \mathcal{D}'(\Omega)$ such that

$$\frac{\partial u}{\partial x_i} = f_i \quad \text{in } \mathcal{D}'(\Omega) \tag{2.2}$$

for all $i \in \{1, 2, \dots, n\}$.

Remark 2.1. (1) In the case where $\Omega = \mathbb{R}^n$, Theorem 2.1 was proved in [16].

(2) If we consider the 1-form on Ω with distributional components

$$\alpha = f_1 \mathrm{d}x_1 + f_2 \mathrm{d}x_2 + \cdots + f_n \mathrm{d}x_n,$$

then the above theorem can be restated as follows. *If α is a closed 1-form on Ω, then α is exact, i.e., there exists a 0-form $\beta \in \Lambda^0(\Omega)$ such that $\alpha = \mathrm{d}\beta$.*

Proof of Theorem 2.1. First, we prove the existence of local solutions to system (2.2), then we prove the existence of a global solution by using the simple-connectedness of the set Ω.

To prove the existence of local solutions to system (2.2), we follow the same ideas as in Schwartz [16]. For the sake of completeness, we give below the whole argument.

Let $\omega := (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \subset \Omega$ and $\varphi \in \mathcal{D}(\omega)$. The idea is to modify the test function φ in order to obtain a derivative of another function in $\mathcal{D}(\omega)$.

For every $i \in \{1, 2, \ldots, n\}$ let $\theta_i \in \mathcal{D}((a_i, b_i))$ be such that $\int_{\mathbb{R}} \theta_i(t)\,\mathrm{d}t = 1$. Define the function $\psi_1^\varphi : \omega \to \mathbb{R}$ by

$$\psi_1^\varphi(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) - \theta_1(x_1) \int_{\mathbb{R}} \varphi(s, x_2, \ldots, x_n)\,\mathrm{d}s \tag{2.3}$$

and note that $\psi_1^\varphi = \frac{\partial \Psi_1^\varphi}{\partial x_1}$, where the function Ψ_1^φ is defined by

$$\Psi_1^\varphi(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \psi_1^\varphi(t, x_2, \ldots, x_n)\,\mathrm{d}t. \tag{2.4}$$

It is easy to check that $\Psi_1^\varphi \in \mathcal{D}(\omega)$. Hence a distribution $u \in \mathcal{D}'(\omega)$ that satisfies $\frac{\partial u}{\partial x_1} = f_1$ in $\mathcal{D}'(\omega)$ must satisfy the relation

$$\langle u, \psi_1^\varphi \rangle = -\langle f_1, \Psi_1^\varphi \rangle$$

or, equivalently, the relation (the variables $x_1, x_2, \ldots, x_n$ appearing in the right-hand side are mute)

$$\langle u, \varphi \rangle = -\langle f_1, \Psi_1^\varphi \rangle + \Big\langle u, \theta_1(x_1) \int_{\mathbb{R}} \varphi(s, x_2, \ldots, x_n)\,\mathrm{d}s \Big\rangle,$$

where ψ_1^φ and Ψ_1^φ are the functions defined by (2.3) and (2.4), respectively.

We apply the same method to construct the functions ψ_2^φ and Ψ_2^φ, but this time we start with the test function

$$(x_1, x_2, \ldots, x_n) \mapsto \theta_1(x_1) \int_{\mathbb{R}} \varphi(s, x_2, \ldots, x_n)\,\mathrm{d}s$$

(instead of φ) and consider the derivative with respect to the variable x_2.

After n iterations of this argument, we obtain for the distribution u the formula

$$\langle u, \varphi \rangle = -\langle f_1, \Psi_1^\varphi \rangle - \langle f_2, \Psi_2^\varphi \rangle - \cdots - \langle f_n, \Psi_n^\varphi \rangle + \langle C, \varphi \rangle, \tag{2.5}$$

where C is a constant and the functions $\Psi_1^\varphi, \ldots, \Psi_n^\varphi$ are defined by

$$\Psi_i^\varphi(x_1, \ldots, x_i, \ldots, x_n) := \int_{-\infty}^{x_i} \psi_i^\varphi(x_1, \ldots, t, \ldots, x_n)\,\mathrm{d}t \tag{2.6}$$

with

$$\psi_i^\varphi = \eta_{i-1}^\varphi - \eta_i^\varphi \tag{2.7}$$

and

$$\eta_0^{\varphi} = \varphi, \tag{2.8}$$

$$\eta_i^{\varphi}(x_1,\ldots,x_n) := \theta_1(x_1)\ldots\theta_i(x_i)\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\varphi(s_1,\ldots,s_i,x_{i+1},\ldots,x_n)\,\mathrm{d}s_1\ldots\mathrm{d}s_i.$$

We claim that the mapping u defined by (2.5) belongs to the space $\mathcal{D}'(\omega)$. Indeed, it is clear that u in linear with respect to φ. It is also easy to check that if a sequence $(\varphi_m)_{m\in\mathbb{N}}$ of test functions satisfies $\varphi_m \to \varphi$ in $\mathcal{D}(\omega)$, then $\Psi_i^{\varphi_m} \to \Psi_i^{\varphi}$ in $\mathcal{D}(\omega)$ for all $i \in \{1,2,\ldots,n\}$. Then the continuity of u (with respect to the usual topology of the space $\mathcal{D}(\omega)$) follows from the continuity of the mappings $f_i : \mathcal{D}(\Omega) \to \mathbb{R}$ (recall that f_i are distributions).

Now, we prove that the distribution u given by (2.5) satisfies the equations

$$\frac{\partial u}{\partial x_k} = f_k \text{ in } \mathcal{D}'(\omega) \text{ for all } k \in \{1,2,\ldots,n\}.$$

Let $k \in \{1,2,\ldots,n\}$ and $\varphi = \frac{\partial\tilde{\varphi}}{\partial x_k}$ for some $\tilde{\varphi} \in \mathcal{D}(\omega)$. Then a straightforward computation shows that the functions ψ_i^{φ} introduced above satisfy

$$\psi_i^{\varphi} = \begin{cases} \dfrac{\partial\psi_i^{\tilde{\varphi}}}{\partial x_k} & \text{if } i < k, \\[2ex] \dfrac{\partial\eta_{k-1}^{\tilde{\varphi}}}{\partial x_k} & \text{if } i = k, \\[2ex] 0 & \text{if } i > k, \end{cases}$$

whence we deduce that

$$\Psi_i^{\varphi} = \begin{cases} \dfrac{\partial\Psi_i^{\tilde{\varphi}}}{\partial x_k} & \text{if } i < k, \\[2ex] \eta_{k-1}^{\tilde{\varphi}} & \text{if } i = k, \\[1ex] 0 & \text{if } i > k. \end{cases}$$

It then follows from formula (2.5) and equations (2.1) that

$$\begin{aligned}\Big\langle u, \frac{\partial\tilde{\varphi}}{\partial x_k}\Big\rangle &= -\sum_{i=1}^{k-1}\Big\langle f_i, \frac{\partial\Psi_i^{\tilde{\varphi}}}{\partial x_k}\Big\rangle - \langle f_k, \eta_{k-1}^{\tilde{\varphi}}\rangle + C\int_{\omega}\frac{\partial\tilde{\varphi}}{\partial x_k}\,\mathrm{d}x \\ &= -\sum_{i=1}^{k-1}\Big\langle f_k, \frac{\partial\Psi_i^{\tilde{\varphi}}}{\partial x_i}\Big\rangle - \langle f_k, \eta_{k-1}^{\tilde{\varphi}}\rangle = -\sum_{i=1}^{k-1}\langle f_k, \psi_i^{\tilde{\varphi}}\rangle - \langle f_k, \eta_{k-1}^{\tilde{\varphi}}\rangle \\ &= -\Big\langle f_k, \sum_{i=1}^{k-1}(\eta_{i-1}^{\tilde{\varphi}} - \eta_i^{\tilde{\varphi}}) + \eta_{k-1}^{\tilde{\varphi}}\Big\rangle = -\langle f_k, \eta_0^{\tilde{\varphi}}\rangle = -\langle f_k, \tilde{\varphi}\rangle.\end{aligned}$$

This ends the proof of the existence of local solutions to Poincaré's system (2.2). Note that if u satisfies (2.2), then $u + C$ also satisfies (2.2), hence we can take C to be any real constant in formula (2.5).

Now, we will construct a global solution to system (2.2) by using the simple-connectedness of the set Ω. To this end, we first note that the local existence result we just proved insures the existence of local solutions defined on open balls that can be included in cubes contained in Ω. More specifically, if x is any point in Ω, then there exists a local solution of system (2.2) in any open ball $B(x, r)$ with radius $r \leq \frac{1}{\sqrt{n}} \operatorname{dist}(x, \Omega^c)$, where $\Omega^c := \mathbb{R}^n \setminus \Omega$.

Let there be given a point $x_0 \in \Omega$ and an open ball $B(x_0, r_0)$ with $r_0 \leq \frac{1}{\sqrt{n}} \operatorname{dist}(x_0, \Omega^c)$. The above local existence result yields a distribution $u^0 \in \mathcal{D}'(B(x_0, r_0))$ that satisfies

$$\frac{\partial u^0}{\partial x_k} = f_k \text{ in } \mathcal{D}'(B(x_0, r_0)), \quad k \in \{1, 2, \dots, n\}.$$

Let $x \in \Omega$ and consider a triple $(\gamma, \Delta, (B_j))$, where $\gamma : [0, 1] \to \Omega$ is a continuous path joining x^0 to x, i.e., $\gamma(0) = x^0$ and $\gamma(1) = x$, $\Delta = (0 = t_0 < t_1 < \dots < t_N < t_{N+1} = 1)$ is a division of the interval $[0, 1]$, and $(B_j)_{j=0}^N$ are open balls contained in cubes that are themselves contained in Ω and satisfy

$$\begin{aligned} &B_0 = B(x_0, r_0), \\ &\gamma([t_j, t_{j+1}]) \subset B_j \quad \text{for all } j \in \{1, 2, \dots, N\}. \end{aligned} \tag{2.9}$$

Note that such triples do exist for any $x \in \Omega$. Indeed, for any continuous path γ joining x_0 to x, the triple $(\gamma, \Delta, (B_j))$ with $B_0 = B(x_0, r_0)$ and $B_j = B(\gamma(t_j), r)$ for $j \in \{1, 2, \dots, N\}$, where $r < \frac{1}{\sqrt{n}} \operatorname{dist}(\gamma([0, 1]), \Omega^c)$ and $|t_{j+1} - t_j| \leq \delta_r$ for all $j \in \{1, 2, \dots, N\}$, satisfies the conditions above if δ_r is chosen in such a way that $|\gamma(t) - \gamma(s)| < \min\{r, r_0\}$ whenever $|t - s| \leq \delta_r$ (the existence of δ_r is given by the uniform continuity of γ).

We construct o global solution to system (2.2) as follows. For $x \in \Omega$ and $(\gamma, \Delta, (B_j))$ satisfying the conditions above, we define recursively the distributions $u^j \in \mathcal{D}'(B_j)$, $j = 1, 2, \dots, N$, such that

$$\begin{aligned} &\frac{\partial u^j}{\partial x_k} = f_k \quad \text{in } \mathcal{D}'(B_j), \\ &u^j = u^{j-1} \quad \text{in } B_j \cap B_{j-1}. \end{aligned} \tag{2.10}$$

Note that this construction is possible since the intersection $B_j \cap B_{j-1}$ is a convex set, in particular connected; hence, if u^{j-1} and u^j respectively satisfy $\frac{\partial u^{j-1}}{\partial x_k} = f_k$ in $\mathcal{D}'(B_{j-1})$ and $\frac{\partial u^j}{\partial x_k} = f_k$ in $\mathcal{D}'(B_j)$, then the distribution $(u^{j-1} - u^j)$ is constant in $B_{j-1} \cap B_j$ and so we can add if necessary a constant to u^j in order to meet the condition that $u^j = u^{j-1}$ in $B_j \cap B_{j-1}$.

From this construction we keep only the last distribution u^N and claim that u^N is independent of the triple $(\gamma, \Delta, (B_j))$ in the following sense. If

$$(\gamma,\ \Delta := (0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = 1),\ (B_j)_{j=0}^N),$$
$$(\tilde{\gamma},\ \tilde{\Delta} := (0 = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_{\tilde{N}} < \tilde{t}_{\tilde{N}+1} = 1),\ (\tilde{B}_j)_{j=0}^{\tilde{N}})$$

are two triples associated with x, then $\tilde{u}^{\tilde{N}} = u^N$ in $B_N \cap \tilde{B}_{\tilde{N}}$.

Let us first prove that u^N is independent of (B_j). To this end, we fix γ and Δ and consider two sequences of open balls $(B_j)_{j=0}^N$ and $(\tilde{B}_j)_{j=0}^{\tilde{N}}$, both satisfying (2.9). We apply an induction argument. First, we have

$$u^0 = \tilde{u}^0 \text{ in } \mathcal{D}'(B_0) \text{ (by definition).}$$

Assume that $u^j = \tilde{u}^j$ in $B_j \cap \tilde{B}_j$. Then from (2.10) we have

$$u^{j+1} = u^j = \tilde{u}^j = \tilde{u}^{j+1} \text{ in } B_j \cap \tilde{B}_j \cap B_{j+1} \cap \tilde{B}_{j+1}.$$

Note that $B_j \cap \tilde{B}_j \cap B_{j+1} \cap \tilde{B}_{j+1}$ is a nonempty open set, since it contains $\gamma(t_{j+1})$. It follows that $u^{j+1} = \tilde{u}^{j+1}$ in $B_{j+1} \cap \tilde{B}_{j+1}$, since the set $B_{j+1} \cap \tilde{B}_{j+1}$ is a connected open set.

Now, we prove that u^N is independent of Δ. To this end, we first consider the case of two divisions Δ and $\tilde{\Delta}$ of the form $\Delta = (0 = t_0 < t_1 < \cdots < t_{N+1} = 1)$ and $\tilde{\Delta} = (0 = t_0 < t_1 < \cdots < t_k < t^* < t_{k+1} < \cdots < t_{N+1} = 1)$. Let $(B_j)_{j=0}^N$ be a family of open balls satisfying (2.9). For $\tilde{\Delta}$, we consider the family $(\tilde{B}_j)_{j=0}^{N+1}$ defined by

$$\begin{aligned} \tilde{B}_j &= B_j \quad \text{for all } j \in \{0, 1, \ldots, k\}, \\ \tilde{B}_{j+1} &= B_j \quad \text{for all } j \in \{k, k+1, \ldots, N\}. \end{aligned}$$

Obviously, the family $(\tilde{B}_j)_{j=0}^{N+1}$ associated with the division $\tilde{\Delta}$ satisfies (2.9). We also have $\tilde{u}^j = u^j$ in $\mathcal{D}'(B_j)$ for all $j \le k$. By (2.10), we then have $\tilde{u}^{k+1} = \tilde{u}^k = u^k$ in $\tilde{B}_{k+1} \cap \tilde{B}_k = B_k$. Using again system (2.10), we obtain by induction that $\tilde{u}^{j+1} = u^j$ in B_j for all $j \ge k$ and in particular for $j = N$.

Let now $\Delta = (0 = t_0 < t_1 < \cdots < t_{N+1} = 1)$ and $\tilde{\Delta} = (0 = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_{\tilde{N}+1} = 1)$ be two arbitrary divisions with the respective associated families of open balls $(B_j)_{j=0}^N$ and $(\tilde{B}_j)_{j=0}^{\tilde{N}}$ satisfying (2.9). Consider the joint division $\overline{\Delta} := (0 = s_0 < s_1 < \cdots < s_{M+1} = 1)$ defined by

$$\{s_0, s_1, \ldots, s_{M+1}\} = \{t_0, t_1, \ldots, t_{N+1}\} \cup \{\tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_{\tilde{N}+1}\}, \quad M \le N + \tilde{N}.$$

Starting with Δ and applying $(M - N)$ times the above argument about divisions differing at one single point, we obtain (with self-explanatory notation)

$$\overline{u}^M = u^N \text{ in } B_N.$$

Similarly, but starting with $\tilde{\Delta}$ and applying $(M-\tilde{N})$ times the same argument, we obtain

$$\overline{u}^M = \tilde{u}^{\tilde{N}} \text{ in } \tilde{B}_{\tilde{N}}.$$

Combining the last two equalities and using the fact that $\overline{u}^M$ is independent of (B_j), we get

$$u^N = \tilde{u}^{\tilde{N}} \text{ in } B_N \cap \tilde{B}_{\tilde{N}}.$$

Finally, we prove that u^N is independent of the path γ. To this end, we use, as expected, the simple-connectedness of Ω. Let $(\gamma, \Delta, (B_j)_{j=0}^N)$ and $(\tilde{\gamma}, \tilde{\Delta}, (\tilde{B}_j)_{j=0}^{\tilde{N}})$ be two triples associated with the point $x \in \Omega$. Since Ω is simply connected, there exists a continuous function $H : [0,1] \times [0,1] \to \Omega$ such that

$$H(0,\cdot) = \gamma, \quad H(1,\cdot) = \tilde{\gamma}, \quad H(\cdot\,,0) = x_0, \quad H(\cdot\,,1) = x.$$

For each $s \in [0,1]$, set $\gamma_s := H(s,\cdot)$ and consider a triple $(\gamma_s, \Delta_s, (B_j^s)_{j=0}^{N_s})$ associated with x. For $s = 0$ we choose the triple $(\gamma, \Delta, (B_j)_{j=0}^N)$, while for $s = 1$ we choose the triple $(\tilde{\gamma}, \tilde{\Delta}, (\tilde{B}_j)_{j=0}^{\tilde{N}})$.

Let (with self-explanatory notation)

$$s^* := \sup\{s \in [0,1];\ u^{s,N_s} = u^N := u^{0,N_0} \text{ in } B_{N_s}^s \cap B_N\}.$$

We wish to prove that

$$1 = s^* = \max\{s \in [0,1];\ u^{s,N_s} = u^N \text{ in } B_{N_s}^s \cap B_N\}.$$

First, we prove that $s^* > 0$. To this end, we show that for s sufficiently small,

$$u^{s,N_s} = u^N \text{ in } B_{N_s}^s \cap B_N.$$

In order to prove this relation, we begin by showing that the triple $(\gamma_s, \Delta, (B_j)_{j=0}^N)$ associated with x is admissible provided that s is sufficiently small. It is enough to prove that, for any fixed $j \in \{0,1,\ldots,N\}$, $\gamma_s([t_j,t_{j+1}]) \subset B_j$ for any sufficiently small s. We argue by contradiction. Suppose that there exist sequences $s^m \to 0$ and $t^m \in [t_j,t_{j+1}]$ ($m \in \mathbb{N}$ is an index going to infinity) such that $\gamma_{s^m}(t^m) = H(s^m,t^m) \notin B_j$. Since $[t_j,t_{j+1}]$ is compact, there exists a subsequence of (t_m), still denoted (t_m), such that $t^m \to t \in [t_j,t_{j+1}]$. The function H being continuous, we have $H(s^m,t^m) \to H(0,t) = \gamma(t) \in B_j$, which contradicts the relation $H(s^m,t^m) \notin B_j$ for all $m \in \mathbb{N}$ (the contradiction follows from the fact that B_j is an open set).

Next, we prove that s^* is a maximum. The above contradiction argument shows that for α sufficiently small and for all $0 \le \varepsilon < \alpha$, the triple

$(\gamma_{s^*-\varepsilon}, \Delta_{s^*}, (B_j^{s^*})_{j=0}^{N_{s^*}})$ is admissible for x. By the independence of $u^{s^*,N_{s^*}}$ with respect to Δ and (B_j), this implies that

$$u^{s^*,N_{s^*}} = u^{s^*-\varepsilon,N_{s^*-\varepsilon}} \text{ on } B_{N^*}^{s^*} \cap B_{N_{s^*-\varepsilon}}^{s^*-\varepsilon}$$

for all $\varepsilon \in [0, \alpha)$. Since s^* is a supremum, we have

$$u^{s^*-\delta,N_{s^*-\delta}} = u^N \text{ on } B_{N_{s^*-\delta}}^{s^*-\delta} \cap B_N$$

for some $\delta \in [0, \alpha)$. By combining the last two equalities, we get

$$u^{s^*,N_{s^*}} = u^N \text{ on } B_{N_{s^*}}^{s^*} \cap B_N. \tag{2.11}$$

Finally, we prove that $s^* = 1$ by a contradiction argument. If $s^* < 1$ then using once again the previous argument shows that for $\varepsilon > 0$ sufficiently small, the triple $(\gamma_{s^*+\varepsilon}, \Delta_{s^*}, (B_j^{s^*})_{j=1}^{N_{s^*}})$ is admissible for x. This contradicts the definition of s^*. Therefore $s^* = 1$ and $u^N = u^{0,N_0} = u^{1,N_1} = \tilde{u}^{\tilde{N}}$.

Now, we are in a position to define a global solution to the Poincaré system (2.2). For any $x \in \Omega$, let us choose $B_x \subset \Omega$ to be an admissible "final" ball, i.e., $B_x = B_N$ for some admissible triple $(\gamma, \Delta, (B_j)_{j=1}^N)$ associated with x. Denote $u^x = u^N$, where the distribution $u^N \in \mathcal{D}'(B_x)$ is constructed as above. Let us prove that for any $x, y \in \Omega$ such that $B_x \cap B_y \neq \emptyset$ we have

$$u^x = u^y \text{ on } B_x \cap B_y. \tag{2.12}$$

Let $(\gamma, \Delta, (B_j)_{j=1}^N)$ and $(\tilde{\gamma}, \tilde{\Delta}, (\tilde{B}_j)_{j=1}^{\tilde{N}})$ be two admissible triples for x and y, respectively. Let $z \in B_z \cap B_y$. We consider the path obtained by joining γ with the segment $[x, z]$, parameterized for instance by (prime is not a symbol for the derivative with respect to t)

$$\gamma'(t) := \begin{cases} \gamma(2t) & \text{if } t \in [0, \frac{1}{2}], \\ (2-2t)x + (2t-1)z & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Then the triple $(\gamma', \Delta' := (0 = \frac{t_0}{2} < \frac{t_1}{2} < \cdots < \frac{t_{N+1}}{2} < t_{N+2} = 1), (B'_j)_{j=1}^{N+1})$, where $B'_j = B_j$ for all $j \leq N$ and $B'_{N+1} = B_N = B_x$, is admissible for z (since B_x is a convex set, $\gamma'([\frac{1}{2}, 1]) = [x, z] \subset B_x$). We make a similar construction for $\tilde{\gamma}$ to find another admissible triple for z. Then one can see that the relation (2.12) is a consequence of the relation (2.11) with $s^* = 1$, since $u'^{N+1} = u^N = u^x$ in B_x and $\tilde{u}'^{N+1} = \tilde{u}^N = u^y$ in B_y.

The property (2.12) allows us to construct a distribution on the set $\bigcup_{x\in\Omega} B_x = \Omega$ by letting, for any given test function $\varphi \in \mathcal{D}(\Omega)$,

$$\langle u, \varphi \rangle := \sum_{i=1}^{m} \langle u^{x_i}, \theta_i \varphi \rangle, \tag{2.13}$$

where $\operatorname{supp}\varphi \subset \bigcup_{i=1}^m B_{x_i}$ (such a finite covering exists because the support of φ is a compact subset of Ω) and the family $(\theta_i)_{i=1}^m$ is a partition of unity subordinated to the covering $(B_{x_i})_{i=1}^m$ of $\operatorname{supp}\varphi$ (this means that $\theta_i \in \mathcal{D}(B_{x_i})$ for all i and $\sum_{i=1}^m \theta_i = 1$ on $\operatorname{supp}\varphi$). By Theorem 1.2, the relation (2.13) defines a distribution u over Ω that satisfies $u = u^x$ in B_x for all $x \in \Omega$.

We now prove that the distribution u defined by (2.13) satisfies the system (2.2). Let $\varphi \in \mathcal{D}(\Omega)$. Then $\frac{\partial\varphi}{\partial x_k} \in \mathcal{D}(\Omega)$ and $\operatorname{supp}\left(\frac{\partial\varphi}{\partial x_k}\right) \subset \operatorname{supp}\varphi$. Let $K \subset \Omega$ be a compact neighborhood of $\operatorname{supp}\varphi$. Consider a family of open balls $(B_{x_i})_{i=1}^m$ such that $K \subset \bigcup_{i=1}^m B_{x_i}$ and a partition of unity $(\theta_i)_{i=1}^m$ subordinated to the covering $(B_{x_i})_{i=1}^m$. In particular, (θ_i) satisfies $\sum_{i=1}^m \theta_i = 1$ in $K \supset \operatorname{supp}\Omega$. Then we have

$$\Big\langle u, \frac{\partial\varphi}{\partial x_k}\Big\rangle = \sum_{i=1}^m \Big\langle u^{x_i}, \theta_i \frac{\partial\varphi}{\partial x_k}\Big\rangle = \sum_{i=1}^m \Big\langle u^{x_i}, \frac{\partial}{\partial x_k}(\theta_i\varphi) - \frac{\partial\theta_i}{\partial x_k}\varphi\Big\rangle$$

$$= \sum_{i=1}^m \Big(\Big\langle u^{x_i}, \frac{\partial}{\partial x_k}(\theta_i\varphi)\Big\rangle - \Big\langle u, \frac{\partial\theta_i}{\partial x_k}\varphi\Big\rangle\Big) = \sum_{i=1}^m \big(-\langle f_k, \theta_i\varphi\rangle\big) - \Big\langle u, \sum_{i=1}^m \frac{\partial\theta_i}{\partial x_k}\varphi\Big\rangle$$

$$= -\Big\langle f_k, \Big(\sum_{i=1}^m \theta_i\Big)\varphi\Big\rangle - \Big\langle u, \frac{\partial\Big(\sum_{i=1}^m \theta_i\Big)}{\partial x_k}\varphi\Big\rangle = -\langle f_k, \varphi\rangle.$$

Note that we used in the third equality the fact that $u = u^{x_i}$ in B_{x_i}. In the forth equality, we used the relation $\frac{\partial u^{x_i}}{\partial x_k} = f_k$ in B_{x_i}. In the last equality, we used the fact that $\sum_{i=1}^m \theta_i = 1$ in the neighborhood K of $\operatorname{supp}\varphi$, which implies that $\frac{\partial\left(\sum_{i=1}^m \theta_i\right)}{\partial x_k} = 0$ on $\operatorname{supp}\varphi$, hence $\frac{\partial\left(\sum_{i=1}^m \theta_i\right)}{\partial x_k}\varphi = 0$ in Ω. This completes the proof. □

Remark 2.2. By following the argument used to construct a local solution to system (2.2) (see formula (2.5)), we can see that any u satisfying (2.2) in ω is given by the formula (the variables $x_1, \ldots, x_n$ appearing in the formula below are mute)

$$\langle u, \varphi\rangle = -\langle f_1, \Psi_1^\varphi\rangle - \cdots - \langle f_n, \Psi_n^\varphi\rangle + \langle u, \theta_1(x_1)\ldots\theta_n(x_n)\rangle \int_\omega \varphi(x)\,\mathrm{d}x,$$

where $\Psi_1^\varphi, \ldots, \Psi_n^\varphi$ are constructed as in the proof of Theorem 2.1. In particular, if $f_1 = f_2 = \cdots = f_n = 0$, then u is a constant (equal to $\langle u, \theta_1(x_1)\ldots\theta_n(x_n)\rangle$). This argument proves that on a connected open set, a distribution has null partial derivatives if and only if it is a constant. We have used this well-known "uniqueness up to a constant" result (Theorem 1.1) in the construction of a global solution u. See (2.10).

3. REGULARITY OF THE SOLUTION TO POINCARÉ'S SYSTEM

Now we address the question of regularity for the solution u to system (2.2). More specifically, assuming that $f_1, f_2, \ldots, f_n$ all belong to some Sobolev space $W^{m,p}(\Omega)$, we investigate whether the solution u defined in the space of distributions by Theorem 2.1 possesses some regularity properties. Intuitively, we expect u to be of class $W^{m+1,p}(\Omega)$, at least if the boundary of Ω is sufficiently smooth.

If $m \geq 0$ and $p \in [1, \infty]$, it is well known that any solution $u \in \mathcal{D}'(\Omega)$ to system (2.2) satisfies $u \in W^{m+1,p}_{\mathrm{loc}}(\Omega)$. This result is valid for any open set Ω, irrespectively of the regularity of its boundary (see, e.g., Maz'ja [11, Theorem at page 7]). The usual proof of this regularity theorem consists in proving that u is of class $W^{m+1,p}$ in any open ball or hypercube contained in Ω. In order to prove this local regularity property of u, one could use either a regularizing argument (as in [4]), or an integral formula of u on well chosen sections of a hypercube contained in Ω (as in [12]). If in addition Ω is a bounded domain with a Lipschitz-continuous boundary, then $u \in W^{m+1,p}(\Omega)$. This global regularity result is a consequence of the formula (obtained after a change of variables, if necessary)

$$u(x', x_n) = u(x', 0) + \int_0^{x_n} \frac{\partial u}{\partial x_n}(x', t)\, \mathrm{d}t,$$

which is valid for almost all points (x', x_n), where $x' := (x_1, \ldots, x_{n-1})$, in a "good" subset of Ω whose precise definition depends on the regularity of Ω. In fact, one can see from this argument that a much weaker regularity of Ω is really needed (we recall that $m \geq 0$). For instance, in the case where $m \geq 0$ and $p \in [1, \infty)$, it is enough that Ω be a bounded domain with continuous boundary such that Ω lies locally on the same side of its boundary. Note that the regularity assumption on the boundary cannot be dropped altogether, because counterexamples (see Maz'ja [11]) show that the global regularity result fails if Ω is an arbitrary bounded domain.

The regularity of u in the case $m < 0$ is much more difficult to study. The one dimensional case ($n = 1$) is however easy to study thanks to formula (2.5). To see this, let $\Omega = (a, b)$ be a bounded open interval in $\mathbb{R}$ and assume that $f_1 \in W^{m,p}(\Omega)$, $m < 0$, $p \in (1, \infty)$. Then for any $\varphi \in \mathcal{D}(\Omega)$, we have (without any loss in generality, we choose $C = 0$ in (2.5))

$$|\langle u, \varphi\rangle| = |\langle f_1, \Psi_1^\varphi\rangle| \leq \|f_1\|_{W^{m,p}(\Omega)} \|\Psi_1^\varphi\|_{W_0^{-m,q}(\Omega)},$$

where q is defined by $\frac{1}{p}+\frac{1}{q}=1$. Then Poincaré's inequality (or simply the explicit definition (2.4) of Ψ_1^φ) implies that

$$|\langle u,\varphi\rangle| \le C\|f_1\|_{W^{m,p}(\Omega)}\|(\Psi_1^\varphi)'\|_{W_0^{-m-1,q}(\Omega)}.$$

Or $(\Psi_1^\varphi)'(x_1) = \psi_1^\varphi(x_1) = \varphi(x_1) - \theta_1(x_1)\int_{\mathbb{R}}\varphi\,\mathrm{d}x_1$ for some fixed function $\theta_1 \in \mathcal{D}(\Omega)$ such that $\int_{\mathbb{R}}\theta_1\,\mathrm{d}x_1 = 1$. Hence

$$\begin{aligned}(3.1)\quad |\langle u,\varphi\rangle| &\le C\|f_1\|_{W^{m,p}(\Omega)}\Big(\|\varphi\|_{W_0^{-m-1,q}(\Omega)} + \|\theta_1\|_{W_0^{-m-1,q}(\Omega)}\Big|\int_{\mathbb{R}}\varphi\,\mathrm{d}x_1\Big|\Big)\\ &\le C\|f_1\|_{W^{m,p}(\Omega)}\big(1+\|\theta_1\|_{W_0^{-m-1,q}(\Omega)}\big)\|\varphi\|_{W_0^{-m-1,q}(\Omega)},\end{aligned}$$

the last inequality being a consequence of Hölder's inequality

$$\Big|\int_{\mathbb{R}}\varphi\,\mathrm{d}x_1\Big| = \Big|\int_{\Omega}\varphi\,\mathrm{d}x_1\Big| \le (b-a)^{\frac{1}{p}}\|\varphi\|_{L^q(\Omega)}.$$

Inequality (3.1) implies that u is a continuous linear functional over the space $W_0^{-m-1,q}(\Omega)$ and therefore $u \in W^{m+1,p}(\Omega)$. Moreover, the chosen distribution u, i.e., the one defined by (2.5) with $C=0$ (in other words, the solution to Poincaré's system (2.2) satisfying $\langle u,\theta_1\rangle = 0$), satisfies the inequality

$$\|u\|_{W^{m+1,p}(\Omega)} \le C\|f_1\|_{W^{m,p}(\Omega)}$$

for some constant C depending only on Ω.

Note that the above argument does not apply in higher dimensions ($n \ge 2$), since the term $\|\nabla\Psi_i^\varphi\|_{W_0^{-m-1,q}(\Omega)}$ cannot be controlled by $\|\varphi\|_{W_0^{-m-1,q}(\Omega)}$. To see this, consider for instance the case $m=-1$. Then

$$\begin{aligned}\frac{\partial\Psi_1^\varphi}{\partial x_2}(x_1,x_2,\dots,x_n) &= \int_{-\infty}^{x_1}\frac{\partial\psi_1^\varphi}{\partial x_2}(t,x_2,\dots,x_n)\,\mathrm{d}t\\ &= \int_{-\infty}^{x_1}\Big(\frac{\partial\varphi}{\partial x_2}(t,x_2,\dots,x_n) - \theta_1(t)\int_{\mathbb{R}}\frac{\partial\varphi}{\partial x_2}(s,x_2,\dots,x_n)\,\mathrm{d}s\Big)\,\mathrm{d}t,\end{aligned}$$

which means that in order to control the norm $\|\nabla\Psi_1^\varphi\|_{L^q(\Omega)}$, one needs partial derivatives of φ. On the other hand, this argument does not take into account the compatibility conditions satisfied by the functions (f_i), because it relies only on formula (2.5), which is valid for any n-tuple $(f_1,\dots,f_n)$. Thus a different argument is needed in dimension $n \ge 2$.

Our aim is to obtain a generalization of Poincaré's lemma in Sobolev spaces $W^{m,p}(\Omega)$ valid for any integer m (not necessarily nonnegative) and for any simply-connected Lipschitz domains. In the case $m=0$ and $p=2$ (i.e., if $f_1,\dots,f_n \in W^{0,2}(\Omega) = L^2(\Omega)$), this was done by Girault and Raviart [9] in dimension three ($n=3$) and by Bourgain, Brezis and Mironescu [4] in arbitrary dimension under an additional regularity assumption on Ω. The case $m=-1$, $p=2$ was studied by Ciarlet and Ciarlet, Jr. [7] in dimension three ($n = 3$) and was generalized by Kesavan [10] in arbitrary dimension. More recently, Amrouche, Ciarlet and Ciarlet, Jr. [1] solved the case $p = 2$ and $m \in \mathbb{Z}$ arbitrary, for three-dimensional domains. As noted in [10], in a simply-connected bounded domain Ω with a Lipschitz-continuous boundary, the Poincaré lemma in $W^{-1,2}(\Omega)$ is equivalent to a well-known Lions lemma stating that $f \in \mathcal{D}'(\Omega)$ and $\nabla f \in (W^{-1,2}(\Omega))^n$ implies $f \in L^2(\Omega)$. This lemma was proved in Duvaut and Lions [8] in the case of smooth domains and in Borchers and Sohr [6] (see also Amrouche and Girault [2, Proposition 2.10]) in the case of Lipschitz domains. In fact, Amrouche and Girault [2] proved the following generalization of Lions' lemma.

LEMMA 3.1 (Proposition 2.10 in [2]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz-continuous boundary. If the distribution $f \in \mathcal{D}'(\Omega)$ satisfies $\nabla f \in (W^{m,p}(\Omega))^n$ with $m \in \mathbb{Z}$ and $p \in (1, \infty)$, then $f \in W^{m+1,p}(\Omega)$.*

Now, we establish the Poincaré theorem in the setting of Sobolev spaces by combining Lemma 3.1 with Theorem 2.1. Furthermore, we establish an estimate of the solution to the Poincaré system (2.2) in the corresponding Sobolev norm (see Theorem 3.1).

To begin with, we introduce some notation. If $u \in W^{m,p}(\Omega)$, we denote by $\hat{u}$ the equivalence class of u in the space $W^{m,p}(\Omega)/\mathbb{R}$ and define its norm by

$$\|\hat{u}\|_{W^{m,p}(\Omega)/\mathbb{R}} := \inf_{c \in \mathbb{R}} \|u + c\|_{W^{m,p}(\Omega)}.$$

Note that if u is a solution to system (2.2), then any element of the class $\hat{u}$ is also solution to this system.

The announced result is as follows.

THEOREM 3.1. *Let Ω be a simply-connected open set in $\mathbb{R}^n$ and let $f_1, f_2, \dots, f_n \in W^{m,p}(\Omega)$ for some $m \in \mathbb{Z}$ and $p \in (1, \infty)$. Assume that the functions (f_i) satisfy equations (2.1). Then there exists $u \in W^{m+1,p}_{\text{loc}}(\Omega)$ such that $\nabla u = \boldsymbol{f} := (f_1, \dots, f_n)$ in $\mathcal{D}'(\Omega)$.*

If in addition Ω is connected, bounded, with a Lipschitz-continuous boundary, then $u \in W^{m+1,p}(\Omega)$ and there exists a constant C depending only on Ω such that

$$\|\hat{u}\|_{W^{m,p}(\Omega)/\mathbb{R}} \le C \|\boldsymbol{f}\|_{W^{m,p}(\Omega)}.$$

Remark 3.1. If $m \geq 0$ then Theorem 3.1 also holds for $p = 1$ and $p = \infty$ (see the comments at the beginning of this section).

4. DE RHAM'S THEOREM FOR 1-DIMENSIONAL FLOWS

We give an elementary proof of de Rham's theorem in the case of homogeneous flows (or currents) of dimension one on a Euclidean space (for the definition of these notions, see, e.g., de Rham [15]). Since an Euclidean space is oriented, we will only consider positively oriented local charts, so the problem of the parity (for forms or flows) is not posed here. In fact, by choosing as manifold an open set $\Omega \subset \mathbb{R}^n$, we can always take the identity as local chart around every point. In this setting, we are able to prove the pure analytic form of the de Rham's theorem below.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $\boldsymbol{f} = (f_1, \ldots, f_n) \in (\mathcal{D}'(\Omega)^n$ be a vector field that satisfies*

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle := \sum_{i=1}^{n} \langle f_i, \varphi_i \rangle = 0 \tag{4.1}$$

for all $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n) \in (\mathcal{D}(\Omega))^n$ satisfying $\mathrm{div}\boldsymbol{\varphi} = 0$. Then there exists $u \in \mathcal{D}'(\Omega)$ such that $\boldsymbol{f} = \nabla u$, i.e.,

$$\frac{\partial u}{\partial x_k} = f_k \text{ in } \mathcal{D}'(\Omega) \text{ for all } k \in \{1, \ldots, n\}. \tag{4.2}$$

Remark 4.1. (1) In differential geometry terms, Theorem 4.1 can be restated as follows:

Let T be a homogeneous flow of dimension one and assume that

$$\langle T, \varphi \rangle = 0$$

for all compactly supported 1-forms φ of class $\mathcal{C}^\infty$ satisfying $\delta\varphi = 0$. Then there exists a homogeneous flow S of dimension zero (i.e., a distribution on Ω) such that

$$\mathrm{d}S = T,$$

where $\mathrm{d}S$ is defined by $\langle \mathrm{d}S, \varphi \rangle := \langle S, \delta S \rangle$ for all 1-forms $\varphi \in \Lambda^1(\Omega)$ of class $\mathcal{C}^\infty$ with compact support in Ω.

This statement holds verbatim in the case of oriented Riemannian manifolds. The operator $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ is the codifferential operator and can be defined on any oriented Riemannian manifold; see, e.g., Bleecker [5]. In the particular case of Euclidean spaces, this definition reduces to the expression

$$\delta(\varphi_1 \mathrm{d}x_1 + \varphi_2 \mathrm{d}x_2 + \cdots + \varphi_n \mathrm{d}x_n) = -\mathrm{div}\boldsymbol{\varphi},$$

where $\boldsymbol{\varphi} := (\varphi_1, \varphi_2, \ldots, \varphi_n)$.

The de Rham theorem in the most general setting – i.e., for flows on manifolds which are not necessarily Riemannian – can be found in [15].

(2) Unlike Poincaré's theorem, Theorem 4.1 holds in any connected open set Ω, not necessarily simply-connected. This weaker hypothesis on Ω is possible because condition (4.1) is stronger than condition (2.1). Indeed, we will see in the proof of Theorem 4.1 that (4.1) implies (2.1). The converse is false because otherwise Poincaré's theorem would apply in domains that are not

simply-connected, which is known to be false. Therefore, Theorem 4.1 is an immediate consequence of Theorem 2.1 in the case of simply-connected sets.

Proof of Theorem 4.1. We begin by showing that the vector field $\boldsymbol{f}$ satisfies equations (2.1). For $\varphi \in \mathcal{D}(\Omega)$ and $i < j$, let

$$\boldsymbol{\varphi} := \Big(0, \dots, 0, \frac{\partial \varphi}{\partial x_j}, 0, \dots, 0, -\frac{\partial \varphi}{\partial x_i}, 0, \dots, 0\Big),$$

where the nonzero components are placed on the ith and jth positions. It is clear that $\operatorname{div} \boldsymbol{\varphi} = 0$ and therefore equation (4.1) implies that $\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = 0$. Since

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = \Big\langle f_i, \frac{\partial \varphi}{\partial x_j} \Big\rangle - \Big\langle f_j, \frac{\partial \varphi}{\partial x_i} \Big\rangle,$$

equation (2.1) follows.

Thus, we can construct u locally as in the proof of Theorem 2.1. Note that formula (2.5), which defines u in a hypercube contained in Ω, can be rewritten as

$$\langle u, \varphi \rangle = -\langle \boldsymbol{f}, \boldsymbol{\Psi}^{\varphi} \rangle + \langle C, \phi \rangle,$$

where C is a constant and $\boldsymbol{\Psi}^{\varphi} = (\Psi_1^{\varphi}, \dots, \Psi_n^{\varphi})$ is defined as in the proof of Theorem 2.1.

Then we define a global solution to (4.2) as follows. Since Ω is connected, we use the construction of the global solution described in the proof of Theorem 2.1 (see (2.10)) to associate with every triple $(\gamma, \Delta, (B_j)_{j=0}^N)$ a distribution u^N satisfying equations (4.2) in the open ball B_N.

Recall that in the proof of Theorem 2.1 the simple-connectedness of the domain Ω has only been used to prove that u^N is independent of the path γ. We now have to prove the independence of u^N with respect to the triple $(\gamma, \Delta, (B_j)_{j=0}^N)$ by other means. Specifically, this will be done by using equation (4.1) in its full generality. In fact, we will establish an explicit formula for u^N (see (4.6)–(4.8)) in terms of the triple $(\gamma, \Delta, (B_j)_{j=0}^N)$ from which the independence property can be easily deduced.

Let B_0 and u^0 be fixed as in the proof of Theorem 2.1. In particular, the distribution u^0 is defined by the formula (taking $C = 0$ in (2.5))

$$\langle u^0, \varphi \rangle = -\langle f_1, \Psi_1^{0,\varphi} \rangle - \langle f_2, \Psi_2^{0,\varphi} \rangle - \dots - \langle f_n, \Psi_n^{0,\varphi} \rangle, \tag{4.3}$$

where (see (2.6)–(2.8))

$$\Psi_k^{0,\varphi}(x_1, \dots, x_k, \dots, x_n) := \int_{-\infty}^{x_k} \psi_k^{0,\varphi}(x_1, \dots, t, \dots, x_n)\, \mathrm{d}t, \tag{4.4}$$

$$\psi_k^{0,\varphi} := \eta_{k-1}^{0,\varphi} - \eta_k^{0,\varphi}, \quad \eta_0^{0,\varphi} := \varphi,$$

$$\eta_k^{0,\varphi}(x_1, \dots, x_n) := \theta_1^0(x_1) \dots \theta_k^0(x_k) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \varphi(s_1, \dots, s_k, x_{k+1}, \dots, x_n) \mathrm{d}s_1 \dots \mathrm{d}s_k.$$

Here, $(\theta_i^0)_{i=1}^n$ are given functions that satisfy $\theta_i^0 \in \mathcal{D}((a_i, b_i))$ and $\int_{\mathbb{R}} \theta_i^0(t)\,\mathrm{d}t = 1$, where $(a_1, b_1) \times \cdots \times (a_n, b_n) \subset \Omega$ is any given cube containing B_0.

Now, let $x \in \Omega$ and let $(\gamma, \Delta, (B_j)_{j=0}^N)$ be an admissible triple for x. For each $j \in \{1, 2, \ldots, n\}$, we choose an n-tuple $(\theta_i^j)_{i=1}^n$ associated with a hypercube contained in Ω and containing B_j, whose components θ_i^j satisfy the same properties as the functions θ_i^0. Next, we define the distributions u^j, $j = 1, 2, \ldots, N$, by

$$\langle u^j, \varphi\rangle = \langle C_j, \varphi\rangle - \langle f_1, \Psi_1^{j,\varphi}\rangle - \langle f_2, \Psi_2^{j,\varphi}\rangle - \cdots - \langle f_n, \Psi_n^{j,\varphi}\rangle, \tag{4.5}$$

where $C_j \in \mathbb{R}$ are constants insuring that $u^j = u^{j-1}$ on $B_j \cap B_{j-1}$ and the functions $\Psi_k^{j,\varphi}$, $\psi_k^{j,\varphi}$ and $\eta_k^{j,\varphi}$ are defined as in (4.4). In what follows, these functions are renamed $\Psi_k(\boldsymbol{\theta}^j, \varphi)$, $\psi_k(\boldsymbol{\theta}^j, \varphi)$ and $\eta_k(\boldsymbol{\theta}^j, \varphi)$ to emphasize the fact that they are completely determined by $\boldsymbol{\theta}^j = (\theta_1^j, \ldots, \theta_n^j)$ and $\varphi \in \mathcal{D}(B_j)$.

By relations (4.3) and (4.5), for all $\varphi \in \mathcal{D}(B_0 \cap B_1)$ we have

$$\langle u^1, \varphi\rangle = -\langle \boldsymbol{f}, \boldsymbol{\Psi}(\boldsymbol{\theta}^1, \varphi)\rangle + C_1 \int_\Omega \varphi(x)\,\mathrm{d}x = \langle u^0, \varphi\rangle = -\langle \boldsymbol{f}, \boldsymbol{\Psi}(\boldsymbol{\theta}^0, \varphi)\rangle,$$

By choosing a function $\zeta_1 \in \mathcal{D}(B_0 \cap B_1)$ such that $\int_\Omega \zeta_1(x)\,\mathrm{d}x = 1$, we get an explicit value of C_1, namely

$$C_1 = \langle \boldsymbol{f}, \boldsymbol{\Psi}(\boldsymbol{\theta}^1, \zeta_1) - \boldsymbol{\Psi}(\boldsymbol{\theta}^0, \zeta_1)\rangle.$$

To sum up, we obtained for u^1 the formula

$$\langle u^1, \varphi\rangle = \Big\langle \boldsymbol{f}, -\boldsymbol{\Psi}(\boldsymbol{\theta}^1, \varphi) + \big(\boldsymbol{\Psi}(\boldsymbol{\theta}^1, \zeta_1) - \boldsymbol{\Psi}(\boldsymbol{\theta}^0, \zeta_1)\big) \int_\Omega \varphi(x)\,\mathrm{d}x \Big\rangle$$

for all $\varphi \in \mathcal{D}(B_1)$.

Iterating N times the previous argument for the construction of u^1, for u^N we obtain the formula

$$\langle u^N, \varphi\rangle = \big\langle \boldsymbol{f}, \boldsymbol{\Psi}((\gamma, \Delta, (B_j)), \varphi)\big\rangle \quad \text{for all } \varphi \in \mathcal{D}(B_N), \tag{4.6}$$

where

$$\boldsymbol{\Psi}((\gamma, \Delta, (B_j)), \varphi) := -\boldsymbol{\Psi}(\boldsymbol{\theta}^N, \varphi) + \sum_{i=1}^N \big(\boldsymbol{\Psi}(\boldsymbol{\theta}^i, \zeta_i) - \boldsymbol{\Psi}(\boldsymbol{\theta}^{i-1}, \zeta_i)\big) \int_\Omega \varphi(x)\,\mathrm{d}x \tag{4.7}$$

and

$$\zeta_i \in \mathcal{D}(B_i \cap B_{i-1}) \quad \text{and} \quad \int_\Omega \zeta_i(x)\,\mathrm{d}x = 1 \quad \text{for all } i \in \{1, \ldots, N\}. \tag{4.8}$$

On the other hand, note that for any function $\zeta \in \mathcal{D}(\Omega)$ and any n-tuple $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ associated with a hypercube ω contained in Ω, we have

$$\begin{aligned}\text{(4.9)} \qquad \operatorname{div} \boldsymbol{\Psi}(\boldsymbol{\theta}, \zeta) &= \sum_{k=1}^{n} \psi_k(\boldsymbol{\theta}, \zeta) = \sum_{k=1}^{n} \big(\eta_{k-1}(\boldsymbol{\theta}, \zeta) - \eta_k(\boldsymbol{\theta}, \zeta)\big) \\ &= \eta_0(\boldsymbol{\theta}, \zeta) - \eta_n(\boldsymbol{\theta}, \zeta) = \zeta - \theta_1(x_1)\dots\theta_n(x_n)\int_\Omega \zeta(x)\,\mathrm{d}x.\end{aligned}$$

Then it follows from relations (4.8) and (4.9) that

$$\begin{aligned}\operatorname{div} \boldsymbol{\Psi}((\gamma, \Delta, (B_j)), \varphi) &= -\varphi + \theta_1^N(x_1)\dots\theta_n^N(x_n)\int_\Omega \varphi(x)\,\mathrm{d}x \\ &+ \sum_{i=1}^{n} \big(- \theta_1^i(x_1)\dots\theta_n^i(x_n) + \theta_1^{i-1}(x_1)\dots\theta_n^{i-1}(x_n)\big)\int_\Omega \varphi(x)\,\mathrm{d}x \\ &= -\varphi + \theta_1^0(x_1)\dots\theta_n^0(x_n)\int_\Omega \varphi(x)\,\mathrm{d}x\end{aligned}$$

for all $\varphi \in \mathcal{D}(B_N)$.

Note that the right-hand side of the last equality depends only on $\varphi \in \mathcal{D}(B_n)$ and on the n-tuple $\boldsymbol{\theta}^0$ which has been fixed once for all at the beginning of the proof. Hence, if $(\tilde{\gamma}, \tilde{\Delta}, (\tilde{B}_j)_{j=0}^{\tilde{N}})$ is another admissible triple for x, then

$$\begin{aligned}\operatorname{div} \boldsymbol{\Psi}((\tilde{\gamma}, \tilde{\Delta}, (\tilde{B}_j)), \varphi) &= -\varphi + \theta_1^0(x_1)\dots\theta_n^0(x_n)\int_\Omega \varphi(x)\,\mathrm{d}x \\ &= \operatorname{div} \boldsymbol{\Psi}((\gamma, \Delta, (B_j)), \varphi).\end{aligned}$$

for all $\varphi \in \mathcal{D}(B_N \cap \tilde{B}_{\tilde{N}})$.

Now, we use assumption (4.1). Combined with the previous relation, it shows that

$$\big\langle \boldsymbol{f}, \boldsymbol{\Psi}((\gamma, \Delta, (B_j)), \varphi) - \boldsymbol{\Psi}((\tilde{\gamma}, \tilde{\Delta}, (\tilde{B}_j)), \varphi)\big\rangle = 0$$

for all $\varphi \in \mathcal{D}(B_N \cap \tilde{B}_{\tilde{N}})$. Hence, by (4.6) we have

$$\langle u^N - \tilde{u}^{\tilde{N}}, \varphi\rangle = 0 \quad \text{for all } \varphi \in \mathcal{D}(B_N \cap \tilde{B}_{\tilde{N}}).$$

This proves that u^N is independent of the choice of the triple $(\gamma, \Delta, (B_j))$. The remaining part of the proof is identical with the last part of the proof of Theorem 2.1. □

Remark 4.2. If the distribution $\boldsymbol{f}$ appearing in the statement of Theorem 4.1 belongs to some Sobolev space, then Theorem 4.1 is a consequence of a well-known result in functional analysis. More specifically, one can prove that if $m \leq -1$, $p \in (1, \infty)$, and $\boldsymbol{f} \in (W^{m,p}(\Omega))^n$ satisfies (4.1), then there exists $u \in W^{m+1,p}(\Omega)$ such that $\nabla u = \boldsymbol{f}$. It is enough to prove that the gradient operator $\nabla : W^{m+1,p}(\Omega) \to (W^{m,p}(\Omega))^n$ has closed range. Combined with the fact that $-\nabla$ is the dual operator of the operator div : $(W_0^{-m,q}(\Omega))^n \to W_0^{-m-1,q}(\Omega)$, where q is defined by $1/p + 1/q = 1$, this implies that (see e.g. Brezis [3, Theorem II.18])

$$\mathrm{Im}(\nabla) = (\ker(\mathrm{div}))^{\perp},$$

which is exactly the desired result. The difficult part of the proof is to show that $\mathrm{Im}(\nabla)$ is a closed subspace of $(W^{m,p}(\Omega))^n$. This result, whose proof can be found in Amrouche and Girault [2, Corollary 2.5], relies on a theorem of Peetre and Tartar (see [14] and [17]) and on an inequality due to Nečas [13]. The Nečas inequality states that in a bounded Lipschitz domain, one can control the $W^{m,p}$-norm of a function by the $W^{m-1,p}$-norms of itself and of its gradient. This is obvious if $m \geq 1$, but it is by no means trivial in the case $m \leq 0$.

As in Section 3, we can recover de Rham's theorem in Sobolev spaces by combining Theorem 4.1 and Lemma 3.1 (as mentioned earlier, this lemma is due to Borchers and Sohr [6] and can be found in Amrouche and Girault [2, Proposition 2.10]):

THEOREM 4.2. *Let Ω be an open set in $\mathbb{R}^n$ and let $\boldsymbol{f} \in (W^{m,p}(\Omega))^n$ for some $m \in \mathbb{Z}$, $p \in (1,\infty)$ if $m < 0$, $p \in [1,\infty]$ if $m \geq 0$. Assume that $\boldsymbol{f}$ satisfy equations* (4.1). *Then there exists $u \in W^{m+1,p}_{\mathrm{loc}}(\Omega)$ such that $\nabla u = \boldsymbol{f}$ in $\mathcal{D}'(\Omega)$.*

If in addition Ω is connected, bounded, with a Lipschitz-continuous boundary, then $u \in W^{m+1,p}(\Omega)$ and there exists a constant C depending only on Ω such that

$$\|\hat{u}\|_{W^{m,p}(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{W^{m,p}(\Omega)}.$$

Remark 4.3. The proof of Theorem 4.2 based on the idea described in Remark 4.2 can be found in Amrouche and Girault [2, Theorem 2.8].

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