THE SET OF INCOMPLETE SUMS OF THE FIRST OSTROGRADSKY SERIES AND ANOMALOUSLY FRACTAL PROBABILITY DISTRIBUTIONS ON IT

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We investigate topological, metric and fractal properties of the set of incomplete sums of the first Ostrogradsky series (also called Pierce expansion or Sierpiński expansion) and of the family of such sets in a metric space of compact subsets with Hausdorff metric. The structure and fractal properties of the spectrum of random incomplete sum of the Ostrogradsky series with independent addenda are investigated. We also study the asymptotic behaviour of the absolute value of the characteristic function of this random variable at infinity, and prove that the random variable has an anomalously fractal singular distribution of the Cantor type. Fine fractal properties of this probability measure are studied in details. Conditions for the Hausdorff–Billingsley dimension preservation on the topological support by its distribution function are also obtained.

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1. **INTRODUCTION**

There exist many different methods for the representation of real numbers via a finite as well as an infinite alphabet [11, 12, 25, 30, 34]. Continued fraction expansions, $\tilde{Q}$-representation, Cantor and Lüroth expansions are among them. Each method has its own area of applications and some advantages. Each representation generates its own metric relations and its own "geometry", which are the basis for the development of metric, fractal and probabilistic number theories.

Continued fraction expansions (due to a highly developed theory and different applications) "occupied" a special important place in mathematics. There exist other theories (in some sense similar to the theory of continued fractions but more complicated in a metric sense), connected with expansions

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of real numbers via alternating series. They are less developed, but give us a wide perspective for the investigation of mathematical objects with a complicated local structure (fractals, singular continuous measures, continuous nowhere differentiable functions etc.). One of them is the theory of the representation of real numbers via "the Ostrogradsky–Sierpiński–Pierce series".

About 1861, M.V. Ostrogradsky considered two algorithms for expanding positive real numbers in alternating series:

$$\sum_k \frac{(-1)^{k-1}}{q_1 q_2 \dots q_k}, \quad \text{where } q_k \in \mathbb{N},\ q_{k+1} > q_k, \tag{1}$$

$$\sum_k \frac{(-1)^{k-1}}{q_k}, \quad \text{where } q_k \in \mathbb{N},\ q_{k+1} \geq q_k(q_k+1) \tag{2}$$

(the first and the second Ostrogradsky series, respectively). They were found by E.Ya. Remez [28] among manuscripts and unpublished papers of M.V. Ostrogradsky. Independently of these results, the series of form (1) appeared in works of other authors. Among them we would like to mention papers by W. Sierpiński [34] and T.A. Pierce [24]. In the Western mathematical literature [20, 21, 30–32] such expansions are called Pierce expansions, which is unfair since the paper by Sierpiński was published essentially earlier. In the former Soviet mathematical literature [6, 18, 19, 28, 37] the series of the form (1) are called first Ostrogradsky series. The authors of the present paper will use the latter variant, but it would be more correct to say "the Ostrogradsky–Sierpiński–Pierce series". Let us mention that the history of the discovery and the study of such series was discussed in the paper [21].

In the present paper we study geometrical objects connected with the first Ostrogradsky series and probability distributions on them. The main aim of the paper is to investigate properties of infinite Bernoulli convolutions generated by series of the form (1).

Let us recall that an infinite *symmetric* Bernoulli convolution is the probability distribution of a random variable $\xi = \sum\limits_{k=1}^{\infty} \xi_k b_k$, where ξ_k are independent random variables taking values -1 and 1 with probabilities $1/2$ and $1/2$, and $b_k \in \mathbb{R}$ with $\sum\limits_{k=1}^{\infty} b_k^2 < \infty$. The study of (general non-symmetric) Bernoulli convolutions with bounded spectra is equivalent to the investigation of the distributions of random series $\sum\limits_{k=1}^{\infty} \xi_k b_k$, where $\sum\limits_{k=1}^{\infty} |b_k| < \infty$, and ξ_k are independent random variables taking values 0 and 1 with probabilities p_{0k} and p_{1k}, respectively. Measures of this form have been studied since 1930s from the pure probabilistic point of view as well as for their applications in harmonic analysis, in the theory of dynamical systems and in fractal analysis. We shall not describe in details the history of the investigation of these measures, because the paper [22] contains a good survey of Bernoulli convolutions, corresponding historical notes and a discussion of some applications, generalizations and problems.

For the general situation, necessary and sufficient conditions for the random variable ξ to be absolutely continuous are still unknown (see, e.g., [22, 23, 26, 27]) even for the simplest symmetric case of random power series where $b_k = \lambda^k$, $\lambda \in (0,1)$ and $p_{0k} = \frac{1}{2}$. Properties of the corresponding probability measure μ_λ, *depending on the only parameter* λ, are well investigated for $\lambda \in (0, 1/2]$. In the case $\lambda \in (1/2, 1)$ the dependence of properties of μ_λ on λ is rather mysterious. In the latter case the spectrum S_{μ_λ} is the whole interval $\left[0, \frac{\lambda}{1-\lambda}\right]$, and one could think that μ_λ has a density. Nevertheless, in 1939, P. Erdös [9] proved that the Fourier–Stieltjes transform of the measure μ_λ does not tend to zero if and only if λ is the reciprocal of a Pisot number in $(1, 2)$. So, in such a case μ_λ is singular w.r.t. Lebesgue measure. Up to now we do not know any other examples of λ leading to the singularity of μ_λ. "Almost sure results" in the opposite direction were obtained by B. Solomyak and Y. Peres [23, 25] who proved the long standing Erdös–Garsia conjecture: for Lebesgue almost all $\lambda \in (1/2, 1)$ the corresponding probability measure μ_λ is absolutely continuous w.r.t. Lebesgue measure.

There exist other interesting subclasses of infinite Bernoulli convolutions, which depend on only one parameter. Actually, any expansion of real numbers via absolutely convergent series (see, e.g., [11, 30]) generates a proper subclass of Bernoulli convolutions depending on a real parameter. Based on Theorem 1 (Section 2), one can establish the following one-to-one correspondence between the family of all infinite first Ostrogradsky series and the set I of irrational numbers r from the unit interval: for any $r \in I$ there exists a unique sequence $\{q_k\} = \{q_k(r)\}$ such that $q_{k+1} > q_k$ and

$$r = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{q_1 q_2 \cdots q_k}.$$

The main purpose of the paper is to study properties of symmetric as well as general non-symmetric Bernoulli convolutions ψ_r defined by infinite first Ostrogradsky series, i.e., the probability distributions of random variables

$$\psi = \psi_r = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varepsilon_k}{q_1 q_2 \cdots q_k},$$

where $r \in I$, $q_k = q_k(r)$, and $\{\varepsilon_k\}$ is a sequence of independent random variables taking the values 0 and 1 with probabilities p_{0k} and p_{1k}, respectively ($p_{0k} + p_{1k} = 1$). The random variable ψ belongs to the class of random variables of the Jessen–Wintner type (i.e., it is the sum of a convergent series of independent discrete random variables), and, therefore, is of pure type, i.e., it is either discrete or continuous and in the latter case it is not a mixture of singularly continuous and absolutely continuous distributions.

For the symmetric case ($p_{0k} = \frac{1}{2}$) as well as for the case $0 < p_{0k} < 1$ (for any $k \in \mathbb{N}$) the spectrum (minimal closed support of ψ) coincides (Lemma 2) with the set

$$C_r = \left\{ x : x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a_k}{q_1(r) q_2(r) \dots q_k(r)},\ a_k \in \{0,1\} \right\}$$

of all incomplete sums of the first Ostrogradsky series generated by an irrational number r. That is why in Section 2 we study in details topological, metric and fractal properties of sets C_r. As a result, in Section 4 we show that for any choice of $r \in I$ the probability distribution of ψ is either discrete or singularly continuous, and the Hausdorff–Besicovitch dimension of the corresponding spectra is equal to zero independently of the choice of the stochastic matrix $P = \|p_{ik}\|$, $i \in \{0,1\}$, $k \in \mathbb{N}$, and $r \in I$.

We also show (Section 5) that this class of Bernoulli convolutions does not contain any Rajchman measure, i.e., the characteristic function $f_\psi(t)$ of the random variable ψ (the Fourier–Stieltjes transform of the corresponding probability measure) does not tend to zero as t tends to infinity. Moreover, we prove that $L_\psi = \limsup_{|t|\to\infty} |f_\psi(t)| = 1$ for all values of the parameter $r \in I$ and for any choice of the matrix P. So, it follows from $\prod_{k=1}^{\infty} \max\{p_{0k}, p_{1k}\} = 0$ that the distribution of ψ is continuous but it is "similar" to a discrete one (it is supported by a set of zero Hausdorff–Besicovitch dimension and $L_\psi = 1$).

The independence of the above mentioned properties of ψ of both r and P can be explained by a rather fast convergence of the first Ostrogradsky series (this fact also demonstrates that these series give good rational approximations of real numbers). On the other hand, for a given $r \in I$, if $p'_{0k} = p' \in (0,1)$ and $p''_{0k} = p'' \in (0,1)$, $p'' \neq p'$, then the corresponding random variables ψ'_r and ψ''_r are mutually singularly distributed (this follows directly from the Kakutani theorem [15]) on the common spectrum C_r, and all of them are singularly continuous w.r.t. Lebesgue measure. To emphasize essential differences between these distributions, in Section 6 we study their fine fractal properties (see, e.g., [36] for details). Since the spectra of all random variables ψ are "very poor" in both the metric and the fractal sense (their Hausdorff–Besicovitch dimensions are equal to zero), to compare their "massivity" it is necessary to use an appropriate dimension function (gauge function) for the Hausdorff measure (see, e.g., [10, p. 37]) or an appropriate probability measure for the Hausdorff–Billingsley dimension (see Section 6 for definitions). As an adequate example of such a measure we consider the measure ν^* corresponding to the uniform distribution on C_r. We prove a formula for the calculation of the Hausdorff–Billingsley dimension of the spectrum of ψ with respect to the measure ν^*. Moreover, we study internally fractal properties of ψ. In particular, we find the Hausdorff–Billingsley dimension of the distribution of ψ w.r.t. ν^*, i.e., the infimum of the Hausdorff–Billingsley dimension of all Borel supports of ψ w.r.t. ν^*. We would also remark that for a more exact characterization of internally fractal properties of ψ one can use the Hausdorff–Billingsley dimension with respect to the probability measure which is uniformly distributed on the spectrum of ψ.

The last section of the paper is devoted to the problem of the preservation of the Hausdorff–Billingsley dimension of subsets of the spectrum under the distribution function F_ψ. For the case where elements of the matrix P are bounded away from zero we find necessary and sufficient conditions for the dimension preservation in terms of the relative entropy of the distribution.

A considerable part of our results can be easily extended to other classes of Bernoulli convolutions generated by fast convergent series.

2. TOPOLOGICAL, METRIC AND FRACTAL PROPERTIES OF THE SET OF INCOMPLETE SUMS OF THE FIRST OSTROGRADSKY SERIES

An expression (1) is called *a first Ostrogradsky series*. We denote this expression by $\mathrm{O}^1(q_1, q_2, \ldots, q_n, \ldots)$. In particular, the latter notation symbolizes that the sequence $\{q_n\}$ determinates the series (1). The number q_n is called the *n-th element of the first Ostrogradsky series* (1).

Any partial sum

$$\frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n-1}}{q_1 q_2 \cdots q_n} \equiv \mathrm{O}^1(q_1, q_2, \ldots, q_n) \tag{3}$$

of the series (1) is called *a finite first Ostrogradsky series.*

By Leibniz' well-known theorem on alternating series, the series (1) converges and its sum is

$$r = r(\{q_n\}) \in \left[\sum_{k=1}^{2m-1} \frac{(-1)^{k-1}}{q_1 q_2 \cdots q_k} - \frac{1}{q_1 q_2 \cdots q_{2m-1} q_{2m}}, \sum_{k=1}^{2m-1} \frac{(-1)^{k-1}}{q_1 q_2 \cdots q_k}\right) \subset [0, 1].$$

In fact, any Ostrogradsky series converges absolutely, because $q_{n+1} > q_n$.

One can prove that $r(\{q_n\}) \neq r(\{q'_n\})$ if the infinite sequences $\{q_n\}$ and $\{q'_n\}$ do not coincide. However, for a finite first Ostrogradsky series we have

$$\mathrm{O}^1(q_1, q_2, \ldots, q_{n-1}, q_n, q_n + 1) = \mathrm{O}^1(q_1, q_2, \ldots, q_{n-1}, q_n + 1),$$

since

$$\frac{1}{q_1q_2\dots q_{n-1}q_n} - \frac{1}{q_1q_2\dots q_{n-1}q_n(q_n+1)} = \frac{1}{q_1q_2\dots q_{n-1}(q_n+1)}.$$

THEOREM 1 ([28]). *Any real number $x \in (0,1)$ can be represented in the form* (1). *If x is irrational, then the expression* (1) *is unique and has an infinite number of terms. If x is rational, then it can be represented in the form* (3) *in two different ways:*

$$x = \mathrm{O}^1(q_1, q_2, \dots, q_{n-1}, q_n, q_n+1) = \mathrm{O}^1(q_1, q_2, \dots, q_{n-1}, q_n+1).$$

The Ostrogradsky series have a rather high rate of convergence (they converge "not slower" than the series $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} = \frac{e-1}{e}$). So, any irrational number can be well approximated by rational numbers which are partial sums of the Ostrogradsky series corresponding to the initial number. Since the Ostrogradsky series is alternating, we have

$$\left| x - \sum_{k=1}^{m} \frac{(-1)^{k-1}}{q_1(x)q_2(x)\dots q_k(x)} \right| = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{q_1(x)q_2(x)\dots q_{m+j}(x)} <$$

$$< \frac{1}{q_1(x)q_2(x)\dots q_{m+1}(x)}.$$

Let $\{q_k\}$ be a *fixed* sequence of positive integers with $q_{k+1} > q_k$ for any $k \in \mathbb{N}$ and let r be the sum of the corresponding Ostrogradsky series (1). Then $r = d - b$, where

$$d = \sum_{i=1}^{\infty} \frac{1}{q_1q_2\dots q_{2i-1}}, \quad b = \sum_{i=1}^{\infty} \frac{1}{q_1q_2\dots q_{2i}}.$$

Since the Ostrogradsky series converges absolutely, we have

$$\sum_{k=1}^{\infty} \frac{1}{q_1q_2\dots q_k} = d + b.$$

Let $A = \{0,1\}$, and let L be the space of infinite sequences of symbols 0 and 1, i.e.,

$$L = A^{\infty} = \{a : a = (a_1, a_2, \dots, a_k, \dots),\ a_k \in A\}.$$

The sum $s = s(\{a_k\})$ of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}a_k}{q_1q_2\dots q_k}, \tag{4}$$

where $\{a_k\} \in L$, is called the *incomplete sum of the series* (1). It is clear that s depends on the whole *infinite* sequence $\{a_k\}$. We formally denote the expression (4) and its sum s by $\Delta_{a_1a_2\ldots a_k\ldots}$ (the reason for such a notation will be clarified later in Lemma 1, property 5). Any partial sum of the series (1) is its incomplete sum. It is evident that the set of incomplete sums of the series (1) has the cardinality of continuum.

One can define the notion of the set of incomplete sums for any absolutely convergent series $\sum_{k=1}^{\infty} b_k$ in a completely similar way, and such a set can be considered as a "geometrical image" of the corresponding series. This set can be nowhere dense or can be a finite union of intervals. It can be of positive resp. zero Lebesgue measure. In the latter case, its "massivity" can by characterized by using different fractal dimensions (see, e.g., [10, Ch. 2]). Topological, metric and fractal properties of the set of incomplete sums reflect the "rate of convergence" of the series, and in some cases play an important role in the analysis of mathematical objects connected with the initial series. For instance, if $b_k = \lambda^k$, then the corresponding set of incomplete sum coincides with the interval $\left[0, \frac{\lambda}{1-\lambda}\right]$ for $\lambda \in [1/2, 1)$, and is a perfect nowhere dense set of zero Lebesgue measure for $\lambda \in [0, 1/2)$. In the latter case, its Hausdorff–Besicovitch dimension is equal to $\frac{\log 2}{-\log \lambda}$ (for $\lambda = \frac{1}{3}$ we have a copy of the classical Cantor set on the interval $[0, 1/2]$). It is worth to mention here that for $\lambda \in [0, 1/2)$ the set of incomplete sums is a set of uniqueness for trigonometric series if and only if the Fourier–Stieltjes transform of the corresponding above defined Bernoulli convolution μ_λ does not vanish at infinity, i.e., if λ is a Pisot number (see, e.g., [29] for details). More recent investigations on the topic can be found in [3, 4, 7].

We shall study topological, metric and fractal properties of the set C_r of all incomplete sums of the series (1). It is clear that $C_r \subset [-b, d]$, and for any $s \in C_r$ there exists a sequence $\{a_k\} = \{a_k(s)\} \in L$ such that

$$s = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a_k(s)}{q_1 q_2 \cdots q_k}.$$

Let c_1, c_2, ..., c_m be a fixed sequence consisting of zeroes and ones. The set $\Delta'_{c_1c_2\ldots c_m}$ of all incomplete sums $\Delta_{c_1c_2\ldots c_m a_{m+1}\ldots a_{m+k}\ldots}$, where $a_{m+j} \in A$ for any $j \in \mathbb{N}$, is called the *cylinder* of rank m with base $c_1c_2\ldots c_m$. It is evident that

$$\Delta'_{c_1c_2\ldots c_m a} \subset \Delta'_{c_1c_2\ldots c_m}, \quad a \in A.$$

The intervals

$$\Delta_0 = \left[-b, d - \frac{1}{q_1}\right] \quad \text{and} \quad \Delta_1 = \left[-b + \frac{1}{q_1}, d\right]$$

are called the *cylindrical intervals of rank* 1. It is easy to see that

$$|\Delta_0| = |\Delta_1| = d + b - \frac{1}{q_1},$$
$$\Delta'_0 \subset \Delta_0, \quad \Delta'_1 \subset \Delta_1, \quad C_r \subset \Delta'_0 \cup \Delta'_1 \subset \Delta_0 \cup \Delta_1.$$

Setting

$$k_1 = \left(-b + \frac{1}{q_1}\right) - \left(d - \frac{1}{q_1}\right) = \frac{2}{q_1} - b - d =$$
$$= \frac{1}{q_1} - \frac{1}{q_1 q_2} - \frac{1}{q_1 q_2 q_3} - \dots = \frac{1}{q_1}\left(1 - \frac{1}{q_2} - \frac{1}{q_2 q_3} - \dots\right) \geq$$
$$\geq \frac{1}{q_1}\left(1 - \frac{1}{2} - \frac{1}{2 \cdot 3} - \dots - \frac{1}{k!} - \dots\right) = \frac{1}{q_1}(3 - e),$$

we have

$$\Delta_0 \cap \Delta_1 = \varnothing \quad \text{and} \quad |[-b, d] \setminus (\Delta_0 \cup \Delta_1)]| = k_1 \geq \frac{3-e}{q_1}.$$

The intervals

$$\Delta_{00} = \left[-b + \frac{1}{q_1 q_2}, d - \frac{1}{q_1}\right], \quad \Delta_{01} = \left[-b, d - \frac{1}{q_1} - \frac{1}{q_1 q_2}\right],$$
$$\Delta_{10} = \left[-b + \frac{1}{q_1} + \frac{1}{q_1 q_2}, d\right], \quad \Delta_{11} = \left[-b + \frac{1}{q_1}, d - \frac{1}{q_1 q_2}\right]$$

are called the *cylindrical intervals of rank* 2. We have

$$|\Delta_{00}| = |\Delta_{01}| = |\Delta_{10}| = |\Delta_{11}| = d + b - \frac{1}{q_1} - \frac{1}{q_1 q_2} = \sum_{k=3}^{\infty} \frac{1}{q_1 q_2 \dots q_k},$$
$$\Delta'_{00} \subset \Delta_{00} \subset \Delta_0, \quad \Delta'_{01} \subset \Delta_{01} \subset \Delta_0, \quad \Delta'_{10} \subset \Delta_{10} \subset \Delta_1,$$
$$\Delta'_{11} \subset \Delta_{11} \subset \Delta_1, \quad \Delta_{00} \cap \Delta_{01} = \varnothing, \quad \Delta_{10} \cap \Delta_{11} = \varnothing,$$
$$\inf \Delta'_{01} = \inf \Delta'_0 = -b, \quad \sup \Delta'_{00} = \sup \Delta'_0 = d - \frac{1}{q_1},$$
$$\inf \Delta'_{11} = \inf \Delta'_1 = -b + \frac{1}{q_1}, \quad \sup \Delta'_{10} = \sup \Delta'_1 = d,$$

$$k_2 = |\Delta_0 \setminus (\Delta_{01} \cup \Delta_{00})| = |\Delta_1 \setminus (\Delta_{11} \cup \Delta_{10})| = |\Delta_0| - 2\,|\Delta_{00}| =$$
$$= \left(d + b - \frac{1}{q_1}\right) - 2\sum_{k=3}^{\infty} \frac{1}{q_1 q_2 \dots q_k} = \frac{1}{q_1 q_2} - \sum_{k=3}^{\infty} \frac{1}{q_1 q_2 \dots q_k} =$$
$$= \frac{1}{q_1 q_2}\left(1 - \frac{1}{q_3} - \frac{1}{q_3 q_4} - \dots\right) \geq \frac{2 \cdot (3-e)}{q_1 q_2}.$$

The *cylindrical interval of rank* m *with base* $c_1c_2\ldots c_m$ ($c_i \in \{0,1\}$) coincides with the interval

$$\left[s_m - \sum_{i:2i>m}\frac{1}{q_1q_2\cdots q_{2i}}, s_m + \sum_{i:2i-1>m}\frac{1}{q_1q_2\cdots q_{2i-1}}\right] = [-b+u_m, d-v_m],$$

where

$$s_m = \sum_{k=1}^{m}\frac{(-1)^{k-1}c_k}{q_1q_2\cdots q_k}, \quad u_m = \sum_{i:2i-1\le m}\frac{c_{2i-1}}{q_1q_2\cdots q_{2i-1}} + \sum_{i:2i\le m}\frac{1-c_{2i}}{q_1q_2\cdots q_{2i}},$$

$$v_m = \sum_{i:2i-1\le m}\frac{1-c_{2i-1}}{q_1q_2\cdots q_{2i-1}} + \sum_{i:2i\le m}\frac{c_{2i}}{q_1q_2\cdots q_{2i}}.$$

We denote it by $\Delta_{c_1c_2\ldots c_m}$.

Lemma 1. *The cylindrical intervals have the following properties.*

1. $\inf\Delta_{c_1c_2\ldots c_m} = \inf\Delta'_{c_1c_2\ldots c_m} = -b+u_m$, $\sup\Delta_{c_1c_2\ldots c_m} = \sup\Delta'_{c_1c_2\ldots c_m} = d-v_m$, $\Delta'_{c_1c_2\ldots c_m} \subset \Delta_{c_1c_2\ldots c_m}$.

2. *The length of* $\Delta_{c_1c_2\ldots c_m}$ *is equal to*

$$|\Delta_{c_1c_2\ldots c_m}| = \operatorname{diam}\Delta'_{c_1c_2\ldots c_m} d+b-\sum_{k=1}^{m}\frac{1}{q_1q_2\cdots q_k} = \sum_{k=m+1}^{\infty}\frac{1}{q_1q_2\cdots q_k} =$$

$$= \frac{1}{q_1q_2\cdots q_m}\sum_{i=1}^{\infty}\frac{1}{q_{m+1}q_{m+2}\cdots q_{m+i}} \to 0 \quad (m\to\infty).$$

3. $\Delta_{c_1c_2\ldots c_m a} \subset \Delta_{c_1c_2\ldots c_m}$, $a \in A$.

4. $\Delta_{c_1c_2\ldots c_m 0} \cap \Delta_{c_1c_2\ldots c_m 1} = \varnothing$, *moreover*

$$k_{m+1} = |\Delta_{c_1c_2\ldots c_m} \setminus (\Delta_{c_1c_2\ldots c_m 0} \cup \Delta_{c_1c_2\ldots c_m 1})| =$$

$$= \frac{1}{q_1q_2\cdots q_{m+1}} - \sum_{k=m+2}^{\infty}\frac{1}{q_1q_2\cdots q_k} =$$

$$= \frac{1}{q_1q_2\cdots q_{m+1}}\left(1 - \sum_{i=1}^{\infty}\frac{1}{q_{m+2}q_{m+3}\cdots q_{m+1+i}}\right).$$

5. $\bigcap\limits_{m=1}^{\infty}\Delta_{c_1c_2\ldots c_m} = \bigcap\limits_{m=1}^{\infty}\Delta'_{c_1c_2\ldots c_m} = x =: \Delta_{c_1c_2\ldots c_m\ldots}$ *for any* $\{c_k\} \in L$, *i.e., any point from the set* C_r *can be considered as a "cylinder of infinite rank" as well as a "cylindrical interval of infinite rank".*

6. $C_r = \bigcap\limits_{m=1}^{\infty}\bigcup\limits_{c_i\in A,\, i=\overline{1,m}}\Delta_{c_1c_2\ldots c_m}$, $\Delta'_{c_1c_2\ldots c_m} = \Delta_{c_1c_2\ldots c_m} \cap C_r$.

The set C_r is a Cantor-like set, whose properties are stated below.

Theorem 2. *The set C_r of incomplete sums of the series* (1) *is*
1) *a nowhere dense set;*
2) *a perfect set;*
3) *a set of zero Lebesgue measure λ;*
4) *a set of zero Hausdorff–Besicovitch dimension, i.e., $\alpha_0(C_r) = 0$.*

Proof. 1. Property 1 is evident. It follows from Lemma 1, namely from properties 6, 4, and 2.

2. Let x be a limit point of the set C_r. Then for any $k \in \mathbb{N}$ there exists a cylindrical interval $\Delta_{a_1(x)a_2(x)\ldots a_k(x)} =: \Delta_k(x)$ of rank k such that $x \in \Delta_k(x)$. If this is not the case we would deduce that x belongs to an interval adjacent to C_r and there would exist $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap C_r = \varnothing$, which is impossible because x is a limit point of C_r. Then the intersection $\bigcap\limits_{k=1}^{\infty} \Delta_k(x)$ contains a unique point, which belongs to C_r and coincides with x. So, C_r is a closed set.

Let us now assume that C_r has isolated points and $x = \Delta_{a_1 a_2 \ldots a_k \ldots}$ is one of them. Then there exists $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \cap [C_r \setminus \{x\}] = \varnothing. \tag{5}$$

Let us choose k sufficiently large for $|\Delta_{a_1 a_2 \ldots a_k}| < \varepsilon$. Then $\Delta_{a_1 a_2 \ldots a_k} \subset (x - \varepsilon, x + \varepsilon)$ and $x \neq x' = \Delta_{a_1 a_2 \ldots a_k (1-a_{k+1}) a_{k+2} a_{k+3} \ldots} \in (x - \varepsilon, x + \varepsilon) \cap C_r$, thus contradicting (5). So, C_r is a closed set without isolated points, i.e., it is a perfect set.

3. It follows from Lemma 1 (namely, from properties 2 and 6) that

$$\lambda(C_r) = \lim_{m \to \infty} 2^m \left(\frac{1}{q_1 q_2 \cdots q_m} \sum_{i=1}^{\infty} \frac{1}{q_{m+1} q_{m+2} \cdots q_{m+i}} \right) \leq$$
$$\leq \lim_{m \to \infty} \frac{2^m}{q_1 q_2 \cdots q_m} = \lim_{m \to \infty} \left(\frac{2}{q_1} \cdot \frac{2}{q_2} \cdot \ldots \cdot \frac{2}{q_m} \right) = 0.$$

So, $\lambda(C_r) = 0$.

4. Let $H^\alpha(\cdot)$ be the Hausdorff measure and let $H^\alpha_\varepsilon(\cdot)$ be the Hausdorff premeasure (see, e.g., [10, p. 27]). The family of all cylindrical intervals of rank m forms an ε_m-covering of C_r with $\varepsilon_m = \frac{1}{q_1 q_2 \cdots q_m}$. It is clear that

$$H^\alpha_{\varepsilon_m}(C_r) \leq 2^m \left(\frac{1}{q_1 q_2 \cdots q_m} \sum_{i=1}^{\infty} \frac{1}{q_{m+1} q_{m+2} \cdots q_{m+i}} \right)^\alpha \leq$$
$$\leq \frac{2}{q_1^\alpha} \cdot \frac{2}{q_2^\alpha} \cdot \ldots \cdot \frac{2}{q_m^\alpha} \to 0 \quad (m \to \infty)$$

for any $\alpha > 0$. So, $H^\alpha(C_r) = 0$ for any $\alpha > 0$ and, therefore, the Hausdorff–Besicovitch dimension of C_r is equal to 0. $\square$

The uniformity of the above properties of sets C_r (and their independence of the choice of r) is "relatively unexpected" because, for instance, for the "binary series" $\sum\limits_{i\in M\subset\mathbb{N}} 2^{-i}$ properties of sets of incomplete sums can differ essentially (in the topological sense as well as in the sense of the Lebesgue measure and the Hausdorff–Besicovitch dimension). On the other hand, such a uniformity is, in some sense, natural because of a rather high rate of convergence of the first Ostrogradsky series, which plays an important role in Diophantine approximation and in the metric theory of the corresponding expansion (see, e.g., [1]).

3. THE FAMILY OF SETS OF INCOMPLETE SUMS OF ALL OSTROGRADSKY SERIES

As mentioned in Theorem 1, the sum of any infinite Ostrogradsky series is an irrational number from $[0,1]$. Any irrational number $r\in[0,1]$ uniquely determines an infinite first Ostrogradsky series $\mathrm{O}^1(q_1,q_2,\dots,q_k,\dots)$, $q_k=q_k(r)$. This series generates the set of its incomplete sums C_r, which is a compact anomalously fractal set (i.e., a continuum set of zero Hausdorff–Besicovitch dimension) with

$$\inf C_r=-b_r=-\sum_{i=1}^{\infty}\frac{1}{q_1q_2\cdots q_{2i}},\quad \sup C_r=d_r=\sum_{i=1}^{\infty}\frac{1}{q_1q_2\cdots q_{2i-1}}.$$

So, there exists a natural one-to-one correspondence between the set I of all irrational numbers from $[0,1]$ and the set of all Ostrogradsky series, hence a one-to-one correspondence between the set I and the family

$$G=\{C_r : r\in I\}$$

of sets of incomplete sums.

Since $q_{k+1}>q_k$, it is evident that

$$\inf\{-b_r : r\in I\}=-\sum_{i=1}^{\infty}\frac{1}{(2i)!}=:-b_0,\quad \sup\{d_r : r\in I\}=\sum_{i=1}^{\infty}\frac{1}{(2i-1)!}=:d_0.$$

Moreover, for $r_0=\mathrm{O}^1(q_1,q_2,\dots,q_k,\dots)$ with $q_k=k$, we have $\inf C_{r_0}=-b_0$, $\sup C_{r_0}=d_0$. So, $C_r\subset[-b_0,d_0]$ for any $r\in I$.

Let K be the family of all compact subsets of $[-b_0,d_0]$ and let ρ_H be the Hausdorff metric on K, i.e., for any $A,B\in K$,

$$\rho_H(A,B)=\inf\{\varepsilon : A_\varepsilon\supset B,\ B_\varepsilon\supset A\},$$

where

$$E_\varepsilon=\left\{x : x\in\mathbb{R}^1,\ \rho(x,E)=\inf\{|x-y| : y\in E\}<\varepsilon\right\}$$

is an ε-neighbourhood of the set E.

It is well known [8, p. 67] that (K, ρ_H) is a complete metric space. So, the Ostrogradsky series determine the continuum family G of compact anomalously fractal subsets and G is a subspace of K.

THEOREM 3. *The family G of sets of incomplete sums of all first Ostrogradsky series is a nowhere dense set in the metric space (K, ρ_H) of all compact subsets of $[-b_0, d_0]$ with Hausdorff metric.*

Proof. Let $r = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{q_1 q_2 \dots q_k}$ be an irrational number, and let C_r be the corresponding perfect nowhere dense set from the family G. For any $\varepsilon > 0$, let us choose a positive integer $n = n(\varepsilon)$ such that

$$\sum_{k=1}^{\infty} \frac{1}{q_1 q_2 \dots q_n \dots q_{n+k}} < \frac{\varepsilon}{2}$$

(it is sufficient to choose n such that $\sum\limits_{k=1}^{\infty} \frac{1}{(n+k)!} < \frac{\varepsilon}{2}$, because $q_k \geq k$).

By Lemma 1 (property 2), the length of any of the 2^n cylindrical intervals of rank n is less than $\frac{\varepsilon}{2}$.

Let us order cylindrical intervals of rank n, and let us choose the set $V_n = \{v_1^{(n)}, v_2^{(n)}, \dots, v_{2^n}^{(n)}\}$ such that $v_k^{(n)}$ belongs to the kth cylindrical interval of rank n and $v_k^{(n)} \neq 0$ for any $k = 1, 2, \dots, 2^n$. Let

$$\delta_0^{(n)} = \rho(0, V_n) = \min_k \{v_k^{(n)}\} > 0, \quad \delta^{(n)} = \min\left\{\delta_0^{(n)}, \frac{\varepsilon}{2}\right\}.$$

The set V_n belongs to the ε-neighbourhood of the set C_r, since the distances between a point $v_k^{(n)}$ and the endpoints of the corresponding cylindrical intervals are less than $\frac{\varepsilon}{2}$. On the other hand, C_r belongs to the ε-neighbourhood of the set V_n since the corresponding cylindrical interval belongs to the ε-neighbourhood of the point $v_k^{(n)}$. Therefore, $\rho_H(C_r, V_n) < \varepsilon$, hence in the metric space K the point V_n belongs to the open ball with the centre C_r and radius ε.

Now, let us consider the ball $O_R(V_n)$ in (K, ρ_H) centered at the point V_n and radius $R = \frac{\delta^{(n)}}{3}$. Let us show that it does not contain any point from G. Suppose, contrary to our claim, that there exists an irrational number $u \in [0, 1]$ such that the corresponding point $C_u \in O_R(V_n)$, i.e.,

$$\rho_H(C_u, V_n) \leq \frac{\delta^{(n)}}{3} \Leftrightarrow \inf\{\varepsilon : (C_u)_\varepsilon \supset V_n, (V_n)_\varepsilon \supset C_u\} \leq \frac{\delta^{(n)}}{3}.$$

In particular, the condition $(V_n)_{\frac{\delta^{(n)}}{2}} \supset C_u$ should hold, i.e., all points of the set C_u must belong to the $\frac{\delta^{(n)}}{2}$-neighbourhood of the set V_n, which is impossible, because $0 \in C_u$ and $\rho(V_n, 0) = \delta_0^{(n)} \geq \delta^{(n)} > \frac{\delta^{(n)}}{2}$. Therefore, $O_R(V_n) \cap G = \varnothing$. So, for any irrational number $r \in [0, 1]$ and for any positive ε in the metric space (K, ρ_H) the ball $O_\varepsilon(C_r)$ contains a point V_n having the neighbourhood $O_{\frac{\delta(n)}{3}}(V_n)$ which does not contain points from G. Therefore, G is a nowhere dense subset of the metric space (K, ρ_H). □

It is clear that for any compact subset $E \subset [-b_0, d_0]$ there exists a probability measure μ_E whose spectrum coincides with E. On the other hand, any set C_r is the spectrum of an infinite Bernoulli convolution defined by (6). So, the latter theorem reflects a "massivity" of the family of Bernoulli convolutions generated by the first Ostrogradsky series in the family of all probability distributions on $[-b_0, d_0]$.

4. PROBABILITY DISTRIBUTIONS ON THE SET OF INCOMPLETE SUMS OF A GIVEN OSTROGRADSKY SERIES

Let $\mathrm{O}^1(q_1, q_2, \ldots, q_k, \ldots)$ be any fixed first Ostrogradsky series. Let us consider the random variable

$$\psi = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varepsilon_k}{q_1 q_2 \cdots q_k}, \tag{6}$$

where $\{\varepsilon_k\}$ is a sequence of independent random variables taking the values 0 and 1 with probabilities p_{0k} and p_{1k}, respectively ($p_{0k} + p_{1k} = 1$).

According to the Jessen–Wintner theorem [14], the random variable ψ is of pure type, i.e., it is either discrete, absolutely continuous or singular continuous (with respect to Lebesgue measure).

The following proposition follows directly from the P. Lévy theorem [16] and gives necessary and sufficient conditions for the discreteness (and so, for the continuity) of ψ.

Theorem 4. *The random variable ψ has a discrete distribution if and only if*

$$P_{\max} = \prod_{k=1}^{\infty} \max\{p_{0k}, p_{1k}\} > 0.$$

Corollary 1. *The random variable ψ has a continuous distribution if and only if $P_{\max} = 0$.*

In the sequel we shall be interested in continuous distributions only.

The *spectrum* (*topological support*) S_ζ of the distribution of a random variable ζ is the minimal closed support of ζ, which coincides with the set

$$\{x : \mathsf{P}\{\zeta \in (x-\varepsilon, x+\varepsilon)\} > 0 \ \forall \varepsilon > 0\} = \{x : F_\zeta(x+\varepsilon) - F_\zeta(x-\varepsilon) > 0 \ \forall \varepsilon > 0\},$$

where F_ζ is the distribution function of the random variable ζ.

LEMMA 2. *The spectrum* (*topological support*) *of the distribution* ψ *is the set*

$$E = \{x : x = \Delta_{a_1 a_2 \ldots a_k \ldots},\ p_{a_k k} > 0 \ \forall k \in \mathbb{N}\}$$

which is a subset of the set of incomplete sums of series (1).

Proof. Let us first show that $E \subset S_\psi$. Let $\Delta_{a_1 a_2 \ldots a_k \ldots} = x \in E$. Then

$$\mathsf{P}\{\psi \in \Delta_{a_1 a_2 \ldots a_k}\} = \prod_{i=1}^{k} p_{a_i i} > 0$$

for any $k \in \mathbb{N}$. Let ε be any positive number. It follows from Lemma 1 (property 2) that there exists a positive integer k such that

$$\Delta_{a_1 a_2 \ldots a_k} \subset (x - \varepsilon, x + \varepsilon).$$

Therefore,

$$\mathsf{P}\{\psi \in (x - \varepsilon, x + \varepsilon)\} \ge \mathsf{P}\{\psi \in \Delta_{a_1 a_2 \ldots a_k}\} > 0,$$

i.e., $x \in S_\psi$ and $E \subset S_\psi$.

Next, let us show that $S_\psi \subset E$. Let $x \in S_\psi$, i.e.,

$$\mathsf{P}\{\psi \in (x - \varepsilon, x + \varepsilon)\} > 0 \quad \forall \varepsilon > 0. \tag{7}$$

Let us assume that there exists a positive integer k such that $p_{a_k k} = 0$, where $\Delta_{a_1 a_2 \ldots a_k \ldots} = x$. Then

$$\mathsf{P}\{\psi \in \Delta_{a_1 a_2 \ldots a_k}\} = \prod_{i=1}^{k} p_{a_i i} = 0.$$

Let us consider two mutually exclusive cases:

1) there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset \Delta_{a_1 a_2 \ldots a_k}$;

2) $(x - \varepsilon, x + \varepsilon) \not\subset \Delta_{a_1 a_2 \ldots a_k}$ for any $\varepsilon > 0$.

In the first case,

$$\mathsf{P}\{\psi \in (x - \varepsilon, x + \varepsilon)\} \le \mathsf{P}\{\psi \in \Delta_{a_1 a_2 \ldots a_k}\} = 0,$$

which is impossible by (7).

In the second case, x must be a one-sided limit point of the set C_r. Without loss of generality, let us assume that x is a left-sided limit point. Then there exists $\varepsilon > 0$ such that $(x - \varepsilon, x) \subset \Delta_{a_1 a_2 \ldots a_k}$ and $\mathsf{P}\{\psi \in (x, x + \varepsilon)\} = 0$. In this case,

$$\mathsf{P}\{\psi \in (x - \varepsilon, x + \varepsilon)\} = \mathsf{P}\{\psi \in (x - \varepsilon, x)\} \le \mathsf{P}\{\psi \in \Delta_{a_1 a_2 \ldots a_k}\} = 0,$$

which contradicts condition (7).

These contradictions prove that $p_{a_k k} > 0$ for any $k \in \mathbb{N}$, i.e., $x \in E$. So, $S_\psi = E$, which proves the lemma. □

THEOREM 5. *If $P_{\max} = 0$ then ψ has a singular distribution of the Cantor type with an anomalously fractal spectrum (topological support).*

Proof. If $P_{\max} = 0$ then ψ has a continuous distribution, by Corollary 1 after Theorem 4. Since the Hausdorff–Besicovitch dimension $\alpha_0(C_r) = 0$ and $S_\psi \subset C_r$, the topological support S_ψ is an anomalously fractal set. Since the Lebesgue measure $\lambda(C_r) = 0$, we have $\lambda(S_\psi) = 0$. So, ψ has a distribution of the Cantor type. □

It is clear that to define the distribution function F_ψ of the random variable ψ, it is sufficient to define it at the points of the topological support S_ψ of the distribution, since it is defined by continuity and monotonicity at the remaining points: if $x \notin S_\psi$ then $F_\psi(x) = F_\psi(x_1)$, where $x_1 = \sup\{u : u < x, u \in S_\psi\}$.

LEMMA 3. *At a point $\Delta_{a_1 a_2 \ldots a_k \ldots} = x \in S_\psi$ the distribution function F_ψ of the random incomplete sum (6) has the value*

$$F_\psi(x) = \beta_{a_1 1} + \sum_{k=2}^{\infty}\Bigl(\beta_{a_k k}\prod_{j=1}^{k-1} p_{a_j j}\Bigr), \tag{8}$$

where

$$\beta_{a_k k} = \begin{cases} a_k p_{0k} & \text{if } k \text{ is odd,} \\ (1 - a_k)p_{1k} & \text{if } k \text{ is even.} \end{cases}$$

Proof. The event $\{\psi < x\}$ can be represented as

$$\{\psi < x\} = \{\varepsilon_1 < a_1\} \cup \{\varepsilon_1 = a_1,\ \varepsilon_2 > a_2\} \cup \{\varepsilon_1 = a_1,\ \varepsilon_2 = a_2,\ \varepsilon_3 < a_3\} \cup \cdots$$
$$\cdots \cup \{\varepsilon_1 = a_1,\ \varepsilon_2 = a_2,\ \ldots,\ \varepsilon_{k-1} = a_{k-1},\ \varepsilon_k \vee a_k\} \cup \cdots,$$

where

$$\vee = \begin{cases} < & \text{if } k \text{ is odd,} \\ > & \text{if } k \text{ is even.} \end{cases}$$

Then

$$\mathsf{P}\{\psi < x\} = \mathsf{P}\{\varepsilon_1 < a_1\} + \mathsf{P}\{\varepsilon_1 = a_1,\ \varepsilon_2 > a_2\} +$$
$$+\mathsf{P}\{\varepsilon_1 = a_1,\ \varepsilon_2 = a_2,\ \varepsilon_3 < a_3\} + \cdots$$
$$\cdots + \mathsf{P}\{\varepsilon_1 = a_1,\ \varepsilon_2 = a_2,\ \ldots,\ \varepsilon_{k-1} = a_{k-1},\ \varepsilon_k \vee a_k\} + \cdots.$$

It follows from the independence of the ε_i that

$$\mathsf{P}\{\varepsilon_1 = a_1,\ \varepsilon_2 = a_2,\ \ldots,\ \varepsilon_{k-1} = a_{k-1},\ \varepsilon_k \vee a_k\} =$$
$$= \left(\prod_{j=1}^{k-1} \mathsf{P}\{\varepsilon_j = a_j\}\right) \cdot \mathsf{P}\{\varepsilon_k \vee a_k\} = \beta_{a_k k} \prod_{j=1}^{k-1} p_{a_j j}.$$

Since ψ is a continuous random variable, $F_\psi(x) = \mathsf{P}\{\psi < x\}$ is of the form (8). $\square$

5. THE FOURIER–STIELTJES COEFFICIENTS AND THE CHARACTERISTIC FUNCTION OF THE RANDOM INCOMPLETE SUM OF THE OSTROGRADSKY SERIES WITH INDEPENDENT TERMS

Let us consider the *characteristic function* $f_\zeta(t)$ *of a random variable* ζ, i.e.,

$$f_\zeta(t) = \mathsf{E}\big(\mathrm{e}^{\mathrm{i}t\zeta}\big).$$

The theory of characteristic functions is convenient for the investigation of the structure and properties of distributions of real-valued random variables. For a given random variable ζ one can define

$$L_\zeta = \limsup_{|t|\to\infty} |f_\zeta(t)|.$$

It is well known [17, p. 28] that $L_\zeta = 1$ for any discretely distributed random variable ζ, and $L_\zeta = 0$ for the case of absolute continuity of the distribution of ζ. For a singularly continuously distributed random variable ζ the value L_ζ can be equal to any number from $[0,1]$, i.e., for any $\beta \in [0,1]$ there exists a singularly distributed random variable ζ_β such that $L_{\zeta_\beta} = \beta$. Let us study the asymptotic behaviour at infinity of the absolute value of the characteristic function of the random variable ψ.

LEMMA 4. *The characteristic function of the random variable* ψ *defined by* (6) *is of the form*

$$f_\psi(t) = \prod_{k=1}^{\infty} f_k(t) \quad \textit{with} \quad f_k(t) = p_{0k} + p_{1k} \exp \frac{(-1)^{k-1}\mathrm{i}t}{q_1 q_2 \ldots q_k}, \tag{9}$$

and its absolute value is of the form

$$|f_\psi(t)| = \prod_{k=1}^{\infty} |f_k(t)|, \quad \textit{where} \quad |f_k(t)| = \sqrt{1 - 4p_{0k}p_{1k} \sin^2 \frac{t}{2q_1 q_2 \ldots q_k}}.$$

Proof. By the definition of a characteristic function and properties of expectations, we have

$$f_\psi(t)=\mathsf{E}\big(\mathrm{e}^{\mathrm{i}t\psi}\big)=\mathsf{E}\left(\exp\left(\mathrm{i}t\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\varepsilon_k}{q_1q_2\ldots q_k}\right)\right)=$$
$$=\mathsf{E}\left(\exp\frac{\mathrm{i}t\varepsilon_1}{q_1}\cdot\exp\frac{-\mathrm{i}t\varepsilon_2}{q_1q_2}\cdot\ldots\cdot\exp\frac{(-1)^{k-1}\mathrm{i}t\varepsilon_k}{q_1q_2\ldots q_k}\cdot\ldots\right)=$$
$$=\prod_{k=1}^{\infty}\mathsf{E}\left(\exp\frac{(-1)^{k-1}\mathrm{i}t\varepsilon_k}{q_1q_2\ldots q_k}\right)=\prod_{k=1}^{\infty}\left(p_{0k}+p_{1k}\exp\frac{(-1)^{k-1}\mathrm{i}t}{q_1q_2\ldots q_k}\right)=$$
$$=\prod_{k=1}^{\infty}\left[\left(p_{0k}+p_{1k}\cos\frac{(-1)^{k-1}t}{q_1q_2\ldots q_k}\right)+\mathrm{i}p_{1k}\sin\frac{(-1)^{k-1}t}{q_1q_2\ldots q_k}\right]=\prod_{k=1}^{\infty}f_k(t),$$

and

$$|f_k(t)|=\sqrt{p_{0k}^2+2p_{0k}p_{1k}\cos\frac{t}{q_1q_2\ldots q_k}+p_{1k}^2}=\sqrt{1-4p_{0k}p_{1k}\sin^2\frac{t}{2q_1q_2\ldots q_k}},$$

which proves the lemma. □

THEOREM 6. *For any increasing sequence $\{q_n\}$ we have*

$$L_\psi=\limsup_{|t|\to\infty}|f_\psi(t)|=1.$$

Proof. Let us consider the sequence $t_n=2\pi q_1q_2\ldots q_n$, and let us estimate

$$|f_\psi(t)|=\prod_{k=1}^{\infty}\sqrt{1-4p_{0k}p_{1k}\sin^2\frac{t}{2q_1q_2\ldots q_k}}\ge$$
$$\ge\prod_{k=1}^{\infty}\sqrt{1-\sin^2\frac{t}{2q_1q_2\ldots q_k}}=\prod_{k=1}^{\infty}\left|\cos\frac{t}{2q_1q_2\ldots q_k}\right|.$$

It is clear that

$$L_\psi\ge\lim_{n\to\infty}|f_\psi(t_n)|=\lim_{n\to\infty}\prod_{k=1}^{\infty}|f_k(t_n)|\ge\lim_{n\to\infty}\prod_{k=1}^{\infty}\left|\cos\frac{t_n}{2q_1q_2\ldots q_k}\right|.$$

We have

$$\left|\cos\frac{t_n}{2q_1q_2\ldots q_k}\right|=\begin{cases}1 & \text{if } k\le n,\\ \cos\frac{\pi}{q_{n+1}q_{n+2}\cdots q_k} & \text{if } k>n.\end{cases}$$

So,

$$\prod_{k=1}^{\infty}|f_k(t_n)|\ge\prod_{k=n+1}^{\infty}\cos\frac{\pi}{q_{n+1}q_{n+2}\ldots q_k}.$$

But, for $k > n$,

$$\cos\frac{\pi}{q_{n+1}q_{n+2}\cdots q_k} \geq \cos\frac{\pi n!}{k!} = 1-2\sin^2\frac{\pi n!}{2k!} > 1-2\left(\frac{\pi n!}{2k!}\right)^2 = 1-\left(\frac{\pi n!}{\sqrt{2}k!}\right)^2.$$

Then

$$\prod_{k=n+1}^{\infty}\cos\frac{\pi}{q_{n+1}q_{n+2}\cdots q_k} \geq \prod_{k=n+1}^{\infty}\left(1-\left(\frac{\pi n!}{\sqrt{2}k!}\right)^2\right).$$

Since the series

$$\sum_{k=n+1}^{\infty}\left(\frac{\pi n!}{\sqrt{2}k!}\right)^2$$

converges, the latter infinite product also converges to some positive constant, so that

$$\prod_{k=m}^{\infty}\cos\frac{\pi}{q_{n+1}q_{n+2}\cdots q_k} \to 1 \text{ as } m\to\infty.$$

Therefore,

$$\lim_{n\to\infty}\prod_{k=n+1}^{\infty}\cos\frac{\pi}{q_{n+1}q_{n+2}\cdots q_k} = \lim_{m\to\infty}\prod_{k=m}^{\infty}\cos\frac{\pi}{q_{n+1}q_{n+2}\cdots q_k} = 1,$$

hence $L_\psi = 1$. □

So, the distribution of ψ is "similar" to a discrete one (it is supported by a set of zero Hausdorff–Besicovitch dimension and $L_\psi = 1$).

Moreover, even for the case $p_{0k} = p_{1k} = \frac{1}{2}$ for any $a \in [0,1]$ we can construct a sequence $t_n(a) \to \infty$ such that $\lim\limits_{n\to\infty}|f_\psi(t_n(a))| = a$. Asymptotic properties of the corresponding Fourier–Stieltjes coefficients are also studied in this section.

THEOREM 7. *Let $f_\psi(t)$ be the characteristic function of the random variable ψ and let ε_k be independent identically distributed random variables with $p_{0k} = p_{1k} = \frac{1}{2}$. For any number $a \in [0,1]$ there exists a sequence $t_n(a) \to \infty$ as $n \to \infty$ such that*

$$\lim_{n\to\infty}|f_\psi(t_n(a))| = a.$$

Proof. It follows from the proof of Theorem 6 that

$$|f_\psi(t)| = \prod_{k=1}^{\infty}\left|\cos\frac{t}{2q_1q_2\cdots q_k}\right|.$$

For a given number a let $A = \arccos a \in \left[0,\frac{\pi}{2}\right]$. For each $n\in\mathbb{N}$, let $s_n(a)$ be the unique non-negative integer for which

$$0 \leq \frac{\pi s_n(a)}{q_{n+1}} \leq A < \frac{\pi(s_n(a)+1)}{q_{n+1}}.$$

The sequence $\left\{\frac{\pi s_n(a)}{q_{n+1}}\right\}$ converges to A.

Let us consider the sequence of real numbers $t_n(a) = 2\pi q_1 q_2 \dots q_n s_n(a)$. Then

$$|f_\psi(t_n(a))| = \prod_{k=1}^{\infty} \left|\cos \frac{2\pi q_1 q_2 \dots q_n s_n(a)}{2q_1 q_2 \dots q_k}\right| =$$

$$= \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{n} \cdot \cos \frac{\pi s_n(a)}{q_{n+1}} \cdot \prod_{m=2}^{\infty} \cos \frac{\pi s_n(a)}{q_{n+1} q_{n+2} \cdots q_{n+m}}.$$

Since

$$\lim_{n\to\infty} \cos \frac{\pi s_n(a)}{q_{n+1}} = a \quad \text{and} \quad \lim_{n\to\infty} \prod_{m=2}^{\infty} \cos \frac{\pi s_n(a)}{q_{n+1} q_{n+2} \cdots q_{n+m}} = 1,$$

we have

$$\lim_{n\to\infty} |f_\psi(t_n(a))| = a. \quad \square$$

It follows from the well known Fisher–Riesz theorem (see, e.g., [13, p. 107]) that for a singular continuous distribution with distribution function $F(x)$ the condition

$$\sum_{m=1}^{\infty} c_m^2 = \infty,$$

holds, where c_m are the Fourier–Stieltjes coefficients of $F(x)$, i.e.,

$$c_m = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi m i x} \mathrm{d}F(x).$$

Nevertheless, for some classes of singular measures, c_m can tend to 0, as for absolutely continuous distributions.

Theorem 8. *The Fourier–Stieltjes coefficients of the distribution function of the random variable ψ defined by* (6) *are of the form*

$$c_m = \prod_{k=1}^{\infty} \left(p_{0k} + p_{1k} \exp \frac{(-1)^{k-1} 2\pi m \mathrm{i}}{q_1 q_2 \cdots q_k} \right) \tag{10}$$

and they do not converge to 0 *as* $m \to \infty$. *Moreover,*

$$\tilde{L}_\psi = \limsup_{m\to\infty} |c_m| = 1. \tag{11}$$

Proof. Equation (10) follows from (9) and from $c_m = f_\psi(2\pi m)$. Then

$$|c_m| = \prod_{k=1}^{\infty} |f_k(2\pi m)|, \quad \text{where} \quad |f_k(2\pi m)| = \sqrt{1 - 4p_{0k}p_{1k} \sin^2 \frac{\pi m}{q_1 q_2 \cdots q_k}}.$$

The proof of (11) is similar to the proof of the fact that $L_\psi = 1$ in Theorem 6. Choosing the sequence $m_n = q_1 q_2 \dots q_n$, we have

$$\tilde{L}_\psi \geq \lim_{n\to\infty} |c_{m_n}| \geq \lim_{n\to\infty} \prod_{k=1}^{\infty} \left| \cos \frac{\pi m_n}{q_1 q_2 \dots q_k} \right| = 1.$$

Since $\tilde{L}_\psi \leq 1$, we have $\tilde{L}_\psi = 1$, which proves the theorem. □

6. FRACTAL PROPERTIES OF THE DISTRIBUTION OF THE RANDOM VARIABLE ψ

The investigation of fractal properties of the spectrum (topological support) S_η of a given random variable η is the first step of the multilevel fractal analysis of the corresponding probability measure (see, e.g., [36] for details). Since S_η is the minimal closed support of η, the Hausdorff–Besicovitch dimension $\alpha_0(S_\eta)$ gives us rather "rough external" information about η. For instance, for any closed set E there exists a discretely distributed random variable η such that $S_\eta = E$. So, $\alpha_0(S_\eta) > 0$ does not imply the continuity of η. The topological support does not reflect essential properties even for Cantor-like singular distributions. For instance, the classical one-third Cantor set is the topological support of the random variable

$$\eta_1 = \sum_{k=1}^{\infty} \frac{\xi_k}{3^k},$$

where ξ_k are independent identically distributed random variables taking values 0 and 2 with probabilities $\frac{1}{2}$ and $\frac{1}{2}$, as well as for the random variable

$$\eta_2 = \sum_{k=1}^{\infty} \frac{\xi'_k}{3^k},$$

where ξ'_k are independent identically distributed random variables taking values 0 and 2 with probabilities $\frac{1}{3}$ and $\frac{2}{3}$. On the other hand, it follows from the Kakutani theorem [15] that the distributions of η_1 and η_2 are mutually singular.

Another fractal characteristic of a given probability distribution η is the Hausdorff dimension of itself, which can be introduced as

$$\alpha_0(\eta) = \inf_{E\in B_\eta} \{\alpha_0(E),\ E \in \mathcal{B}\},$$

where $\mathcal{B}$ is the σ-algebra of Borel subsets of the real line, B_η is the class of all possible "supports" of the random variable η, i.e.,

$$B_\eta = \{E : E \in \mathcal{B},\ \mathsf{P}_\eta(E) = 1\}.$$

The Hausdorff dimension of a given random variable gives us richer information about its distribution. In particular, $\alpha_0(\eta) = 0$ for any discrete distribution η. If η has an absolutely continuous distribution then, obviously, $\alpha_0(\eta) = 1$. For a singularly distributed random variable η, the number $\alpha_0(\eta)$ can be any real number from the unit interval.

It is clear that

$$\alpha_0(\eta) \leq \alpha_0(S_\eta).$$

Let us thus consider the Hausdorff dimension of the random variable ψ defined by (6). Since $\alpha_0(S_\psi) = 0$, we have $\alpha_0(\psi) = 0$. In such a case the classical Hausdorff–Besicovitch dimension does not reflect the difference between the topological support and essential supports (see, e.g., [36]) of the distribution. To stress this difference we shall use a generalization of the Hausdorff–Besicovitch dimension – the so-called Hausdorff–Billingsley dimension.

Let us define some notions we shall use in the sequel. Let M be a fixed bounded subset of the real line. A family Φ_M of intervals is said to be a *fine covering family* for M if for any subset $E \subset M$, and for any $\varepsilon > 0$ there exists an at most countable ε-covering $\{E_j\}$ of E, $E_j \in \Phi_M$, i.e., $\forall E \subset M$, $\forall \varepsilon > 0$ $\exists \{E_j\}$ $(E_j \in \Phi_M,\ |E_j| \leq \varepsilon)$: $E \subset \bigcup\limits_j E_j$. A fine covering family Φ_M is said to be *fractal* if for the determination of the Hausdorff–Besicovitch dimension of any subset $E \subset M$ it is enough to consider only coverings from Φ_M.

For a given bounded subset M of the real line, let Φ_M be a fine covering family for M, let α be a positive number and let ν be a continuous probability measure. The ν-α-Hausdorff–Billingsley measure of a subset $E \subset M$ w.r.t. Φ_M is defined as

$$H^\alpha(E, \nu, \Phi_M) = \lim_{\varepsilon \to 0} \left\{ \inf_{\nu(E_j) \leq \varepsilon} \sum_j (\nu(E_j))^\alpha \right\},$$

where $E_j \in \Phi_M$ and $\bigcup\limits_j E_j \supset E$.

Definition. The number $\alpha_\nu(E, \Phi_M) = \inf\{\alpha : H^\alpha(E, \nu, \Phi_M) = 0\}$ is called the *Hausdorff–Billingsley dimension of the set E with respect to the measure ν and the family of coverings Φ_M.*

Remark. 1) If Φ_M is the family of all closed subintervals of the minimal closed interval $[a, b]$ containing M, then for any $E \subset M$ the number $\alpha_\nu(E, \Phi_M)$ coincides with the classical Hausdorff–Billingsley dimension $\alpha_\nu(E)$ of the subset E w.r.t. the measure ν.

2) If $M = [0, 1]$, ν is the Lebesgue measure on $[0, 1]$ and Φ_M is a fractal family of coverings, then for any $E \subset M$ the number $\alpha_\nu(E, \Phi_M)$ coincides with the classical Hausdorff–Besicovitch dimension $\alpha_0(E)$ of the subset E.

Let Φ^{ν}_M be the image of a fine covering family under the distribution function of a probability measure ν, i.e., $\Phi^{\nu}_M = \{E' : E' = F_{\nu}(E),\ E \in \Phi_M\}$.

LEMMA 5. *A fine covering family Φ_M can be used for the equivalent definition of the Hausdorff–Billingsley dimension of any subset $E \subset M$ w.r.t. a measure ν if and only if the covering family Φ^{ν}_M can be used for the equivalent definition of the classical Hausdorff–Besicovitch dimension of any subset $E' = F_{\nu}(E)$, $E \subset M$, that is, $\alpha_{\nu}(E, \Phi_M) = \alpha_{\nu}(E)$ for any $E \subset M$ if and only if the covering family Φ^{ν}_M is fractal.*

Proof. Because of the countable stability of the above mentioned dimensions, we may assume without loss of generality that the set M is bounded. Let Φ be the family of all closed subintervals of the minimal closed interval $[a, b]$ containing M and let Φ_M be a fine covering family. Since $\nu(E_j) = |E'_j|$ and $\bigcup_j E'_j \supset E'$, we have

$$H^{\alpha}(E, \nu, \Phi_M) = \lim_{\varepsilon \to 0} \left\{ \inf_{\nu(E_j) \le \varepsilon} \sum_j (\nu(E_j))^{\alpha} \right\} =$$

$$= \lim_{\varepsilon \to 0} \left\{ \inf_{|E'_j| \le \varepsilon} \sum_j |E'_j|^{\alpha} \right\} = H^{\alpha}(E', \Phi^{\nu}_M).$$

Therefore,

$$\alpha_{\nu}(E, \Phi_M) = \alpha_0(E', \Phi^{\nu}_M).$$

The family $\Phi^{\nu} = F_{\nu}(\Phi)$ coincides with the family of all closed subintervals of $[a', b'] \supset M'$. Therefore,

$$H^{\alpha}(E, \nu) = H^{\alpha}(E, \nu, \Phi) = H^{\alpha}(E', \Phi^{\nu}),$$

and

$$\alpha_{\nu}(E) = \alpha_{\nu}(E, \Phi) = \alpha_0(E', \Phi^{\nu})\alpha_0(E').$$

If the family Φ^{ν}_M is fractal, then $\alpha_{\nu}(E, \Phi_M) = \alpha_0(E', \Phi^{\nu}_M) = \alpha_0(E') = \alpha_{\nu}(E)$.

If the family Φ^{ν}_M is not fractal, then there exists a subset $E' \subset M'$ such that $\alpha_0(E') < \alpha_0(E', \Phi^{\nu}_M)$. In such a case we have

$$\alpha_{\nu}(E) = \alpha_0(E') < \alpha_0(E', \Phi^{\nu}_M) = \alpha_{\nu}(E, \Phi_M),$$

which proves the lemma. □

Similarly to the above mentioned definition of the Hausdorff dimension of η, we introduce the following notion.

Definition. The number

$$\alpha_{\nu}(\eta) = \inf_{E \in B_{\eta}} \{\alpha_{\nu}(E),\ E \in \mathcal{B}\}$$

is said to be the *Hausdorff–Billingsley dimension of the random variable* η *with respect to the measure* ν, where $\mathcal{B}$ is the σ-algebra of Borel subsets of the real line, B_η is the class of all possible "supports" of the random variable η.

To stress the difference between the topological support and essential supports of the distribution of the above mentioned random variable ψ, we shall use the Hausdorff–Billingsley dimension with respect to the measure ν^*, where ν^* is the probability measure that is "uniformly distributed" on the set of incomplete sums, i.e., ν^* is the probability measure corresponding to the random variable

$$\psi^* = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varepsilon_k^*}{q_1 q_2 \dots q_k},$$

where ε_k^* are independent identically distributed random variables taking the values 0 and 1 with probabilities $p_{0k} = \frac{1}{2}$ and $p_{1k} = \frac{1}{2}$, respectively.

Theorem 9. *Let* $h_n = -(p_{0n} \ln p_{0n} + p_{1n} \ln p_{1n})$ *be the entropy of the random variable* ε_n *and let* $H_n = h_1 + h_2 + \dots + h_n$. *Then the Hausdorff–Billingsley dimension of the distribution of the random variable* ψ *with respect to the measure* ν^* *is equal to*

$$\alpha_{\nu^*}(\psi) = \liminf_{n\to\infty} \frac{H_n}{n \ln 2}.$$

Proof. Let $M \equiv C_r$ with $r = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{q_1 q_2 \dots q_k}$, and let Φ_M be the family of the above mentioned cylindrical intervals (see Section 2), i.e., $\Phi_M = \{\Delta_{c_1 c_2 \dots c_k}, k \in \mathbb{N}\}$ and $C_r = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{c_i \in \{0,1\}} \Delta_{c_1 c_2 \dots c_k}$. Since $\nu^*(\Delta_{c_1 c_2 \dots c_k}) = 2^{-k}$, the image $\Phi_M^{\nu^*} = F_{\nu^*}(\Phi_M)$ coincides with the fractal fine covering family consisting of binary closed subintervals of $[0, 1]$. So, it follows from Lemma 5 that for the determination of the Hausdorff–Billingsley dimension of an arbitrary subset of $C_r \supset S_\psi$ w.r.t. ν^* it is enough to consider only coverings consisting of cylindrical intervals.

Let ν be the probability measure corresponding to the random variable ψ, and let $\Delta_n(x) = \Delta_{c_1(x) c_2(x) \dots c_n(x)}$ be a cylindrical interval of nth rank containing a point x. Then we have

$$\nu(\Delta_n(x)) = p_{c_1(x)1} \cdot p_{c_2(x)2} \cdot \dots \cdot p_{c_n(x)n}, \quad \nu^*(\Delta_n(x)) = \frac{1}{2^n}.$$

Let us consider the quantity

$$\frac{\ln \nu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} = \frac{\sum\limits_{j=1}^{n} \ln p_{c_j(x)j}}{-n \ln 2}.$$

If $x = \Delta_{c_1(x)c_2(x)\dots c_n(x)\dots}$ is chosen at random such that $\mathsf{P}\{c_j(x) = i\} = p_{ij}$ (i.e., the distribution of the random variable x corresponds to the measure ν), then $\{\eta_j\} = \{\eta_j(x)\} = \{\ln p_{c_j(x)j}\}$ is a sequence of independent random variables taking the values $\ln p_{0j}$ and $\ln p_{1j}$ with probabilities p_{0j} and p_{1j}, respectively. Hence

$$\mathsf{E}(\eta_j) = p_{0j}\ln p_{0j} + p_{1j}\ln p_{1j} = -h_j, \quad h_j \le \ln 2,$$
$$\mathsf{E}(\eta_j^2) = p_{0j}\ln^2 p_{0j} + p_{1j}\ln^2 p_{1j} \le c_0 < \infty,$$

and the constant c_0 does not depend on j. For instance, one can choose $c_0 = \frac{4}{e^2}$. By the strong law of large numbers (Kolmogorov's theorem, see, e.g., [33, Ch. III. 3.2]), for ν-almost all points $x \in C_r$ we have

$$\lim_{n\to\infty} \frac{(\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)) - \mathsf{E}(\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x))}{n} = 0.$$

Remark that $\mathsf{E}(\eta_1 + \eta_2 + \cdots + \eta_n) = -H_n$.

Let us consider the set

$$A = \left\{x : \lim_{n\to\infty}\left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-n\ln 2} - \frac{H_n}{n\ln 2}\right) = 0\right\} =$$
$$= \left\{x : \lim_{n\to\infty} \frac{(\eta_1(x)+\eta_2(x)+\cdots+\eta_n(x)) - \mathsf{E}(\eta_1(x)+\eta_2(x)+\cdots+\eta_n(x))}{-n\ln 2} = 0\right\}.$$

Since $\nu(A) = 1$, we have $\alpha_\nu(A) = 1$ and, therefore, $\alpha_\nu(A, \Phi_M) = 1$.

Let us consider the sets

$$A_1 = \left\{x : \liminf_{n\to\infty}\left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-n\ln 2} - \frac{H_n}{n\ln 2}\right) = 0\right\};$$

$$A_2 = \left\{x : \liminf_{n\to\infty} \frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-n\ln 2} \le \liminf_{n\to\infty} \frac{H_n}{n\ln 2}\right\} =$$
$$= \left\{x : \liminf_{n\to\infty} \frac{\ln\nu(\Delta_n(x))}{\ln\nu^*(\Delta_n(x))} \le \liminf_{n\to\infty} \frac{H_n}{n\ln 2}\right\};$$

$$A_3 = \left\{x : \liminf_{n\to\infty} \frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-n\ln 2} \ge \liminf_{n\to\infty} \frac{H_n}{n\ln 2}\right\} =$$
$$= \left\{x : \liminf_{n\to\infty} \frac{\ln\nu(\Delta_n(x))}{\ln\nu^*(\Delta_n(x))} \ge \liminf_{n\to\infty} \frac{H_n}{n\ln 2}\right\}.$$

It is obvious that $A \subset A_1$. We now prove the inclusions $A_1 \subset A_3$ and $A \subset A_2$. To this end we use the well-known inequality

$$\liminf_{n\to\infty}(x_n - y_n) \le \liminf_{n\to\infty} x_n - \liminf_{n\to\infty} y_n$$

holding for arbitrary sequences $\{x_n\}$ and $\{y_n\}$ of real numbers. If $x \in A_1$ then

$$\liminf_{n\to\infty} \frac{\eta_1(x)+\eta_2(x)+\cdots+\eta_n(x)}{-n\ln 2} - \liminf_{n\to\infty} \frac{H_n}{n\ln 2} \geq$$
$$\geq \liminf_{n\to\infty} \left(\frac{\eta_1(x)+\eta_2(x)+\cdots+\eta_n(x)}{-n\ln 2} - \frac{H_n}{n\ln 2} \right) = 0.$$

So, $x \in A_3$. If $x \in A$ then

$$\liminf_{n\to\infty} \frac{H_n}{n\ln 2} - \liminf_{n\to\infty} \frac{\eta_1(x)+\eta_2(x)+\cdots+\eta_n(x)}{-n\ln 2} \geq$$
$$\geq \liminf_{n\to\infty} \left(\frac{H_n}{n\ln 2} - \frac{\eta_1(x)+\eta_2(x)+\cdots+\eta_n(x)}{-n\ln 2} \right) = 0.$$

So, $x \in A_2$.

Let $D = \liminf_{n\to\infty} \frac{H_n}{n\ln 2}$. Since $A \subset A_2 = \left\{x : \liminf_{n\to\infty} \frac{\ln \nu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} \leq D\right\}$, we have

$$\alpha_{\nu^*}(A, \Phi_M) \leq D.$$

Since $A \subset A_3 = \left\{x : \liminf_{n\to\infty} \frac{\ln \nu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} \geq D\right\}$, we deduce that

$$\alpha_{\nu^*}(A, \Phi_M) \geq D \cdot \alpha_\nu(A, \Phi_M) = D$$

since $\alpha_\nu(A, \Phi_M) = 1$. Therefore,

$$\alpha_{\nu^*}(A, \Phi_M) = \alpha_{\nu^*}(A) = D.$$

We now prove that the above constructed set A is the "smallest" support of the measure ν in the sense of the Hausdorff–Billingsley dimension w.r.t. ν^*. Let C be an arbitrary support of the measure ν. It is easily seen that the set $C_1 = C \cap A$ also is a support of the same measure ν, and $C_1 \subset C$. It follows from $C_1 \subset A$ that $\alpha_{\nu^*}(C_1) \leq D$, and

$$C_1 \subset A \subset A_3 = \left\{x : \liminf_{n\to\infty} \frac{\ln \nu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} \geq D\right\}.$$

Therefore,

$$\alpha_{\nu^*}(C_1) = \alpha_{\nu^*}(C_1, \Phi_M) \geq D \cdot \alpha_\nu(C_1, \Phi_M) = D \cdot 1 = D.$$

So, $\alpha_{\nu^*}(C) \geq \alpha_{\nu^*}(C_1) = D = \alpha_{\nu^*}(A)$. □

THEOREM 10. *The Hausdorff–Billingsley dimension* (*with respect to the measure* ν^*) *of the spectrum* (*topological support*) S_ψ *of the random variable* ψ *is equal to*

$$\alpha_{\nu^*}(S_\psi) = \liminf_{k\to\infty} \frac{N_k}{k},$$

where $N_k = \#\{j : j \leq k,\ p_{0j}p_{1j} > 0\}$, *i.e.,* N_k *is the number of columns* (*in the matrix* $\|p_{ij}\|$) *without zero elements until the* k*th column.*

Proof. Let us consider an auxiliary probability measure $\tilde{\nu}$ which corresponds to the random variable

$$\tilde{\psi} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\tilde{\varepsilon}_k}{q_1 q_2 \dots q_k}, \tag{12}$$

where $\tilde{\varepsilon}_k$ are independent random variables taking the values 0 and 1 with probabilities $\tilde{p}_{0k}$ and $\tilde{p}_{1k}$ such that

$$\tilde{p}_{ik} = \begin{cases} 0 & \text{iff } p_{ik} = 0, \\ 1 & \text{iff } p_{ik} = 1, \\ \frac{1}{2} & \text{in all other cases.} \end{cases}$$

Generally speaking, the random variables ψ and $\tilde{\psi}$ have different distributions, but their topological supports coincide. The measure $\tilde{\nu}$ is "uniformly distributed" on the topological supports of the initial measure. Therefore,

$$\alpha_0(\tilde{\psi}) = \alpha_0(S_{\tilde{\psi}}) = \alpha_0(S_\psi).$$

So, to determine the Hausdorff–Besicovitch dimension of the topological support of the initial measure w.r.t. the measure ν^*, it is enough to apply Theorem 9 to the measure $\tilde{\nu}$, taking into account that $\tilde{h}_k = 0$, if $p_{0k}p_{1k} = 0$, and $\tilde{h}_k = \ln 2$, if $p_{0k}p_{1k} \neq 0$. □

Corollary 2. *If* $\lim_{k\to\infty} p_{0k} = p_0 > 0$ *then*

$$\alpha_{\nu^*}(S_\psi) = 1, \quad \alpha_{\nu^*}(\psi) = \frac{p_0 \ln p_0 + p_1 \ln p_1}{\ln 2}.$$

7. FRACTAL DIMENSION PRESERVATION

Let F be a distribution function and let $\mu = \mu_F$ be the corresponding probability measure with the topological support S_F. Let $\tilde{\mu}$ be the probability measure which gives uniform distribution on S_F.

We say that a distribution function F preserves the Hausdorff–Billingsley dimension (w.r.t. $\tilde{\mu}$) on a set A if the Hausdorff–Billingsley dimension $\alpha_\mu(E)$ of any subset $E \subseteq A$ is equal to the Hausdorff–Billingsley dimension $\alpha_{\tilde{\mu}}(E)$.

If the distribution function F strictly increases (i.e., the topological support S_F is a closed interval), then the above definition reduces to the usual definition of a transformation preserving the Hausdorff–Besicovitch dimension (see, e.g., [2] for details).

Let ψ and $\tilde{\psi}$ be random variables defined by equations (6) and (12), and let ν and $\tilde{\nu}$ be the corresponding probability measures. It is easily seen that their topological supports coincide. If, in addition, $p_{ik} > 0$ for any $i \in \{0,1\}$ and $k \in \mathbb{N}$, then $S_{\tilde{\nu}} = S_\nu = C_r$ with $r = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{q_1 q_2 \ldots q_k}$.

Of course, the distribution function F_ν does not preserve the classical Hausdorff–Besicovitch dimension because $\alpha_0(S_\nu) = 0$ and $\alpha_0(F_\nu(S_\nu)) = 1$. But F_ν can preserve the Hausdorff–Billingsley dimension on its topological support.

THEOREM 11. *If there exists a positive constant p_0 such that $p_{ik} > p_0$, then the distribution function F_ψ of the random variable ψ preserves the Hausdorff–Billingsley dimension $\alpha_{\tilde{\nu}}(\cdot)$ on the topological support of ψ if and only if*

$$\liminf_{n\to\infty} \frac{H_n}{n \ln 2} = 1. \tag{13}$$

Proof. *Necessity.* Let $N(\nu)$ be the set of all possible "supports" of the measure ν. Let $\alpha_{\tilde{\nu}}(\nu) = \inf\limits_{E \in N(\nu)} \{\alpha_{\tilde{\nu}}(E)\}$ be the Hausdorff–Billingsley dimension of the measure ν w.r.t. the measure $\tilde{\nu}$. It follows from Theorem 9 that $\alpha_{\tilde{\nu}}(\nu) = \liminf\limits_{n\to\infty} \frac{H_n}{n \ln 2}$ with $H_n = h_1 + h_2 + \cdots + h_n$ and $h_k = -(p_{0k} \ln p_{0k} + p_{1k} \ln p_{1k})$. If $\alpha_{\tilde{\nu}}(\nu) < 1$ then there exists a set $E \in N(\nu)$ such that $\alpha_{\tilde{\nu}}(\nu) \le \alpha_{\tilde{\nu}}(E) < 1$. Since $\nu(E) = 1$, we conclude that $\alpha_\nu(E) = 1 \neq \alpha_{\tilde{\nu}}(E)$, which contradicts the assumption of the theorem.

Sufficiency. It is well known that $h_k \le \ln 2$ for any $k \in \mathbb{N}$ and the equality holds if and only if $p_{0k} = p_{1k} = \frac{1}{2}$. Therefore, condition (13) is equivalent to the existence of the limit

$$\lim_{k\to\infty} \frac{h_1 + h_2 + \cdots + h_k}{k \ln 2} = 1. \tag{14}$$

Let $\varepsilon < \frac{1}{3}$ be an arbitrary positive real number and let us consider the sets

$$M^+_{\varepsilon,k} = \left\{ j : j \in \mathbb{N},\, j \le k,\, \left|p_{0j} - \tfrac{1}{2}\right| \le \varepsilon \right\},$$
$$M^-_{\varepsilon,k} = \{1, 2, \ldots, k\} \setminus M^+_{\varepsilon,k} = \left\{ j : j \in \mathbb{N},\, j \le k,\, \left|p_{0j} - \tfrac{1}{2}\right| > \varepsilon \right\}.$$

It follows from $h_k \le \ln 2$ and condition (14) that

$$\lim_{k\to\infty} \frac{|M^+_{\varepsilon,k}|}{k} = 1, \quad \lim_{k\to\infty} \frac{|M^-_{\varepsilon,k}|}{k} = 0.$$

Let $\Delta_{\alpha_1(x)\ldots\alpha_k(x)}$ be the k-rank cylindrical interval containing the point x. For any $x \in S_\nu$ we consider the limit

$$V(x) = \lim_{k\to\infty} \frac{\ln \nu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \tilde{\nu}(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}.$$

We shall show that $p_{ij} > p_0$ and condition (13) imply that the above limit exists and $V(x) = 1$ for any $x \in S_\nu$. We have

$$\frac{\ln \nu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \tilde{\nu}(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} = \frac{\ln(p_{\alpha_1(x)1} \cdot p_{\alpha_2(x)2} \cdot \ldots \cdot p_{\alpha_k(x)k})}{\ln 2^{-k}} =$$

$$= \frac{\sum\limits_{j \in M^+_{\varepsilon,k}} \ln p_{\alpha_j(x)j} + \sum\limits_{j \in M^-_{\varepsilon,k}} \ln p_{\alpha_j(x)j}}{-k \ln 2}.$$

If $j \in M^+_{\varepsilon,k}$, then $\frac{1}{2} - \varepsilon \leq p_{\alpha_j(x)j} \leq \frac{1}{2} + \varepsilon$ and, consequently, there exists a number $C_{\varepsilon,k} \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ such that

$$\sum_{j \in M^+_{\varepsilon,k}} \ln p_{\alpha_j(x)j} = |M^+_{\varepsilon,k}| \cdot \ln C_{\varepsilon,k}.$$

If $j \in M^-_{\varepsilon,k}$ then $p_0 \leq p_{\alpha_j(x)j} \leq 1 - p_0$. Therefore, there exists a number $d_{\varepsilon,k} \in [p_0, 1 - p_0]$ such that

$$\sum_{j \in M^-_{\varepsilon,k}} \ln p_{\alpha_j(x)j} = |M^-_{\varepsilon,k}| \cdot \ln d_{\varepsilon,k}.$$

Let us consider the upper limit

$$V^*(x) = \limsup_{k\to\infty} \frac{\ln \nu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \tilde{\nu}(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} =$$

$$= \limsup_{k\to\infty} \left(\frac{|M^+_{\varepsilon,k}|}{k} \cdot \frac{\ln C_{\varepsilon,k}}{-\ln 2} + \frac{|M^-_{\varepsilon,k}|}{k} \cdot \frac{\ln d_{\varepsilon,k}}{-\ln 2} \right).$$

We have $V^*(x) \leq \frac{\ln\left(\frac{1}{2} - \varepsilon\right)}{-\ln 2}$ because $\frac{|M^+_{\varepsilon,k}|}{k} \to 1$, $\frac{1}{2} - \varepsilon \leq C_{\varepsilon,k} \leq \frac{1}{2} + \varepsilon$, $\frac{|M^-_{\varepsilon,k}|}{k} \to 0$, $p_0 \leq d_{\varepsilon,k} \leq 1 - p_0$. In the same manner we can show that

$$V_*(x) = \liminf_{k\to\infty} \frac{\ln \nu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \tilde{\nu}(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} \geq \frac{\ln\left(\frac{1}{2} + \varepsilon\right)}{-\ln 2}, \quad \forall x \in S_\nu,\ \forall \varepsilon \in \left(0, \tfrac{1}{3}\right).$$

Therefore, $V(x) = 1$ for any $x \in S_\nu$ and, by using Billingsley's theorem [5], we have

$$\alpha_{\tilde{\nu}}(E) = 1 \cdot \alpha_\nu(E)$$

for all $E \subset S_\nu$, which proves the theorem. □

Corollary 3. *If*

$$\lim_{k\to\infty} p_{0k} = \frac{1}{2}$$

then the distribution function F_ψ *of the random variable* ψ *defined by* (6) *preserves the Hausdorff–Billingsley dimension on the topological support of* ψ.

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REFERENCES

[1] S. Albeverio, O. Baranovskyi, M. Pratsiovytyi and G. Torbin, *The Ostrogradsky series and related Cantor-like sets*. Acta Arith. **130** (2007), *3*, 215–230.

[2] S. Albeverio, M. Pratsiovytyi and G. Torbin, *Fractal probability distributions and transformations preserving the Hausdorff–Besicovitch dimension*. Ergodic Theory Dynam. Systems **24** (2004), *1*, 1–16.

[3] S. Albeverio and G. Torbin, *Image measures of infinite product measures and generalized Bernoulli convolutions*. Transactions of the National Pedagogical University of Ukraine. Mathematics **5** (2004), 228–241.

[4] S. Albeverio and G. Torbin, *On fine fractal properties of generalized infinite Bernoulli convolutions*. Bull. Sci. Math. **132** (2008), *8*, 711–727.

[5] P. Billingsley, *Hausdorff dimension in probability theory*. II. Illinois J. Math. **5** (1961), 291–298.

[6] P.I. Bodnarchuk and V.Ya. Skorobogat'ko, *Branching Continued Fractions and their Applications*. Naukova Dumka, Kyiv, 1974. (Ukrainian)

[7] M.J.P. Cooper, *Dimension, measure and infinite Bernoulli convolutions*. Math. Proc. Cambridge Philos. Soc. **124** (1998), *1*, 135–149.

[8] G.A. Edgar, *Measure, Topology, and Fractal Geometry*. Springer-Verlag, New York, 1990.

[9] P. Erdös, *On a family of symmetric Bernoulli convolutions*. Amer. J. Math. **61** (1939), *4*, 974–976.

[10] K.J. Falconer, *Fractal Geometry – Mathematical Foundations and Applications*, 2nd Edition. John Wiley, Chichester, 2003.

[11] J. Galambos, *Representations of Real Numbers by Infinite Series*. Lecture Notes in Math. **502**. Springer-Verlag, Berlin, 1976.

[12] M. Iosifescu and C. Kraaikamp, *Metrical Theory of Continued Fractions*. Kluwer Acad. Publ., Dordrecht, 2002.

[13] O.S. Ivašev-Musatov, *On Fourier–Stieltjes coefficients of singular functions*. Amer. Math. Soc. Transl. Ser. 2 **10** (1958), 107–124.

[14] B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*. Trans. Amer. Math. Soc. **38** (1935), *1*, 48–88.

[15] S. Kakutani, *On equivalence of infinite product measures*. Ann. of Math. (2) **49** (1948), *1*, 214–224.

[16] P. Lévy, *Sur les séries dont les termes sont des variables éventuelles indépendantes*. Studia Math. **3** (1931), 119–155.

[17] E. Lukacs, *Characteristic Functions*, 2nd Edition. Hafner, New York, 1970.

[18] Yu.V. Melnichuk, *On representation of real numbers by fast converging series.* In: *Continued Fractions and Their Applications*, pp. 77–78. Inst. Mat., Akad. Nauk Ukrain. SSR, Kiev, 1976. (Russian)

[19] Yu.V. Melnichuk, *Fast converging series representations of real numbers and their implementations in digital processing.* In: A. Pethö et al. (Eds.), *Computational Number Theory* (*Debrecen,* 1989), pp. 27–29. de Gruyter, Berlin, 1991.

[20] J. Paradís, Ll. Bibiloni and P. Viader, *On actually computable bijections between* $\mathbb{N}$ *and* $\mathbb{Q}^+$. Order **13** (1996), *4*, 369–377.

[21] J. Paradís, P. Viader and Ll. Bibiloni, *A mathematical excursion: from the three-door problem to a Cantor-type set.* Amer. Math. Monthly **106** (1999), *3*, 241–251.

[22] Y. Peres, W. Schlag and B. Solomyak, *Sixty years of Bernoulli convolutions.* In: *Fractal Geometry and Stochastics*, II, pp. 39–65. Progr. Probab. **46**, Birkhäuser, Basel, 2000.

[23] Y. Peres and B. Solomyak, *Absolute continuity of Bernoulli convolutions, a simple proof.* Math. Res. Lett. **3** (1996), *2*, 231–239.

[24] T.A. Pierce, *On an algorithm and its use in approximating roots of algebraic equations.* Amer. Math. Monthly **36** (1929), *10*, 523–525.

[25] M.V. Pratsiovytyi, *Fractal Approach to Investigations of Singular Probability Distributions.* National Pedagogical Univ., Kyiv, 1998. (Ukrainian)

[26] J.I. Reich, *Some results on distributions arising from coin tossing.* Ann. Probab. **10** (1982), *3*, 780–786.

[27] J.I. Reich, *When do weighted sums of independent random variables have a density—some results and examples.* Ann. Probab. **10** (1982), *3*, 787–798.

[28] E.Ya. Remez, *On series with alternating sign which may be connected with two algorithms of M.V. Ostrogradskiĭ for the approximation of irrational numbers.* Uspehi Matem. Nauk (N.S.) **6** (1951), *5* (*45*), 33–42. (Russian)

[29] R. Salem, *Sets of uniqueness and sets of multiplicity.* Trans. Amer. Math. Soc. **54** (1943), *2*, 218–228.

[30] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory.* Oxford Sci. Publ., Oxford Univ. Press, New York, 1995.

[31] J.O. Shallit, *Some predictable Pierce expansions.* Fibonacci Quart. **22** (1984), *4*, 332–335.

[32] J.O. Shallit, *Metric theory of Pierce expansions.* Fibonacci Quart. **24** (1986), *1*, 22–40.

[33] A.N. Shiryaev, *Probability.* Springer-Verlag, New York, 1996.

[34] W. Sierpiński, *O kilku algorytmach dla rozwijania liczb rzeczywistych na szeregi.* Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego, Wydział III **4** (1911), 56–77. French translation *Sur quelques algorithmes pour développer les nombres réels en séries.* In: W. Sierpiński, *Oeuvres choisies*, **I**, pp. 236–254. PWN, Warszawa, 1974.

[35] B. Solomyak, *On the random series* $\sum \pm\lambda^n$ (*an Erdös problem*). Ann. of Math. (2) **142** (1995), *3*, 611–625.

[36] G.M. Torbin, *Multifractal analysis of singularly continuous probability measures.* Ukrainian Math. J. **57** (2005), *5*, 837–857.

[37] K.G. Valēēv and E.D. Zlēbov, *The metric theory of the Ostrogradskiĭ algorithm*. Ukrainian Math. J. **27** (1975), *1*, 47–51.

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