ON GEOMETRIC MEASURE THEORY
ON RECTIFIABLE METRIC SPACES

MARIA KARMANOVA

We give a survey of the area and coarea formulas for mappings of metric spaces. We formulate some recent results regarding this topic.


Key words: metric space, metric differential, area formula, coarea formula, level set.

It is well-known that a mapping \( \varphi : \mathbb{R}^n \to \mathbb{R}^k \) is said to be differentiable at a point \( x \) if there exists a linear mapping \( L : \mathbb{R}^n \to \mathbb{R}^k \) such that
\[
\varphi(y) - \varphi(x) = L(y - x) + o(y - x).
\]
Set \( L = D\varphi(x) \). In the case when \( n \leq k \) and \( \varphi \) is a homeomorphism of the class \( C^1 \) and rank \( D\varphi(x) = n \), in some neighborhood of each point \( x \) we have
\[
(1 - o(1))|D\varphi(x)(y - z)| \leq |\varphi(y) - \varphi(z)| \leq (1 + o(1))|D\varphi(x)(y - z)|
\]
as \( y, z \to x \). The Jacobian
\[
J(x, \varphi) \overset{\text{def}}{=} \lim_{r \to 0} \frac{\mathcal{H}^n(\varphi(Q(x, r)))}{|Q(x, r)|} = \sqrt{\det(D\varphi^*(x)D\varphi(x))}
\]
characterizes the local distortion of \( \mathcal{H}^n \)-measure (here \( D\varphi^*(x) \) is the adjoint operator of the differential \( D\varphi(x) \)), and for injective mappings the area and change-of-variable formulas below hold:
\[
\int_A J(x, \varphi) \, dx = \mathcal{H}^n(\varphi(A));
\]
\[
\int_A u(\varphi(x))J(x, \varphi) \, dx = \int_{\varphi(U)} u(y) \, d\mathcal{H}^n(y),
\]
where the set \( A \subset \mathbb{R}^n \) is measurable, and \( u \) is a nonnegative measurable function on \( \varphi(U) \). (Here, \( \mathcal{H}^n \) denotes the \( n \)-dimensional Hausdorff measure.)

REV. ROUMAINE MATH. PURES APPL., 54 (2009), 5–6, 461–472
In the case when $n \geq k$ and $\varphi$ is a $C^1$-mapping, the coarea formula:

\begin{equation}
\int_{U} \varphi(x) \sqrt{\det(D\varphi(x)D\varphi^*(x))} \, d\mathcal{H}^n(x) = \int_{\mathbb{R}^k} d\mathcal{H}^k(s) \int_{\varphi^{-1}(s)} \varphi(u) \, d\mathcal{H}^{n-k}(u),
\end{equation}

does hold. Here, $U$ is an open set in $\mathbb{R}^n$. The coarea factor $\sqrt{\det(D\varphi(x)D\varphi^*(x))}$ in (2) characterizes the distortion of the $k$-dimensional Hausdorff measure in the direction orthogonal to the kernel of the differential $D\varphi(x)$ under the mapping $\varphi$. Formula (2) has applications, in particular, in the theory of exterior forms, theory of currents, in minimal surfaces problems, algebraic geometry, etc. (see, for example, [9]). Also, Stokes formula can easily be proved using the coarea formula (see, e.g., [30], [25]). The extension to Lipschitz mappings of Implicit Function Theorem asserts [8] (see also [11]) that the preimage $\varphi^{-1}(z)$ of $\mathcal{H}^k$-almost any point in $z \in \mathbb{R}^k$ is an $\mathcal{H}^{n-k}$-rectifiable metric space. Note that such a statement can be deduced via the coarea formula for Lipschitz mappings.

This formula was first stated in 1950 by Kronrod [19] in a special case of $\varphi : \mathbb{R}^2 \to \mathbb{R}$. Later, in 1959, Federer [7] generalized it to Lipschitz mappings of Riemannian manifolds, and in 1969 this result was extended to Lipschitz mappings of rectifiable sets of Euclidean spaces [8]. Results on the coarea formula can also be found in [6], [10], [20]. In 1978, M. Ohtsuka proved the coarea formula in the case of $\varphi \in \text{Lip}(\mathbb{R}^n, \mathbb{R}^m)$, $n, m \geq k$, when the image $\varphi(\mathbb{R}^n)$ is $\mathcal{H}^k$-$\sigma$-finite [26]. An infinite-dimensional analog of the coarea formula was proved in 1988 by Airault and Malliavin [1] for Wiener spaces. This result can be found in [21].

In 2000-2003, different coarea properties of Sobolev mappings were investigated in [11], [22], [23], [24].

As noted earlier, (1) and (2) were generalized to the case of Lipschitz mappings of Riemannian manifolds. One of the simplest example of a Riemannian manifold is a surface in a Euclidean space which is defined as the image of an open subset $U \subset \mathbb{R}^n$ under a regular mapping, that is, a one-to-one mapping of the class $C^1$, the differential of which has maximal rank.

The images of measurable sets in $\mathbb{R}^n$ are generalizations of such manifolds. $\mathcal{H}^n$-rectifiable metric spaces are of independent interest. A metric space $\mathbb{Y}$ is called $\mathcal{H}^n$-rectifiable if there exists an at most countable collection $\{\beta_i\}$ of Lipschitz mappings defined on measurable subsets $\{B_i\} \subset \mathbb{R}^n$ such that $\mathcal{H}^n(\mathbb{Y} \setminus \bigcup_i \beta_i(B_i)) = 0$. In other words, the space $\mathbb{Y}$ up to a set of $\mathcal{H}^n$-measure zero equals a union of images of sets $B_i$ under Lipschitz mappings $\beta_i : B_i \to \mathbb{Y}$. An example of $\mathcal{H}^n$-rectifiable metric space is a Riemannian manifold with measurable (but not necessarily continuous) Riemann tensor enjoying some non-degeneracy condition; a manifold with singularities is also an example of a rectifiable metric space.
Modern trends of the development of geometric measure theory lead to necessity of extension of (1) and (2) to metric structures of quite general nature. Such structures are, e.g., manifolds with singularities. In this direction, there were obtained [12, 14, 16] the results below.

**Theorem 1 (The Area Formula [12, 14, 16]).** Suppose that \((\mathbb{Y}, d_\mathbb{Y})\) is an \(\mathcal{H}^n\)-rectifiable metric space, \((\mathcal{X}, d_\mathcal{X})\) is an arbitrary metric space, and \(\varphi \in \text{Lip}(\mathbb{Y}, \mathcal{X})\). Then the formula

\[
\int_{\mathbb{Y}} J(\varphi, x) \, d\mathcal{H}^n(x) = \int_{\mathcal{X}} N(\varphi, y, \mathbb{Y}) \, d\mathcal{H}^n(y),
\]

where

\[
J(\varphi, x) = \frac{J(MD(\varphi \circ \beta_m, \beta_m^{-1}(x)))}{J(MD(\beta_m, \beta_m^{-1}(x)))}
\]

on \(\mathbb{Y}_m\), holds for every function \(f : \mathbb{Y} \to E\) such that \(f(x)J(\varphi, x)\) is integrable. Here, \(E\) is an arbitrary Banach space.

**Theorem 2 (The Coarea Formula [12, 16]).** Suppose that \((\mathbb{Y}, d_\mathbb{Y})\) is an \(\mathcal{H}^n\)-rectifiable metric space, \((\mathcal{X}, d_\mathcal{X})\) is an \(\mathcal{H}^k\)-rectifiable metric space, \(n \geq k\), and \(\varphi \in \text{Lip}(\mathbb{Y}, \mathcal{X})\). Then the formula

\[
\int_{\mathbb{Y}} f(x)J_k(\varphi, x) \, d\mathcal{H}^n(x) = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z)} f(u) \, d\mathcal{H}^{n-k}(u),
\]

where

\[
J_k(\varphi, x) = \frac{J_k(MD(\varphi \circ \beta_m, \beta_m^{-1}(x)))J_{n-k}(MD(\beta_m, \beta_m^{-1}(x)))}{J(MD(\beta_m, \beta_m^{-1}(x)))}
\]

on \(\mathbb{Y}_m\), holds for every function \(f : \mathbb{Y} \to E\) such that \(f(x)J_k(\varphi, x)\) is integrable. Here, \(E\) is an arbitrary Banach space.

In the above statements, \(MD(\varphi, x)\), \(J(\varphi, x)\), \(J_k(\varphi, x)\) are the metric differential, the metric Jacobian and the metric coarea factor; these notions are generalizations of the classical differential, Jacobian and coarea factor, respectively. In the case of \(\mathbb{Y} = E \subset \mathbb{R}^n\), we have \(J(\varphi, x) = J(MD(\varphi, x))\) and \(J_k(\varphi, x) = J_k(MD(\varphi, x))\). The geometric meaning of the metric Jacobian and coarea factor is the distortion of \(\mathcal{H}^n\)-measure and \(\mathcal{H}^k\)-measure, respectively, and the metric differential is a seminorm on \(\mathbb{R}^n\) approximating the initial mapping \(\varphi\).

**Definition 3 ([18]).** Let \(E \subset \mathbb{R}^n\) be a measurable set and let \((\mathcal{X}, d_\mathcal{X})\) be a metric space. A mapping \(\varphi : E \to (\mathcal{X}, d_\mathcal{X})\) is **metrically differentiable**, or
$m$-differentiable, at a point $x \in E$ if there exists a seminorm $L(x)$ on $\mathbb{R}^n$ such that, in some neighborhood of $x$,

$$d_X(\varphi(z), \varphi(y)) = L(x)(z - y) = o(|z - x| + |y - x|)$$

as $z, y \to x$, where $z, y \in E$. The seminorm $L(x)$ is called the metric differential, or the $m$-differential, of the mapping $\varphi$ at the point $x$ and is denoted by $MD(\varphi, x)$.

It is easy to see that for each $u \in \mathbb{S}^{n-1}$ we have

$$MD(\varphi, x)(u) = \lim_{|\alpha - \beta| \to 0 \atop \alpha < 0 < \beta, x + \alpha u, x + \beta u \in E} \frac{d_E(\varphi(x + \beta u), \varphi(x + \alpha u))}{|\alpha - \beta|}.$$

In the case where $X = \mathbb{R}^n$, $MD(\varphi, x)(u)$ is the norm of the derivative in the direction $u$.

The definition of a metric differential, or $m$-differential, was given by Ambrosio [2] in 1990 for curves in a metric space. In the same paper, the metric differentiability of Lipschitz curves was proved. See also results on the metric differentiability and on length of curves in [5]. The notions of metric differential and metric Jacobian for mappings defined on $\mathbb{R}^n$, $n > 1$, were given by Kirchheim [18] for Lipschitz mappings of open subset of $\mathbb{R}^n$ into a metric space. He also solved the problem on the area formula in this case.

Analytic expressions for the classical Jacobian and coarea factor show that the differential of a mapping $\varphi$ is a tool for describing the measure distortion. It is well-known that there is no linear structure in general metric spaces. Thus, it is impossible to introduce the notion of differentiability in the usual sense. Note that for calculating the local measure distortion of a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ it is sufficient to find the values of the seminorm $\| \cdot \|_{D\varphi(x)}$ on the unit sphere $\mathbb{S}^{n-1}$ and to use the relations

$$\lim_{r \to 0} \frac{\mathcal{H}^n(\varphi(Q(x, r)))}{|Q(x, r)|} = \mathcal{J}(\| \cdot \|_{D\varphi(x)}) = \omega_n \left( \int_{\mathbb{S}^{n-1}} \| D\varphi(x)(u) \|^{-n} d\mathcal{H}^{n-1}(u) \right)^{-1},$$

where $Q(x, r)$ is a cube in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Note also that

$$\| D\varphi(x)(u) \| = \lim_{|\alpha - \beta| \to 0 \atop \alpha < 0 < \beta} \frac{d_E(\varphi(x + \beta u), \varphi(x + \alpha u))}{|\alpha - \beta|},$$

where $d_E$ is the Euclidean distance. Obviously, in (5) and (6) the linear structure is not used. Thus, such an approach can be used for metric-valued Lipschitz mappings.
As mentioned earlier, for mappings defined on rectifiable metric spaces, the metric Jacobian and coarea factor are defined in the following way. It is well-known that for mappings \( \varphi : \mathbb{R}^n \to \mathbb{R}^m \) and \( \psi : \varphi(\mathbb{R}^n) \to \mathbb{R}^l \) the chain rule \( D(\psi \circ \varphi)(x) = D\psi(\varphi(x)) \circ D\varphi(x) \) holds. Consequently, the Jacobian of the composition is the product of the corresponding Jacobians. Such a property motivates the definition below of the Jacobian for a mapping \( \varphi \) defined on an \( \mathcal{H}^n \)-rectifiable metric space \( Y \) \cite{12, 16}:

\[
\mathcal{J}(\varphi, x) = \frac{\mathcal{J}(MD(\varphi \circ \beta_j, \beta_j^{-1}(x)))}{\mathcal{J}(MD(\beta_j, \beta_j^{-1}(x)))};
\]

moreover, we show that such a definition does not depend on choice of \( \beta_j \) (here, \( \beta_j \) is one of “parametrizing” mappings).

We use similar arguments to formulate the definition of the metric coarea factor.

In 2000, Ambrosio and Kirchheim \cite{3, 4} discussed questions on proving (3) and (4). They proved the coarea formula in the particular case when the preimage is an \( \mathcal{H}^n \)-rectifiable metric space while the image is a Euclidean space \( \mathbb{R}^k \), \( n \geq k \).

One of main approaches used in these papers is the isometric embedding of a metric space into a Banach one. This allows to reduce the case of a measurable set to the case of an open set in the proof of the area formula, and to linearize the problem in the proof of the coarea formula.

In some sharp problems of analysis, intrinsic methods of investigation of the metric differential without isometric embeddings are of great interest. Such an approach is essentially different from that of \cite{3}. Note that recently new intrinsic methods have been developed for the investigation of complicated geometric structures \cite{29}, \cite{28}. Thus, “direct” methods of working with mappings defined on measurable sets and their metric images instead of isometric embeddings into Banach spaces, are of special interest. The development of “direct” methods allows us to obtain results more general than those in \cite{18}, \cite{3}.

As noted earlier, we prove the coarea formula for mappings defined on an \( \mathcal{H}^n \)-rectifiable metric space with values in an \( \mathcal{H}^k \)-rectifiable metric space. Compare with the result of \cite{3}, where the mapping takes values in a Euclidean space. The method we set up can be considered as a generalization of the approach to geometric measure theory problems developed in \cite{30} for \( C^1 \)-mappings of Euclidean spaces.

The first step in obtaining Theorem 2 is the following result.

**Lemma 4 (The Chain Rule [16]).** Suppose that \( E \) is a measurable set in \( \mathbb{R}^n \), \( A \) a measurable set in \( \mathbb{R}^k \), \( k \leq n \), and \((X, d_X)\) a metric space. Suppose also that \( \varphi : E \to X \) is a Lipschitz continuously \( m \)-differentiable mapping.
such that \( \dim \ker(MD(\varphi, x)) = n - k \), \( \alpha : A \to X \) a bi-Lipschitz continuously \( m \)-differentiable mapping with nondegenerate \( m \)-differential \( MD(\alpha, x)(u), u \in S^{k-1} \), and the mapping \( \alpha^{-1} \circ \varphi \) is continuously differentiable everywhere. Then

\[
|J(\alpha^{-1} \circ \varphi)(x)| = \frac{\mathcal{J}_k(MD(\varphi, x))}{\mathcal{J}(MD(\alpha, \alpha^{-1}(\varphi(x))))}
\]

for almost every \( x \in E \).

Using this metric chain rule, we reduce part of our problem to the coarea formula for mappings of Euclidean spaces with applications to the metric area formula and the chain rule.

Next, one of the main differences between the case considered in [3] and our case is that in an \( H^k \)-rectifiable metric space \( X \) there exists a set of \( H^k \)-measure zero such that it cannot be parametrized by mappings defined on subsets of \( \mathbb{R}^k \). Thus, on this set we cannot apply the chain rule, and the question on the influence of this set of the left-hand part of the coarea formula should be solved independently. We have the following result.

**Theorem 5** ([12, 13, 16, 17]). Suppose that \( (X, d_X) \) is a metric space, \( A \subset X \), and \( H^k(A) = 0 \). Then

\[
\int_{\varphi^{-1}(A)} \mathcal{J}_k(MD(\varphi, x)) \, dx = 0.
\]

From here we obtain the coarea formula for Lipschitz mappings \( \varphi : E \to X \), \( E \subset \mathbb{R}^n \) is a measurable set, where \( X \) is an \( H^k \)-rectifiable metric space.

**Theorem 6** (The Coarea Formula [12, 16]). Suppose that \( E \) is a measurable set in \( \mathbb{R}^n \), \( (X, d_X) \) an \( H^k \)-rectifiable metric space, \( n \geq k \), and \( \varphi : E \to X \) a Lipschitz mapping. Then the formula

\[
\int_E f(x) \mathcal{J}_k(MD(\varphi, x)) \, dx = \int_X d\mathcal{H}^k(s) \int_{\varphi^{-1}(s)} f(u) d\mathcal{H}^{n-k}(u).
\]

holds for each function \( f : E \to \mathbb{E} \) (\( \mathbb{E} \) is an arbitrary Banach space) provided that \( f(x) \mathcal{J}_k(MD(\varphi, x)) \) is integrable.

Note that in (7) the dimension of the kernel of the \( m \)-differential is at least \( n - k \) almost everywhere on \( E \). The coarea \( m \)-factor for a mapping defined on \( \mathbb{R}^n \) is equal to

\[
\mathcal{J}_k(MD(\varphi, x)) = \omega_k k \left( \int_{S^{n-1} \cap (\ker(MD(\varphi, x)))^\perp} [MD(\varphi, x)(u)]^{-k} d\mathcal{H}^{k-1}(u) \right)^{-1},
\]
where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

The final step in obtaining the coarea formula for mappings defined on an $\mathcal{H}^n$-rectifiable metric space is the proof of the area formula on level sets:

**Theorem 7** (The Area Formula for Mappings of Level Sets [16, 17]). Suppose that $z \in X$, $\dim \ker(MD(\varphi \circ \beta_m, y)) = n - k$, and $(\varphi \circ \beta_m)^{-1}(z) \cap B_m$ is a level set. Then, for each $m \in \mathbb{N}$, there exists $B_{m,0} \subset B_m$, $\mathcal{H}^n(B_{m,0}) = 0$, such that the area formula

$$\int_{(\varphi \circ \beta_m)^{-1}(z) \cap B_m \setminus B_{m,0}} \mathcal{J}_{n-k}(MD(\beta_m, y)) \, d\mathcal{H}^{n-k}(y) = \int_{\varphi^{-1}(z) \cap Y \setminus \beta_m(B_{m,0})} d\mathcal{H}^{n-k}(x)$$

holds. Here,

$$\mathcal{J}_{n-k}(MD(\beta_m, y)) = \omega_{n-k}(n-k) \int_{S^{n-k-1}} [MD(\beta_m, y)(u)]^{-(n-k)} \, d\mathcal{H}^{n-k-1}(u),$$

and $S^{n-k-1}$ is the $(n - k - 1)$-dimensional unit sphere in $\ker(MD(\varphi \circ \beta_m, y))$.

Suppose now that (7) is valid for a Lipschitz mapping $\varphi : E \to X$, where $X$ is an arbitrary metric space. A natural question about geometric properties of $X$ caused by the validity of (7), arises. We find necessary and sufficient conditions on the image and the preimage of a Lipschitz mapping for the validity of (7). In particular, we show that the $\mathcal{H}^k$-rectifiability of the image is not only sufficient, but it is also necessary.

We consider Lipschitz mappings defined on a measurable set $E \subset \mathbb{R}^n$ and taking values in an arbitrary metric space $X$, such that $\dim \ker(MD(\varphi, x)) \geq n - k$ for $\mathcal{H}^n$-almost all $x \in E$. Such restrictions on $\varphi$ are minimal since, first, sets of zero $\mathcal{H}^n$-measure do not influence both integrals in the coarea formula, and, second, the definition of the coarea $m$-factor (8) implies that the dimension of the kernel of the $m$-differential should not exceed $k$ almost everywhere on $E$.

The main tool in the investigation is the following metric analog of the Implicit Function Theorem.

**Theorem 8** (The Preimage-of-a-Point Theorem [13, 17]). Let $E \subset \mathbb{R}^n$ be a measurable set and $X$ an arbitrary metric space. Suppose that $\varphi \in \text{Lip}(E, X)$ and $\dim \ker(MD(\varphi, y)) \geq n - k$ for $\mathcal{H}^n$-almost all $y \in E$. Then there exists a set $\Sigma \subset E$ such that $\mathcal{H}^n(\Sigma) = 0$ and $\varphi^{-1}(z) \setminus (Z \cup \Sigma)$ is a subset of a union of countably many images of measurable subsets of $\mathbb{R}^{n-k}$ under Lipschitz mappings. In particular, it is an $\mathcal{H}^{n-k}$-rectifiable metric space for all $z \in X$.

This result was also generalized to mappings defined on an $\mathcal{H}^n$-rectifiable metric space $Y$. The main result is the following (note that it is also valid for mappings defined on $Y$):
Theorem 9 (The Coarea Formula Validity Criterion [13, 17]). Let \((\mathcal{Y}, d_{\mathcal{Y}})\) be an \(\mathcal{H}^n\)-rectifiable metric space and \(\mathcal{X}\) an arbitrary metric space. Let a mapping \(\varphi \in \text{Lip}(\mathcal{Y}, \mathcal{X})\). Suppose that \(\dim \ker(MD(\varphi \circ \beta_m, y)) \geq n - k\) for \(\mathcal{H}^n\)-almost all \(y \in B_m\), \(m \in \mathbb{N}\). Then the following statements are equivalent.

1. The coarea formula
   \[
   \int_{\mathcal{Y}} f(x) J_k(\varphi, x) \, dx = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z)} f(u) \, d\mathcal{H}^{n-k}(u)
   \]
   holds for every function \(f : \mathcal{Y} \to \mathbb{E}\) such that \(f(x) J_k(\varphi, x)\) is integrable, where \(\mathbb{E}\) is an arbitrary Banach space.

2. The coarea formula holds for any \(f(x) = \chi_A(x)\), where \(A \subset \mathcal{Y}\) is such that \(A \cap (\mathcal{Y} \setminus Z)\) is measurable.

3. \(\varphi(\mathcal{Y}) = \mathcal{X}_\Sigma \cup \mathcal{X}_Z \cup \mathcal{X}_0\), where \(\mathcal{X}_\Sigma = \varphi(\Sigma), \mathcal{H}^n(\Sigma) = 0, \mathcal{X}_Z = \varphi(Z)\); moreover,
   \[
   \int_{\varphi^{-1}(z) \cap Z} d\mathcal{H}^{n-k}(u) = 0
   \]
   for \(\mathcal{H}^k\)-almost all \(z \in \mathcal{X}\), and \(\mathcal{X}_0\) is a subset of a union of countably many images of measurable subsets of \(\mathbb{R}^k\) under Lipschitz mappings. In particular, \(\mathcal{X}_0\) is an \(\mathcal{H}^k\)-rectifiable metric space.

4. \(\varphi(\mathcal{Y}) = \mathcal{X}_\Sigma \cup \mathcal{X}_Z \cup \mathcal{X}_0\), where the sets \(\mathcal{X}_\Sigma\) and \(\mathcal{X}_Z\) are the same as in 3, and \(\mathcal{X}_0\) is an arbitrary \(\mathcal{H}^k\)-rectifiable metric space.

5. In the case of \(\mathcal{Y} = E \subset \mathbb{R}^n\), where \(E\) is measurable, the set \(E\) enjoys the following properties:
   
   (a) for each intersection \(E \cap B(0, s), s \in \mathbb{R}\), and each \(\varepsilon > 0\), there exist a measurable set \(\Sigma_\varepsilon \subset E \cap B(0, s)\) and a collection of compact sets \(\{K_i\}_{i \in \mathbb{N}}\) such that \(\mathcal{H}^n(\Sigma_\varepsilon) < \varepsilon\), \((E \cap B(0, s)) \setminus \bigcup_{i \in \mathbb{N}} K_i\), and the equation
   \[
   J_k(MD(\varphi|_{K_i}, x)) = \lim_{r \to 0} \frac{\mathcal{H}^k(\varphi|_{K_i}(B(x, r) \cap (E \setminus (\Sigma_\varepsilon \cup Z))))}{\mathcal{H}^k(B_k(x, r))}
   \]
   holds everywhere on \((E \cap B(0, s) \cap K_i) \setminus (\Sigma_\varepsilon \cup Z), i \in \mathbb{N}\), where \(B_k(x, r)\) is a \(k\)-dimensional ball;
   
   (b) the set \(Z\) is the union of two disjoint measurable sets \(Z_1\) and \(Z_2\), where \(Z_2\) is such that
   \[
   \int_{\varphi^{-1}(z) \cap Z_2} d\mathcal{H}^{n-k}(u) = 0
   \]
   for all \(z \in \mathcal{X}\) and \(\mathcal{H}^k(\varphi(Z_1)) = 0\).

We also obtained the following criterion for mappings defined on measurable subsets of \(\mathbb{R}^n\).
Theorem 10 ([13, 17]). Let $E \subset \mathbb{R}^n$ be a measurable set and $(\mathcal{X}, d_{\mathcal{X}})$ a metric space. Let $\varphi \in \text{Lip}(E, \mathcal{X})$ be such that $\dim \ker(MD(\varphi, x)) \geq n - k$ almost everywhere. Denote by $D_E$ the set on which the functions $\Phi_m$ defined by

$$\Phi_m(x) = \frac{m^{n-k}}{2^{n-k}} \int_{\varphi^{-1}(\varphi(x)) \cap Q(x, 1/m) \cap E} d\mathcal{H}^{n-k}(u), \quad x \in E,$$

converge to unity as $m \to \infty$. Then the coarea formula holds for every measurable set $A \subset E \setminus Z$ if and only if $\mathcal{H}^n(E \setminus D_E) = 0$ for every $E \subset E$.

Note that Theorem 5 implies that if the image $\varphi(E)$ is $\mathcal{H}^k$-σ-finite, then we have $\dim \ker(MD(\varphi, x)) \geq n - k$ for $\mathcal{H}^n$-almost all $x$. Consequently, the class of mappings under our consideration includes that investigated in [26].

As a consequence, we obtain a generalization of a result of [26].

Theorem 11 (The Coarea Formula for Mappings with $\mathcal{H}^k$-σ-Finite Image [13, 17]). Assume that $(\mathcal{Y}, d_{\mathcal{Y}})$ is an $\mathcal{H}^n$-rectifiable metric space, $\mathcal{X}$ is an arbitrary metric space, and $\varphi \in \text{Lip}(\mathcal{Y}, \mathcal{X})$. Suppose also that $\dim \ker(MD(\varphi \circ \beta_m, y)) \leq n - k$ almost everywhere and $\varphi(\mathcal{Y})$ is $\mathcal{H}^k$-σ-finite. Then the formula

$$\int_{\mathcal{Y}} f(x) J_k(\varphi, x) d\mathcal{H}^n(x) = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z)} f(u) d\mathcal{H}^{n-k}(u)$$

holds for every function $f : \mathcal{Y} \to \mathcal{E}$ such that $f(x) J_k(\varphi, x)$ is integrable. Here, $\mathcal{E}$ is an arbitrary Banach space.

As mentioned earlier, our methods have much more applications than those developed earlier. E.g., they were useful in the proof of a Stepanov Type Theorem [12, 15], i.e., the $m$-differentiability of wider classes of mappings in comparison with [18].

Theorem 12 (Stepanov-Type Theorem [12, 15]). Suppose that $E \subset \mathbb{R}^n$ is a measurable set and $(\mathcal{X}, d_{\mathcal{X}})$ is a metric space. Assume that $\varphi : E \to \mathcal{X}$ is such that

$$\lim_{y \to x, y \in E} \frac{d_{\mathcal{X}}(\varphi(x), \varphi(y))}{|x - y|} < \infty$$

for almost all $x \in E$. Then $\varphi$ is metrically differentiable almost everywhere.

We also introduce the notion of approximate metric differentiability and prove the corresponding theorem, i.e., we establish the approximate metric differentiability of mappings for which

$$\lim_{y \to x} \frac{d_{\mathcal{X}}(\varphi(x), \varphi(y))}{|x - y|} < \infty$$
for almost all \( x \in E \) (see [15]). Using these results we proved the area formula
and the change-of-variable formula for large classes of mappings defined on a
measurable set \( E \subset \mathbb{R}^n \) with values in a metric space.

**Definition 13** ([27]). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \((X, d_X)\) a
metric space. A mapping \( \varphi : \Omega \rightarrow X \) belongs to the Sobolev class \( W^{1}_{p,\text{loc}}(\Omega, X) \) if
1) for each \( z \in X \) the function \( \varphi_z : \Omega \ni x \mapsto d_X(\varphi(x), z) \) belongs to the
class \( W^{1}_{p,\text{loc}}(\Omega, \mathbb{R}) \);
2) there exists a function \( w \) of a class \( L^{p,\text{loc}}(\Omega) \) independent of \( z \in X \),
such that \( |\nabla \varphi_z(x)| \leq w(x) \) for almost all \( x \in \Omega \).

This definition is similar to that of metric-valued BV-mappings (see [2]).

**Definition 14** ([29]). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \((X, d_X)\) a
metric space. A continuous mapping \( \varphi : \Omega \rightarrow X \) is called
quasimonotone if there
exists a constant \( K \geq 1 \) such that \( \text{diam}(\varphi(B(x, r))) \leq K \text{diam}(\varphi(S(x, r))) \) for
\( B(x, r) \subset \Omega \).

It is well-known [29] that continuous quasimonotone Sobolev mappings
of class \( W^{1}_{q,\text{loc}}(\Omega, X) \) and continuous Sobolev mappings of class \( W^{1}_{q,\text{loc}}(\Omega, X) \),
\( q > n \), possess both property (9) and Luzin property \( N \). Thus, the area
and change-of-variable formulas for these mappings are of the same type as for
Lipschitz ones mappings. Sobolev mappings of class \( W^{1}_{q}(\Omega; X) \), \( q \geq 1 \) (in the
case of a separable metric space \( X \)) and BV-mappings with values in a metric
space possess property (10) (see [4], [31]), and thus the equation
\[
\int_{A} u(\varphi(x)) J(MD_{\text{ap}}(\varphi, x)) \, dx = \int_{X} u(y) N(\varphi, y, A \setminus \Sigma_{\varphi}) \, d\mathcal{H}^n(y),
\]
holds for them (see [12, 15]). Here, \( \Sigma_{\varphi} \subset E \), \( |\Sigma_{\varphi}| = 0 \) and \( MD_{\text{ap}}(\varphi, x) \) is the
approximate differential of \( \varphi \) at \( x \):

**Definition 15** ([12, 15]). Suppose that \( E \subset \mathbb{R}^n \) is a measurable set, \((X, d_X)\)
a metric space, and \( \varphi : E \rightarrow (X, d_X) \). The mapping \( \varphi \) is approximately metrically
differentiable, or approximately \( m \)-differentiable at a point \( x \in E \) if there
exists a seminorm \( L_x \) on \( \mathbb{R}^n \) such that
\[
\text{ap lim}_{y \to x} \frac{L_x(x - y) - d_X(\varphi(x), \varphi(y))}{|x - y|} = 0,
\]
i.e., the set
\[
A_{\varepsilon} = \{ y \in E : \left| \frac{L_x(x - y) - d_X(\varphi(x), \varphi(y))}{|x - y|} \right| < \varepsilon \}
\]
has density 1 at \( x \) for every \( \varepsilon > 0 \). The seminorm \( L_x \) is called the approximate
metric differential, or the approximate \( m \)-differential of the mapping \( \varphi \) at the
point \( x \) and is denoted by \( MD_{\text{ap}}(\varphi, x) \).
All results on necessary and sufficient conditions have been extended to mappings defined on an $\mathcal{H}^n$-rectifiable metric space with the following property: their domain of definition $\mathcal{Y}$ is representable up to a set of measure zero as a union $\bigcup_{i \in \mathbb{N}} A_i$ of countably many measurable subsets such that on each $A_i$, $i \in \mathbb{N}$, the mapping is Lipschitz. E.g., metric-valued Sobolev and BV-mappings have this property.

**Theorem 16 ([13, 17]).** Let $\mathcal{Y}$ be an $\mathcal{H}^n$-rectifiable metric space and $(\mathcal{X}, d_\mathcal{X})$ an arbitrary metric space. Then, for every mapping $\varphi \in \Phi(\mathcal{Y}, \mathcal{X})$, statements 1–4 and 5(b) of Theorem 8, and Theorem 11 hold, but the coarea formula is

$$\int_{\mathcal{Y}} J_k(\varphi, x) \, d\mathcal{H}^n(x) = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \setminus \Sigma_\varphi} d\mathcal{H}^{n-k}(u).$$

In the case where $\mathcal{Y} = E \subset \mathbb{R}^n$ is a measurable set, all five statements of Theorem 9 hold, but the coarea formula is

$$\int_{\mathcal{A}} J_k(MD_{ap}(\varphi, x)) \, d\mathcal{H}^n(x) = \int_{\mathcal{X}} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z) \setminus \Sigma_\varphi} d\mathcal{H}^{n-k}(u).$$

**REFERENCES**


Received 27 February 2009

Siberian Branch of Russian Academy of Sciences
Sobolev Institute of Mathematics
maryka84@gmail.com