REMOVABILITY, SINGULAR INTEGRALS AND RECTIFIABILITY

PERTTI MATTILA

This is a survey on the interplay between the structure of removable singularities of bounded analytic functions and of Lipschitz harmonic functions, behaviour of Cauchy and Riesz singular integrals on $m$-dimensional subsets of $\mathbb{R}^n$, and rectifiability properties of sets.

AMS 2000 Subject Classification: Primary 28A75, 42B20.

Key words: removable set, singular integral, rectifiable set.

1. INTRODUCTION

In this brief survey I discuss the interplay between the structure of removable singularities of bounded analytic functions and of Lipschitz harmonic functions, behaviour of singular integrals on $m$-dimensional subsets of $\mathbb{R}^n$, and rectifiability. I begin with 1-dimensional sets and kernels. In this case the results are pretty complete for the Cauchy kernel and the removability questions due to a special tool, Menger curvature. For much more of recent developments and further references on this topic, see the lecture notes of Pajot [20] and the survey of Tolsa [23]. Then I shall discuss the much less complete results and open problems for higher dimensional sets and Riesz kernels. I shall mostly consider $m$-dimensional AD- (Ahlfor-David-) regular closed subsets of $\mathbb{R}^n$. This means that there is a constant $C$ such that

$$r^m/C \leq H^m(E \cap B(x, r)) \leq Cr^m \quad \text{for all } x \in E, \ 0 < r < d(E).$$

Here $H^m$ is the $m$-dimensional Hausdorff measure, $B(x, r)$ is the closed ball with centre $x$ and radius $r$, and $d(E)$ is the diameter of $E$. Finally, I shall discuss relations between boundedness and convergence properties of singular integrals for general kernels with a hope that further developments on these would lead to progress on the main open problems.
2. THE ONE-DIMENSIONAL CASE

I start with the following result from [14] for 1-dimensional sets. It characterizes geometrically the 1-dimensional AD-regular sets on which the singular integral operator related to the 1-dimensional Riesz kernel \( x/|x|^2 \) is bounded in \( L^2(E) \). Note that in the complex plane this kernel is essentially the Cauchy kernel \( 1/|z|^2 \).

**Theorem 2.1.** Let \( E \subset \mathbb{R}^n \) be a closed 1-dimensional AD-regular subset of \( \mathbb{R}^n \). The following three conditions are equivalent:

1. \[ \int_E | \int_{B(x,r)} \frac{x-y}{|x-y|^2} g(y) dH^1 y |^2 dH^1 x \leq C \int_E |g|^2 dH^1 \text{ for all } g \in L^2(E) \text{ and all } r > 0. \]
2. \[ \int_{B \cap E} \int_{B \cap E} \int_{B \cap E} c(x,y,z)^2 dH^1 x dH^1 y dH^1 z \leq C d(B) \text{ for all balls } B \subset \mathbb{R}^n \text{ and all } r > 0. \]
3. \( E \subset \Gamma \) where \( \Gamma \) is a curve with \( H^1(\Gamma \cap B(x,r)) \leq Cr \) for all \( x \in \mathbb{R}^n \) and all \( r > 0 \).

The key for the proof was the identity

\[
(2.1) \quad c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})(z_{\sigma(2)} - z_{\sigma(3)})}, \quad z_1, z_2, z_3 \in \mathbb{C},
\]

found by Melnikov in [18]. Here \( \sigma \) runs through all six permutations of 1, 2 and 3, and \( c(z_1, z_2, z_3) \) is the reciprocal of the radius of the circle passing through \( z_1, z_2 \) and \( z_3 \). It is called the Menger curvature of this triple. It vanishes exactly when the three points lie on the same line. In general, it measures how far they are from being collinear. Melnikov and Verdera used this identity to give a new proof for the boundedness of the Cauchy singular integral operator on Lipschitz graphs in [17]. Integrating the above identity with respect to all three variables and using Fubini’s theorem, one can relate the conditions (1) and (2) of Theorem 2.1.

Identity (2.1) connects the sum over permutations, which is a kind of symmetrization over the three variables, to a nice geometric object. But already the fact that this sum is non-negative is unexpected and useful. The proof of the identity is an exercise.

Based on earlier work of many people the above theorem gives the result below.

**Corollary 2.2.** Let \( E \) be a compact 1-dimensional AD-regular subset of the complex plane. The following three conditions are equivalent:

1. \( E \) is removable for bounded analytic functions;
2. \( E \) is removable for Lipschitz harmonic functions;
3. \( E \) is purely unrectifiable.
Here, the pure unrectifiability of $E$ means that $E$ meets every rectifiable curve in zero length. The removability of $E$ for bounded analytic functions means that if $E$ is contained in an open set $U$, any bounded analytic function in $U \setminus E$ can be extended analytically to $U$. The removability for Lipschitz harmonic functions is analogous but, since Lipschitz functions can always be extended as Lipschitz functions, (2) means that any Lipschitz function in $U$ which is harmonic in $U \setminus E$ is harmonic in $U$.

David [3] showed that instead of AD-regularity it is enough to assume that $E$ has finite $H^1$ measure. Still later Tolsa [22] gave a characterization of removability for all compact subsets of the complex plane in terms of Menger curvature. A consequence of this is that (1) and (2) in the above corollary are equivalent for any compact set $E$. An amusing fact is that nobody knows how to prove this without going through the Menger curvature characterization. Tolsa’s result is

**Theorem 2.3.** Let $E$ be a compact subset of the complex plane. The following three conditions are equivalent.

1. $E$ is not removable for bounded analytic functions.
2. $E$ is not removable for Lipschitz harmonic functions.
3. There is a finite Borel measure $\mu$ supported in $E$ such that $\mu(E) > 0$, $\mu(B) \leq d(B)$ for all discs $B$ and $\int \int \int c(x, y, z)^2 d\mu x d\mu y d\mu z < \infty$.

### 3. THE HIGHER DIMENSIONAL CASE

The higher dimensional analogues of the above results are unknown. Let $E$ be an $m$-dimensional AD-regular compact subset of $\mathbb{R}^n$. The natural questions are: is it true that

(a) $\int_E \int_{E \setminus B(x, r)} \left| \frac{x - y}{|x - y|^{m+1}} g(y) dH^m y \right|^2 dH^m x \leq C \int_E |g|^2 dH^m$ for all $g \in L^2(E)$ and all $r > 0$ if and only if $E$ is uniformly rectifiable,

(b) when $m = n - 1$, is $E$ removable for Lipschitz harmonic functions if and only if it is purely unrectifiable?

The reason that the Riesz kernel $|x|^{-n} x$ appears in connection with removable sets of Lipschitz harmonic functions is that it is essentially the gradient of the fundamental solution of the Laplacian.

The $m$-dimensional pure unrectifiability can be defined, for example, as the property that the set intersects every $m$-dimensional $C^1$ surface in a set of zero $m$-dimensional measure. The uniform rectifiability is a quantitative concept of rectifiability due to David and Semmes [5]. For 1-dimensional sets
it means exactly the condition (3) of Theorem 2.1. It is known that the ‘if’-part in (a) and the ‘only if’-part in (b) are true. Some partial results for the converse can be found in [15], [13] and [11]; they are discussed also in the book [12]. The main problem for the converse is to prove that boundedness such as in (a) implies some sort of rectifiability. One characterization of the rectifiability of $E$ is approximation of $E$ with $m$-dimensional planes almost everywhere at all small scales. The partial results referred to above are in the spirit that the boundedness implies such approximation almost everywhere at some, but maybe not all, small scales.

By the so-called Cotlar’s inequality, see, e.g., [12], the first part of (a) can be stated in terms of the maximal operator

$$
T_E^* g(x) = \sup_{r>0} \left| \int_{E \setminus B(x,r)} |x-y|^{m+1} (x-y)g(y)dH^m(y) \right|.
$$

Then the $L^2$-boundedness of (a) is equivalent to

$$
\int_{E} (T_E^* g)^2 dH^m \leq C \int_{E} |g|^2 dH^m
$$

for all $g \in L^2(E)$.

All attempts to modify the Menger curvature method to higher dimensional sets have failed. In fact, Farag [6] has proved that for the Riesz kernel $x/|x|^{m+1}$ with $m > 1$ any reasonable symmetrization, such as the sum in (2.1), takes both positive and negative values and thus seems to be useless.

One attempt to attack the higher dimensional case would be to try to use the following result. It was first proved for AD-regular (and a little more general) sets in [16] (or see [12]). The general case was proved by Tolsa [24].

**Theorem 3.1.** Let $E$ be an $H^m$-measurable subset of $\mathbb{R}^n$ with $H^m(E) < \infty$. Then for $H^m$ almost all $x \in E$ the limit

$$
\lim_{r \to 0} \int_{E \setminus B(x,r)} \frac{x-y}{|x-y|^{m+1}} dH^m y
$$

exists if and only if $E$ is $m$-rectifiable, that is, there exist $m$-dimensional $C^1$ surfaces $S_1, S_2, \ldots$ such that $H^m(E \setminus \bigcup_i S_i) = 0$.

The main problem in that attempt would be to prove that the $L^2$-boundedness of the maximal operator $T^*$ implies the almost everywhere convergence of principal values as in Theorem 3.1. Although it is well-known that with very general Calderón-Zygmund kernels boundedness of singular integrals implies almost everywhere convergence of principal values when the basic measure is the Lebesgue measure, the question is much more problematic with general measures. For the Lebesgue measure it is easy to check the
convergence for smooth functions using the cancellation properties of the kernel, and then the density of smooth functions in $L^2$ and the $L^2$-boundedness give the almost everywhere convergence for integrable functions. For general measures the convergence cannot be easily checked even for any single non-zero function. In the last section I discuss some connections between such boundedness and convergence properties.

Tolsa [25] obtained sufficient conditions for the uniform rectifiability of AD-regular sets in terms of boundedness properties of Riesz kernel operators, however a little different from the $L^2$-boundedness. Lerman and Whitehouse [9, 10] introduced a higher dimensional analogue of Menger curvature and used it to characterize uniform rectifiability. Unfortunately, there does not seem to be any way to relate this curvature to the Riesz kernels.

4. BOUNDEDNESS AND CONVERGENCE OF SINGULAR INTEGRALS

Let $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ be continuous and odd: $K(-x) = -K(x)$. Let $\mu$ be a finite Borel measure in $\mathbb{R}^n$ with compact support. For $\epsilon > 0$, $f \in L^2(\mu)$ and $x \in \mathbb{R}^n$ set

$$T_\epsilon f(x) = \int_{\mathbb{R}^n \setminus B(x, \epsilon)} K(x - y)f(y)d\mu_y$$

and

$$T^* f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|.$$

We say that the operators $T_\epsilon$ converge weakly in $L^2(\mu)$ if the limit

$$\lim_{\epsilon \to 0} \int gT_\epsilon f d\mu$$

exists and is finite for all $f, g \in L^2(\mu)$. It follows easily by the Banach-Steinhaus theorem that then the operators $T_\epsilon$ are uniformly bounded with respect to $\epsilon$ in $L^2(\mu)$, and there exists a linear bounded operator $T : L^2(\mu) \rightarrow L^2(\mu)$ such that

$$\int gT f d\mu = \lim_{\epsilon \to 0} \int gT_\epsilon f d\mu$$

for all $f, g \in L^2(\mu)$. Verdera and I [17] proved that the following converse also holds.

Theorem 4.1. Suppose that there exists a constant $C$ such that

$$\int |T^* f|^2 d\mu \leq C \int |f|^2 d\mu$$

for all $f \in L^2(\mu)$. Then the operators $T_\epsilon$ converge weakly in $L^2(\mu)$. 
Using Menger curvature, Tolsa [21] has proved for the Cauchy kernel in $\mathbb{C}$ that such $L^2$-boundedness implies the almost everywhere convergence of principal values $\lim_{r \to 0} T_\epsilon f(x)$. Although we don’t know this for essentially any other interesting kernels, we can use Theorem 4.1 to get a different average convergence from $L^2$-boundedness. We namely have

**Theorem 4.2.** Suppose that the maximal operator $T^*$ is bounded in $L^2(\mu)$ (as in Theorem 4.1). If $f \in L^2(\mu)$, then for $\mu$ almost all $z \in \mathbb{R}^n$ the limit

$$\lim_{r \to 0} \frac{1}{\mu(B(z,r))} \int_{B(z,r)} \int_{\mathbb{R}^n \setminus B(z,r)} K(x-y) f(y) d\mu y d\mu x$$

exists.

Another proof for this, without going through the weak convergence, can be given using the martingale convergence theorem.

Now, the problem is that so far we have not been able to show that this convergence would imply rectifiability when $K$ is a higher dimensional Riesz kernel.

Some $L^2$-boundedness holds for very general, even fractal type, $(n-1)$-dimensional measures with quite general kernels. But then we split the set into two parts by some Lipschitz graph, restrict one variable to one and the other to the other side. In this sense the following result was proved in [2].

**Theorem 4.3.** Let $\mu$ and $\nu$ be a finite Radon measures on $\mathbb{R}^n$ such that

$$\mu(B(x,r)) \leq Cr^{n-1} \quad \text{for } x \in \mathbb{R}^n \text{ and } r > 0$$

and

$$\nu(B(x,r)) \leq Cr^{n-1} \quad \text{for } x \in \mathbb{R}^n \text{ and } r > 0.$$ 

Let $K$ be an antisymmetric continuously differentiable kernel such that

$$|K(x)| \leq C|x|^{-(n-1)} \quad \text{and} \quad |\nabla K(x)| \leq C|x|^{-n}.$$ 

Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function and suppose that

$$\mu(\{(x,t) : t < f(x)\}) = \nu(\{(x,t) : t > f(x)\}) = 0.$$ 

Then there exist constants $C_p$, $1 \leq p < \infty$, depending only on $p$, $n$, $C_\mu$, $C_\nu$ and $\text{Lip}(f)$ such that

$$\int (T^*_\nu g)^p d\mu \leq C_p \int |g|^p d\nu \quad \text{for } 1 < p < \infty$$

for all $g \in L^1(\nu)$, and

$$\mu(\{x \in \mathbb{R}^n : T^*_\nu g(x) > t\}) \leq \frac{C_1}{t} \int |g| d\nu \quad \text{for } t > 0.$$
Using this one can prove the following general ‘restricted weak convergence’ result in a dense subspace of $L^2(\mu)$; however, it cannot hold in all of $L^2(\mu)$:

**Theorem 4.6.** If $\mu$ and $K$ are as above, the finite limit

$$\lim_{\epsilon \to 0} \int T_\epsilon(f)(x)g(x) \, d\mu x$$

exists whenever $f$ and $g$ are linear combinations of characteristic functions of balls.

Some counter-examples to the ‘rule’ that $L^2$-boundedness should imply almost everywhere convergence of principal values, have also been constructed. For $m$-dimensional sets it is natural to consider kernels $K$ which are smooth outside the origin such that $|K(x)|$ behaves like $|x|^{-m}$ as $x \to 0$. Let us call such kernels $K$ $m$-dimensional Calderón-Zygmund kernels without specifying the conditions precisely.

A standard example of a purely unrectifiable 1-dimensional AD-regular set in the plane is the Cantor set which is obtained by starting with the unit square, taking four squares of side-length $1/4$ inside it in its corners, then taking squares of side-length $1/16$ in the corners of these, and so on. The final Cantor set $C$ is the compact set inside all these squares of all generations. Before the Menger curvature method many proofs were given to show that the Cauchy singular integral operator is not $L^2$-bounded on $C$ and that $C$ is removable for bounded analytic functions. David [4] constructed a 1-dimensional Calderón-Zygmund kernel $K$ such that the truncated operators $T_\epsilon$ related to $K$ are uniformly bounded in $L^2(C)$. However, it is easy to check that in this example the principal values

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus B(x,\epsilon)} K(x-y) \, dH^1 y$$

fail to exist at $H^1$ almost all points $x \in C$. Chousionis [1] had similar results for a larger class of kernels for some $s$-dimensional fractals with $s < 1$.

Huovinen [8] considered the kernel

$$K(z) = \text{Re}(z/|z|^2 - z^3/|z|^4)$$

for $z \in \mathbb{C}$ and closely related kernels. He showed that there exist purely unrectifiable 1-dimensional AD-regular sets on which the related operators are bounded in $L^2$ and the principal values exist almost everywhere. On the other hand, in [7] he showed that for the kernels $z^{2k-1}/|z|^{2k}$, $k = 1, 2, \ldots$, and their linear combinations the almost everywhere convergence of principal values on 1-dimensional AD-regular sets implies their rectifiability.
Acknowledgements. This survey is based on the talk I gave at the XIth Romanian-Finnish Seminar on Complex Analysis and related topics in Alba Iulia, August 2008. I would like to thank the organizers, in particular Professor Cabiria Andreian Cazacu, for the invitation. This was supported by the Academies of Finland and Romania.

REFERENCES


Received 5 February 2009

University of Helsinki
Department of Mathematics and Statistics
FI-00014 Helsinki, Finland
pertti.mattila@helsinki.fi