DANIELL AND RIEMANN INTEGRABILITY

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We introduce the notion of Riemann integrable function with respect to a Daniell integral and prove the approximation theorem of such functions by a monotone sequence of Jordan simple functions. This is a generalization of the famous Lebesgue criterion of Riemann integrability.

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1. PRELIMINARIES

We consider a linear vector space C of real bounded functions over an arbitrary set X, $X \neq \phi$, such that the constant real functions on X belong to C and for any $f, g \in C$ the function $f \vee g : X \to \mathbb{R}$ defined as $(f \vee g)(x) = \max\{f(x), g(x)\}$ also belongs to C.

Obviously, for any $f, g \in C$ the function $f \wedge g$ (respectively $|f|$) defined as

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \quad (\text{resp. } |f|(x) = |f(x)|)$$

belongs to C and we have $f \wedge g + f \vee g = f + g$, $|f| = (-f) \vee f$.

Definition 1. A real linear map $I : C \to \mathbb{R}$ is called a *Daniell integral* (or a *Cauchy-Daniell integral*) if it is increasing and monotone sequentially continuous, i.e.,

(a) $I\,(\alpha f + \beta g) = \alpha I(f) + \beta I(g)\ \forall \alpha, \beta \in \mathbb{R},\ \forall f, g \in C$;

(b) $f, g \in C,\ f \leq g \Rightarrow I(f) \leq I(g)$;

(c) for any decreasing sequence $(f_n)_n$ of C such that $\inf\{f_n(x) \mid n \in \mathbb{N}\} = 0$ for any $x \in X$ we have $\inf\limits_n I(f_n) = 0$.

As is well known, there exists a real positive measure μ on the σ-algebra $\mathcal{B}(C)$, generated by C (the coarsest σ-algebra of sets on X for with respect to which any function $f \in C$ is a measurable real function on X), such that we have

$$I(f) = \int f \mathrm{d}\mu \quad \forall f \in C.$$

There are several steps of extension of the given functional I such that, finally, any characteristic function 1_{A} with $A \in \mathcal{B}(C)$, belongs to the domain of the extension of I.

We recall briefly this procedure. First, we denote by C_i (respectively, C_s), the set of all functions $\varphi : X \to (-\infty, \infty]$ (respectively, $\Psi : X \to [-\infty, \infty)$), for which there exists an increasing sequence $(f_n)_n$ (respectively decreasing sequence $(g_n)_n$) in C, such that $\varphi = \sup\limits_n f_n =: \bigvee\limits_n f_n$ ($\Psi = \inf\limits_n g_n =: \bigwedge\limits_n g_n$).

Using the property of a Daniell integral one can show that the element $I(\varphi)$ (respectively, $I(\Psi)$ of $(-\infty, \infty]$ (respectively, $[-\infty, \infty)$) defined as $I(\varphi) = \sup\limits_n I(f_n)$ ($I(\Psi) = \inf\limits_n I(g_n)$) does not depend on the sequence $(f_n)_n$ (respectively, $(g_n)_n$) which increases to φ (respectively, decreases to Ψ).

We notice the following facts:

(1) C_i and C_s are convex cones, i.e., for any f, g in C_i (respectively, C_s) and for any $\alpha, \beta \in \mathbb{R}, \alpha > 0$, $\beta > 0$, we have $\alpha f + \beta g \in C_i$ (respectively, C_s).

(2) for any f, g in C_i (respectively, C_s) the functions $f \vee g$, $f \wedge g$ belong to C_i (respectively, C_s). Moreover, for any increasing sequence of C_i, its pointwise supremum belongs to C_i while the pointwise infimum of a decreasing sequence of C_s belongs to C_s. In fact, we have $C_s = -C_i$, i.e., $C_s = \{-\varphi \mid \varphi \in C_i\}$ or $C_i = \{-\Psi \mid \Psi \in C_s\}$.

(3) $I\left(\alpha f + \beta g\right) = \alpha I(f) + \beta I(g) \forall \alpha, \beta \in \mathbb{R}_+^*$ and $f, g \ \in C_i$ (respectively, $f, g \ \in C_s$).

(4) $I(f) \leq I(g)$ if $f \leq g$ and $f, g \in C_i$ (respectively, $f, g \in C_s$).

(5) $\sup\limits_n I(f_n) = I\big(\sup\limits_n f_n\big)$ for any increasing sequence $(f_n)_n$ of C_i and $\inf\limits_n I(f_n) = I\big(\inf\limits_n f_n\big)$ for any decreasing sequence $(f_n)_n$ of C_s.

(6) $I\left(-f\right) = -I(f) \ \forall f \in C_i$ or $f \in C_s$.

For any function $h : X \to \overline{\mathbb{R}}$, we denote by $I^*(h)$ (respectively, $I_*(h)$) the element of $\overline{\mathbb{R}}$ defined as $I^*(h) = \inf\left\{I(f) \mid f \in C_i,\ f \geq h\right\}$ (respectively, $I_*(h) = \sup\left\{I(g) \mid g \in C_s,\ g \leq h\right\}$). The assertion $I_*(h) \leq I^*(h)$ does always hold.

These new extensions of I have the properties below:

(7) I^* and I_* are increasing, i.e., $I^*(f) \leq I^*(g)$, respectively, $I_*(f) \leq I_*(g)$ whenever $f \leq g$;

(8) $I^*\left(h + g\right) \leq I^*(h) + I^*(g)$, $I_*\left(h + g\right) \geq I_*(h) + I_*(g)$ whenever the algebraic operations make sense;

(9) $\sup\limits_n I^*(f_n) = I^*\big(\sup\limits_n f_n\big)$ for any increasing sequence $(f_n)_n$ for which $I^*\left(f_1\right) > -\infty$;

(10) $\inf\limits_n I_*(f_n) = I_*\big(\inf\limits_n f_n\big)$ for any decreasing sequence $(f_n)_n$ for which $I_*\left(f_1\right) < \infty$.

It can be shown that the set $L_1(I)$ defined as

$$L_1(I) = \{f : X \to \mathbb{R} \mid I_*(f) = I^*(f) \neq \pm\infty\}$$

is a real linear vector space of functions on X, such that

(11) $C \subset L_1(I)$, $f \vee g$, $f \wedge g$ belong to $L_1(I)$ if $f, g \in L_1(I)$;

(12) for any sequence $(f_n)_n$ of $L_1(I)$, dominated in $L_1(I)$, i.e., $|f_n| \leq g$ for all $n \in \mathbb{N}$ for some function $g \in L_1(I)$, we have $\sup\limits_n f_n \in L_1(I)$, $\inf\limits_n f_n \in L_1(I)$.

Moreover, if the above sequence is pointwise convergent to a function f, then we have $f \in L_1(I)$ and $I^*(f) = \lim\limits_{n\to\infty} I^*(f_n)$, $\lim\limits_{n\to\infty} I^*(|f - f_n|) = 0$.

If we denote by $\mathcal{M}$ the set of all subsets A of X such that the characteristic function 1_{A} of A belongs to $L_1(I)$, then $\mathcal{M}$ is a σ-algebra on X, the map $\mu : \mathcal{M} \to \mathbb{R}_+$ defined as $\mu(A) = I^*(1_{\mathrm{A}}) = I_*(1_{\mathrm{A}})$ is a measure on $\mathcal{M}$, any element $f \in L_1(I)$ is $\mathcal{M}$-measurable and μ-integrable. Moreover, we have $I^*(f) = \int f \mathrm{d}\mu$ and, in particular, the above equality holds for any $f \in C$.

The elements of $\mathcal{M}$ are generally called *Lebesgue measurable* (w.r. to I) (or *Daniell-measurable* w.r. to I).

2. RIEMANN INTEGRABILITY AND RIEMANN MEASURABILITY

In the sequel we develop a theory close to the Riemann theory of integration on the real line with respect to the Lebesgue measure on $\mathbb{R}$. The starting point is a pointwise vector lattice of real functions C on a set X which contains the real constant functions, and a Daniell integral $I : C \to \mathbb{R}$.

Definition 2.1. A real bounded function $h : X \to \mathbb{R}$ is called *Riemann integrable with respect to I* or, simply, Riemann integrable if we have

$$\sup\{I(f') \mid f' \in C,\ f' \leq f\} = \inf\{I(f'') \mid f'' \in C,\ f \leq f''\}.$$

We shall denote by $R_1(I)$ the set of all Riemann integrable functions $f : X \to \mathbb{R}$. For any $f \in R_1(I)$, denote

$$I(f) = \sup\{I(f') \mid f' \in C,\ f' \leq f\} = \inf\{I(f'') \mid f'' \in C,\ f \leq f''\}.$$

Proposition 2.2 (Darboux criterion). *A function $f : X \to \mathbb{R}$ is Riemann integrable iff for any $\varepsilon > 0$ there exist f', f'' in C (or $f', f'' \in R_1(I)$) such that $f' \leq f \leq f''$ and $I(f'') - I(f') < \varepsilon$.*

Proof. The assertion follows directly from the above definition. □

Proposition 2.3. a) *The set $R_1(I)$ is a pointwise vector lattice such that $C \subset R_1(I) \subset L_1(I)$ and for any sequence $(h_n)_n$ from $R_1(I)$ which decreases to zero, i.e., $\bigwedge\limits_n h_n = 0$, the sequence $(I(h_n))_n$ decreases to zero.*

b) *If* $(h_n)_n$ *is a sequence from* $R_1(I)$*, uniformly convergent to a function* h*, we have* $h \in R_1(I)$ *and* $\lim\limits_{n\to\infty} I(h_n) = I(h)$*,* $\lim\limits_{n\to\infty} I(|h_n - h|) = 0$.

Proof. a) The inclusions $C \subset R_1(I) \subset L_1(I)$ follow from the definitions, on account of the fact that $C \subset C_i$, $C \subset C_s$.

Let h be an element of $R_1(I)$ and for any $\varepsilon > 0$ let $f', f'' \in C$ be such that $f' \le h \le f''$ and $I(f'' - f') < \varepsilon$. We consequently have $f' \vee 0 \le h \vee 0 \le f'' \vee 0$, $(f'' \vee 0) - (f' \vee 0) \le f'' - f'$, $I(f'' \vee 0) - I(f' \vee 0) \le I(f'') - I(f') < \varepsilon$. Since the functions $f' \vee 0$ and $f'' \vee 0$ belong to C, we deduce that the function $h^+ = h \vee 0$ belongs to $R_1(I)$. Hence $R_1(I)$ is a pointwise lattice.

If the sequence $(h_n)_n$ from $R_1(I)$ decreases to zero, by the above property (12) of the Daniell integral, we have $\lim\limits_{n\to\infty} I(h_n) = \lim\limits_{n\to\infty} I^*(h_n) = 0$.

Nevertheless, we can directly prove this property. For any $\varepsilon > 0$ and for any $n \in \mathbb{N}$ choose $f'_n, f''_n \in C$ such that $f'_n \le h_n \le f''_n$ and $I(f''_n - f'_n) < \varepsilon/2^n$. Consequently, for any natural integer k we have

$$\mathop{\wedge}_{n=1}^{k} f'_n \le h_k \le \mathop{\wedge}_{n=1}^{k} f''_n, \quad \mathop{\wedge}_{n=1}^{k} f''_n - \mathop{\wedge}_{n=1}^{k} f'_n \le \sum_{n=1}^{k} \left(f''_n - f'_n\right),$$

$$I\left(\mathop{\wedge}_{n=1}^{k} f''_n - \mathop{\wedge}_{n=1}^{k} f'_n\right) \le \sum_{n=1}^{k} I\left(f''_n - f'_n\right) \le \varepsilon.$$

Since the sequence $\left(\mathop{\wedge}\limits_{n=1}^{k} f'_n\right)_k$ from C decreases to zero $\left(\inf\limits_k h_k = 0\right)$, we have $\lim\limits_{k\to\infty} I\left(\mathop{\wedge}\limits_{n=1}^{k} f'_n\right) = 0$.

On the other hand, the sequence $\left(\mathop{\wedge}\limits_{n=1}^{k} f''_n\right)_k$ from C decreases and $h_k \le \sum\limits_{n=1}^{k} f''_n$ for any $k \in \mathbb{N}$. Therefore,

$$I(h_k) \le I\left(\mathop{\wedge}_{n=1}^{k} f''_n\right) \le \varepsilon + I\left(\mathop{\wedge}_{n=1}^{k} f'_n\right), \quad \lim_{k\to\infty} I(h_k) \le \varepsilon.$$

The number $\varepsilon > 0$ being arbitrary, we get $\lim\limits_{k\to\infty} I(h_k) = 0$.

b) Let $\varepsilon > 0$ be arbitrary and let $n_0 \in \mathbb{N}$ be such that $|h_n - h| < \varepsilon$ for $n \ge n_0$. We have $h_{n_0} \pm \varepsilon \in \mathbb{R}_1$ and

$$h_{n_0} - \varepsilon \le h \le h_{n_0} + \varepsilon, \quad I(h_{n_0} + \varepsilon) - I(h_{n_0} - \varepsilon) = 2\varepsilon I(1_X),$$

i.e., $h \in \mathbb{R}_1(I)$. Moreover, from the above considerations we deduce the relations

$$I(|h_n - h|) \le \varepsilon I(1_X), \quad |I(h_n) - I(h)| \le I(|h_n - h|) \le \varepsilon, \quad \forall n \ge n_0,$$

i.e., $\lim_{n\to\infty} I\left(|h_n - h|\right) = 0$, $\lim_{n\to\infty} I\left(h_n\right) = I(h)$. $\square$

Let us denote by R the set of subsets of X defined as $R = \{A \mid A \subset X,\ 1_A \in R_1(I)\}$.

We refer to an element of R as *Jordan* measurable with respect to I. Obviously, we have $R \subset \mathcal{M}$.

Proposition 2.4. *The set R is an algebra of subsets of X.*

Proof. The assertion follows from the fact that $R_1(I)$ is a pointwise vector lattice using the equations $1_{A\cup B} = 1_A \vee 1_B, 1_{X\setminus A} = 1 - 1_A$, $\forall A, B \subset X$. $\square$

Theorem 2.5. *If $f : X \to \mathbb{R}$ is a bounded function, then the following assertions are equivalent:*

1) *f is Riemann integrable w.r. to I.*

2) *There exists an increasing sequence of R-step functions which converges uniformly to f.*

3) *There exists a decreasing sequence of R-step functions which converges uniformly to f.*

4) *For any $\varepsilon > 0$ there exists an R-partition*

$$\Delta_\varepsilon = \left(A_1, A_2, \ldots, A_n\right),\ A_i \in R,\ \bigcup_{i=1}^{n} A_i = X,\ A_i \cap A_j = \phi \ \text{ if } \ i \neq j$$

such that $S\left(f, \Delta_\varepsilon\right) - s\left(f, \Delta_\varepsilon\right) < \varepsilon$, where $S\left(f, \Delta_\varepsilon\right)$, respectively $s\left(f, \Delta_\varepsilon\right)$, is the upper, respectively the lower, Darboux sum associated with f and Δ_ε.

Proof. The implications 2) $\Rightarrow$ 4), 3) $\Rightarrow$ 4) are obvious.

4) $\Rightarrow$ 1). For any natural integer n, $n \neq 0$, we consider a partition $\Delta_n = (A_1, A_2, \ldots, A_{k_n})$ of X, with $A_i \in R$ for any $i \leq k_n$, such that

$$\sum_{i=1}^{k_n} M_i \mu\left(A_i\right) - \sum_{i=1}^{k_n} m_i \mu\left(A_i\right) < \frac{1}{n}, \quad \sup_{x\in A_i} f(x) = M_i, \quad \inf_{x\in A_i} f(x) = m_i.$$

Since $1_{A_{\mathrm{i}}} \in R_1$, we deduce that the functions φ, Ψ defined as $\varphi = \sum_i m_i 1_{A_{\mathrm{i}}}, \Psi = \sum_i M_i 1_{A_{\mathrm{i}}}$ belong to $R_1(I)$, and, moreover, we have

$$\varphi \leq f \leq \Psi, \quad I(\varphi) = \sum_i m_i \mu(\mathrm{A_i}), \quad I(\Psi) = \sum_i M_i \mu(\mathrm{A_i}), \quad I(\Psi) - I(\varphi) < \frac{1}{n}.$$

Now, using Proposition 2.2, we deduce that the function f is Riemann integrable.

1) $\Rightarrow$ 2) Without loss of generality, we may assume that $f \geq 0$. Since for any real number r the set $[f = r]$ belongs to $\mathcal{M}$, the set D of real numbers defined as $D = \{r \in \mathbb{R} \mid \mu\left([f = r]\right) > 0\}$ is at most countable. Let us now show that the set $[f > r]$ belongs to R for any $r \in R$, $r \notin D$. Indeed, we have

$1_{[f>r]} = \sup_n \varphi_n, 1_{[f \geq \mathrm{r}]} = \inf_n \Psi_n$, where, for any natural integer n, the functions φ_n, Ψ_n are defined as $\varphi_n = 1 \wedge n\,(f-r)^+$, $\Psi_n = 1 - 1 \wedge n\,(r-f)^+$. Obviously, $\varphi_n \in R_1(I)$, $\Psi_n \in R_1(I)$, the sequence $(\varphi_n)_n$ is increasing, the sequence $(\Psi_n)_n$ is decreasing and the sequence $(\Psi_n - \varphi_n)_n$ decreases to the function $1_{[f=r]}$. Since $\mu\,([f=r]) = 0$ we have

$$\lim_{n\to\infty} I\,(\Psi_n - \varphi_n) = \lim_{n\to\infty} \int (\Psi_n - \varphi_n)\,\mathrm{d}\mu = \mu\,([f=r]) = 0$$

and, therefore, by Proposition 2.2 again, any function $g : X \to \mathbb{R}$, such that $\sup_n \varphi_n \leq g \leq \inf_n \Psi_n$, belongs $R_1(I)$. In particular, the functions $1_{[f>r]}$, $1_{[f \geq \mathrm{r}]}$ belong to $R_1(I)$. Since D is at most countable, there exists a real number a such that $0 < a \leq 1$ and $r \cdot a \notin D$ for any rational number r. For any natural integer $n \neq 0$ we consider the function φ_n defind as

$$\varphi_n = \frac{a}{2^n} \sum_{k=1}^{2^n \cdot p-1} 1_{[f > \frac{k \cdot a}{2^n}]},$$

where $p \in \mathbb{N}$ is such that $f(x) \leq pa$ for any $x \in X$. Obviously φ_n is an R-step function, we have $\varphi_n \leq f \leq \varphi_n + \frac{a}{2^n}$, and the sequence $(\varphi_n)_n$ is uniformly increasing to f. Now, replacing f by $-f$, we deduce that 2) $\Leftrightarrow$ 3). $\square$

Remark 2.6. In the case where $X = [a,b] \subset \mathbb{R}$ and $C = \{f : [a,b] \to \mathbb{R} \mid f \text{ continuous}\}$, Theorem 2.5 generalizes the famous Lebesgue criterion of Riemann integrability.

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