MULTIPARAMETER MARKOV PROCESSES:  
GENERATORS AND ASSOCIATED MARTINGALES  

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We propose a notion of a generator for multiparameter Feller semigroups and summarise a calculus for these generators. Furthermore for multiparameter Feller processes a martingale is associated with the generator.  

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1. INTRODUCTION  

The subject of this article is multiparameter stochastic processes with independent increments, in literature also referred to as additive multiparameter processes. Among the vast research into two- or multiparameter processes the closest to our merely analytic approach are the papers [5] by E.B. Dynkin who solved a boundary value problem for the operator \((-1)^N \Delta_1 \cdots \Delta_N\) acting on functions defined on \(G_1 \times \cdots \times G_N, G_j \subset \mathbb{R}^{n_j}\), where \(\Delta_j\) acts on the variables of \(G_j\) only, see also [4]. Dynkin’s approach was taken up by Mazziotto [11] for elliptic operators of second order and the associated diffusions.  

For the predominantly probabilistic parts we mention [2] where the Lévy-Khinchin theorem as well as the Lévy-Itô decomposition for Lévy processes indexed by \([0, 1]^N\) are developed. For multivariate subordination, i.e., subordination (in the sense of Bochner) for multiparameter Lévy processes we refer the reader to Barndorff-Nielsen et al. [3]. Moreover we want to mention Khoshnevisan [10] as a standard reference.  

In what follows we introduce an operator which is in some sense the infinitesimal generator \(A\) of multiparameter (Feller) semigroups \((T_{(t_1, \ldots, t_N)})(t_1, \ldots, t_N)\in \mathbb{R}_+^N\) as it resembles the generator of a one-parameter semigroup in the associated (partial) differential equation. Moreover making use of this generator we introduce a martingale for the canonical processes associated with a Feller semigroup \((T_{(t_1, \ldots, t_N)})(t_1, \ldots, t_N)\in \mathbb{R}_+^N\).
It seems that our analytic point of view is new and relates partly to a special type of multiparameter processes. In Section 2 we define multiparameter convolution semigroups and, more generally, multiparameter semigroups of operators. We define their generator and offer the necessary calculus. Finally, Section 3 is devoted to a martingale for multiparameter processes, here we refer to P. Imkeller [7] as a reliable source for two-parameter martingales. For more background material on the analytic part of our theory we refer to [12] as well as our joint paper [9] with A. Potrykus.

2. MULTIPARAMETER CONVOLUTION SEMIGROUPS AND FELLER SEMIGROUPS

In this section we introduce \( N \)-parameter convolution semigroups of probability measures and multiparameter Feller semigroups.

For an arbitrarily fixed natural number \( N \in \mathbb{N} \) a family of probability measures \( (\mu_t)_{t \in \mathbb{R}^N_+} \) indexed by non-negative real \( N \)-dimensional vectors, which satisfies for all \( s, t \in \mathbb{R}^N_+ \) the conditions

1. \( \mu_t(\mathbb{R}^n) = 1 \);
2. \( \mu_s \ast \mu_t = \mu_{s+t} \);
3. \( \mu_t \rightharpoonup \mu_0 \) vaguely for \( t \to 0 \) and \( \mu_0 = \varepsilon_0 \)

is called a multiparameter convolution semigroup of probability measures.

A first example of a two-parameter convolution semigroup \( (\eta(s,t))_{(s,t) \in \mathbb{R}^2_+} \) can be constructed as the product of two one-parameter convolution semigroups \( (\mu_s)_{s \geq 0} \) and \( (\nu_t)_{t \geq 0} \), by defining

\[
\eta(s,t) := \mu_s \otimes \nu_t \quad \text{for all } (s, t) \in \mathbb{R}^2_+.
\]

Then \( (\eta(s,t))_{(s,t) \in \mathbb{R}^2_+} \) is called the product semigroup of \( (\mu_s)_{s \geq 0} \) and \( (\nu_t)_{t \geq 0} \).

Multiparameter convolution semigroups feature the following decomposition property:

**Theorem 2.1.** For an \( N \)-parameter convolution semigroup \( (\mu_t)_{t \in \mathbb{R}^N_+} \) on \( \mathbb{R}^n \) there exist continuous negative definite functions \( \psi_1, \psi_2, \ldots, \psi_N : \mathbb{R}^n \to \mathbb{C} \) such that

\[
\hat{\mu}_t(\xi) = (2\pi)^{\frac{n}{2}} e^{-t_1\psi_1(\xi)-t_2\psi_2(\xi)-\cdots-t_N\psi_N(\xi)}
\]

holds for all \( \xi \in \mathbb{R}^n \) and \( t \geq 0 \), i.e., \( t_j \geq 0 \) for \( j = 1, \ldots, N \).

Equation (1) exhibits that every \( N \)-parameter convolution semigroup can be decomposed into the convolution of \( N \) one-parameter semigroups.
The proof of Theorem 2.1 uses the continuity of the mapping $t \mapsto \mu_t$ and results about generalised Cauchy functional equations, see [1], p. 226. More details are given in [12].

Now we consider $N$-parameter semigroups of strongly continuous operators on a real or complex Banach space $(X, \| \cdot \|_X)$.

**Definition 2.2.**

A. An $N$-parameter family $(T_t)_{t \geq 0}, t \in \mathbb{R}_+^N$, of bounded linear operators $T_t : X \to X$ is called an $N$-parameter semigroup of operators, if $T_0 = \text{id}$ and for all $s, t \in \mathbb{R}_+^N$ we have

$$T_{s+t} = T_s \circ T_t. \quad (2)$$

B. We call $(T_t)_{t \geq 0}$ **strongly continuous** if for all $x \in X$ we have

$$\lim_{t \to 0} \| T_t u - u \|_X = 0.$$

C. The semigroup $(T_t)_{t \geq 0}$ is a **contraction** semigroup, if

$$\| T_t \| \leq 1$$

for all $t \geq 0$, i.e., each operator $T_t$ is a contraction. Here $\| \cdot \|$ denotes the operator norm $\| \cdot \|_{X,X}$.

D. A strongly continuous contraction $N$-parameter semigroup $(T_t)_{t \geq 0}$ on $(C_{\infty}(\mathbb{R}^n), \| \cdot \|_{\infty})$ which is positivity preserving is called an $N$-parameter Feller semigroup.

Multiparameter semigroups feature the commuting property, which turns out to be very useful when introducing the generator but also for the construction of associated stochastic processes. A family of operators $(T_t)_{t \in \mathbb{R}_+^N}$ is said to fulfill the **commuting property**, if for all $s, t \in \mathbb{R}_+^N$ we have

$$[T_t, T_s] := T_t \circ T_s - T_s \circ T_t = 0. \quad (3)$$

This is a direct consequence of (2), since $T_t \circ T_s = T_{t+s} = T_s \circ T_t$, thus the commuting property is fulfilled.

The following construction establishes the connection with convolution semigroups.

**Example 2.3.** Any arbitrary $N$-parameter convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+^N}$ gives rise to an $N$-parameter operator semigroup by

$$T_t u(x) := \int_{\mathbb{R}^n} u(x - y) \mu_t(dy), \quad (4)$$

for all $t \in \mathbb{R}_+^N$ and $u \in C_{\infty}(\mathbb{R}^n)$. Indeed $(T_t)_{t \geq 0}$ is a strongly continuous contraction semigroup on $\mathbb{R}^n$ which is positivity preserving. Hence it is a Feller semigroup.
Now, we want to introduce the generator for multiparameter Feller semigroups. For all $j = 1, \ldots, N$, let $A^{(j)}$ denote the generator of the $j$th (one-parameter) marginal semigroup $(T^{(j)}_t)_{t \geq 0}$, i.e., $T^{(j)}_t = T^{(j)}_{t_j}$ for all $t_j \in \mathbb{R}_+$, where $e_j$ is the $j$th canonical basis vector in $\mathbb{R}^N$. Then we define the operator $A$ called the \textit{infinitesimal generator} of $(T_t)_{t \in \mathbb{R}^N_+}$ by

$$A = A^{(1)} \circ \ldots \circ A^{(N)}.$$

With this newly defined generator $A$ we associate the partial differential equation

$$\frac{\partial^N}{\partial t_1 \ldots \partial t_N} u(t, x) = A u(t, x)$$

which is solved by $u(t, x) = T_t f(x)$ for all $t \in \mathbb{R}^N_+$, $x \in \mathbb{R}^n$, and $f \in S(\mathbb{R}^n)$. This is a generalization of the differential equation associated to the generator in the one-parameter case, see [8], and is one strong motivation for our notion of $A$.

For the generator we now develop a calculus, which is a powerful tool when handling multiparameter operator semigroups and associated stochastic processes. This calculus much resembles the calculus available for generators of one-parameter semigroups, see [6] by Ethier and Kurtz or [8], and is the second motivation for the introduction of the operator $A$ as the generator of multiparameter semigroups.

\textbf{Theorem 2.4} (Calculus for the generator). Let $A$ be the generator of an $N$-parameter Feller semigroup $(T_{(t_1, \ldots, t_N)})_{(t_1, \ldots, t_N) \in \mathbb{R}^N_+}$.

A. If $u \in D(A)$ then $T_t u \in D(A)$, i.e., $D(A)$ is invariant under $T_t$ for all $t \in \mathbb{R}^N_+$.

B. Every marginal semigroup commutes with its own generator and the generator of every other marginal semigroup, moreover the generators of the marginal semigroup commute mutually, i.e., for all $i, j \in \{1, \ldots, N\}$ we have

$$[T^{(i)}, A^{(j)}] = 0 \quad \text{and} \quad [A^{(i)}, A^{(j)}] = 0.$$

C. For all $u \in C^\infty(\mathbb{R}^N)$ and arbitrary $(t_1, \ldots, t_N) \in \mathbb{R}^N_+$ the following integration rules

$$A \int_{(0, \ldots, 0)}^{(t_1, \ldots, t_N)} T_{(s_1, \ldots, s_N)} u \, d(s_1, \ldots, s_N) = \int_{(0, \ldots, 0)}^{(t_1, \ldots, t_N)} A T_{(s_1, \ldots, s_N)} u \, d(s_1, \ldots, s_N)$$

$$= \sum_{s_j \in \{0, t_j\}, j \in \{1, \ldots, N\}} (-1)^N \prod_{j=1}^N (-1)^{s_j} T_{(s_1, \ldots, s_N)} u.$$

We refer the reader to [12] for the proof of Theorem 2.4. Moreover in [12] a detailed and comprehensive investigation of the properties of $(A, D(A))$ is given.
3. A MARTINGALE ASSOCIATED WITH MULTIPARAMETER FELLER SEMIGROUPS

We restrict ourselves now for simplicity to the 2-parameter case and we prove that using $A$ as well as $A^{(1)}$ and $A^{(2)}$ we can associate a martingale with a 2-parameter Feller process extending the 1-parameter case in a natural way.

We define the filtration we will be working with. Let $(F^{(j)}_{t_j})_{t_j \geq 0}$ be the natural filtration of the marginal processes $(X^{(j)}_{t_j})_{t_j \geq 0}$, then for all $t \in \mathbb{R}^2_+$ define

$$\mathcal{F}_t := \bigvee_{j=1}^N \mathcal{F}^{(j)}_{t_j} := \sigma \left( \bigcup_{j=1}^N \mathcal{F}^{(j)}_{t_j} \right).$$

For the existence càdlàg-modification which we invoke in the following theorem we refer the reader to Theorem 2.1 in [11].

**Theorem 3.1.** Let $(T(t_1,t_2))_{(t_1,t_2) \in \mathbb{R}^2_+}$ be a Feller semigroup with generator $(A, D(A))$ and associated càdlàg-modification process $((X_t)_{(t_1,t_2) \in \mathbb{R}^2_+}, (\mathcal{F}(t))_{(t_1,t_2) \in \mathbb{R}^2_+})$. Then for every $u \in D(A)$ we have

$$M^u_{(t_1,t_2)} := u(X_{(t_1,t_2)}) - u(X_{(t_1,0)}) - u(X_{(0,t_2)}) + u(X_{0,0}) -$$

$$\int_0^{t_1} A^{(1)}(u(X_{(r_1,t_2)}) - u(X_{(r_1,0)})) \, dr_1 -$$

$$\int_0^{t_2} A^{(2)}(u(X_{(t_1,r_2)}) - u(X_{(0,r_2)})) \, dr_2 + \int_0^{t_1} \int_0^{t_2} Au(X_{(r_1,r_2)}) \, dr_1 \, dr_2$$

is an $\{\mathcal{F}_t\}_{t \in \mathbb{R}^2_+}$-martingale, i.e., the equality

$$\mathbb{E}[M^u_t | \mathcal{F}_s] = M^u_s$$

holds for all $s \leq t, s,t \in \mathbb{R}^2_+$. 

**Proof.** For $u \in D(A)$ we have $Au \in C_\infty(\mathbb{R}^n)$, and especially $A^{(1)}u, A^{(2)}u \in C_\infty(\mathbb{R}^n)$ such that the integrals in (3.1) are well-defined. For $0 \leq s \leq t,$
\[ s, t \in \mathbb{R}^2_+ \text{ we find} \]
\[
\mathbb{E}[M^{u}_{t_1, t_2} - M^{u}_{s_1, s_2} | \mathcal{F}_{s_1, s_2}] = \mathbb{E} \left[ u(X_{(t_1, t_2)}) - u(X_{(t_1, 0)}) - u(X_{(0, t_2)}) + u(X_{(0, 0)}) 
- \int_{t_0}^{t_1} A^{(1)}(u(X_{(r_1, t_2)}) - u(X_{(r_1, 0)})) \, dr_1 - \int_{0}^{t_2} A^{(2)}(u(X_{(t_1, r_2)}) - u(X_{(0, r_2)})) \, dr_2 
+ \int_{0}^{t_1} \int_{0}^{t_2} A u(X_{(r_1, r_2)}) \, dr_1 dr_2 - u(X_{(s_1, s_2)}) + u(X_{(s_1, 0)}) + u(X_{(0, s_2)}) - u(X_{(0, 0)}) 
+ \int_{0}^{s_1} A^{(1)}(u(X_{(r_1, t_2)}) - u(X_{(r_1, 0)})) \, dr_1 + \int_{0}^{s_2} A^{(2)}(u(X_{(t_1, r_2)}) - u(X_{(0, r_2)})) \, dr_2 
- \int_{0}^{s_1} \int_{0}^{s_2} A u(X_{(r_1, r_2)}) \, dr_1 dr_2 \bigg| \mathcal{F}_{s_1, s_2} \right].
\]

By the Markov Property, we get
\[
= T_{(t_1-s_1, t_2-s_2)} u(X_{(s_1, s_2)}) - T_{t_1-s_1}^{(1)} u(X_{(s_1, 0)}) - T_{t_2-s_2}^{(2)} u(X_{(0, s_2)}) 
- \int_{0}^{s_1} A^{(1)}(T_{t_1-s_1}^{(1)} u(X_{(s_1, s_2)}) - u(X_{(r_1, 0)})) \, dr_1 
- \int_{s_1}^{t_1} A^{(1)}(T_{t_1-s_1}^{(1)} u(X_{(s_1, s_2)}) - T_{t_1-s_1}^{(1)} u(X_{(s_1, 0)})) \, dr_1 
- \int_{0}^{s_2} A^{(2)}(T_{t_1-s_1}^{(1)} u(X_{(s_1, s_2)}) - u(X_{(0, r_2)})) \, dr_2 
- \int_{s_2}^{t_2} A^{(2)}(T_{t_1-s_1}^{(1)} u(X_{(s_1, s_2)}) - T_{t_2-s_2}^{(2)} u(X_{(0, s_2)})) \, dr_2 
+ \int_{s_1}^{t_1} \int_{s_2}^{t_2} A T_{(r_1-s_1, r_2-s_2)} u(X_{(s_1, s_2)}) \, dr_1 dr_2 + \int_{s_1}^{t_1} \int_{0}^{s_2} A T_{r_1-s_1}^{(1)} u(X_{(s_1, r_2)}) \, dr_1 dr_2 
+ \int_{0}^{s_1} \int_{s_2}^{t_2} A T_{r_2-s_2}^{(2)} u(X_{(r_1, s_2)}) \, dr_1 dr_2 - u(X_{(s_1, s_2)}) + u(X_{(s_1, 0)}) + u(X_{(0, s_2)}) 
+ \int_{0}^{s_1} A^{(1)}(u(X_{(r_1, t_2)}) - u(X_{(r_1, 0)})) \, dr_1 + \int_{0}^{s_2} A^{(2)}(u(X_{(t_1, r_2)}) - u(X_{(0, r_2)})) \, dr_2 
= T_{(t_1-s_1, t_2-s_2)} u(X_{(s_1, s_2)}) - T_{t_1-s_1}^{(1)} u(X_{(s_1, 0)}) - T_{t_2-s_2}^{(2)} u(X_{(0, s_2)}) 
- T_{t_2-s_2}^{(2)} \int_{0}^{s_1} A^{(1)}(u(X_{(r_1, s_2)}) \, dr_1 + \int_{0}^{s_1} A^{(1)}(u(X_{(r_1, 0)})) \, dr_1 
- T_{(t_1-s_1, t_2-s_2)} u(X_{(s_1, s_2)}) + T_{t_2-s_2}^{(2)} u(X_{(s_1, s_2)}) + T_{(t_1-s_1)}^{(1)} u(X_{(s_1, 0)})
\[ -u(X(s_1,0)) - T_{t_1-s_1}^{(1)} \int_{0}^{(2)} A^{(2)}u(X(s_1,r_2)) \, dr_2 \\
+ \int_{0}^{s_2} A^{(2)}u(X(0,r_2)) \, dr_2 - T_{t_1-s_1,t_2-s_2}^{(1)} u(X(s_1,s_2)) + T_{(t_1-s_1)}^{(1)} u(X(s_1,s_2)) \\
+ T_{t_2-s_2}^{(2)} u(X(0,s_2)) + u(X(0,s_2)) + T_{(t_1-s_1,t_2-s_2)} u(X(s_1,s_2)) \\
- T_{(t_1-s_1)}^{(1)} u(X(s_1,s_2)) - T_{(t_2,s_2)}^{(2)} u(X(s_1,s_2)) + u(X(s_1,s_2)) \\
+ T_{t_1-s_1}^{(1)} \int_{0}^{s_2} A^{(2)}u(X(s_1,r_2)) \, dr_2 - \int_{0}^{s_2} A^{(2)}u(X(s_1,r_2)) \, dr_2 \\
+ T_{t_2-s_2}^{(2)} \int_{0}^{s_1} A^{(1)}u(X(r_1,s_2)) \, dr_1 - \int_{0}^{s_1} A^{(1)}u(X(r_1,s_2)) \, dr_1 - u(X(s_1,s_2)) \\
+ u(X(s_1,0)) + u(X(0,s_2)) \int_{0}^{s_1} A^{(1)}u(X(r_1,s_2)) \, dr_1 - \int_{0}^{s_1} A^{(1)}u(X(r_1,0)) \, dr_1 \\
+ \int_{0}^{s_2} A^{(2)}u(X(s_1,r_2)) \, dr_2 - \int_{0}^{s_2} A^{(2)}u(X(0,r_2)) \, dr_2 = 0. \quad \square \\
\]

The extension of this martingale to stochastic processes depending on three or more parameters does not make any problem.

REFERENCES


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