LEFT INVARIANT COMPLEX STRUCTURES ON $U(2)$ AND $SU(2) \times SU(2)$ REVISITED

LOUIS MAGNIN

We compute the torsion-free linear maps $J : \mathfrak{su}(2) \to \mathfrak{su}(2)$, deduce a new description of the complex structures and their equivalence classes under the action of the automorphism group for $\mathfrak{u}(2)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and prove that in both cases the set of complex structures is a differentiable manifold. The situations of $\mathfrak{u}(2) \oplus \mathfrak{u}(2)$, $\mathfrak{su}(2)^N$ and $\mathfrak{u}(2)^N$ are also considered. Extension of complex structures from $\mathfrak{u}(2)$ to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ are studied, local holomorphic charts given, and attention is paid to what representations of $\mathfrak{u}(2)$ we can get from a substitute to the regular representation on a space of holomorphic functions for the complex structure.

AMS 2000 Subject Classification: 17B30.

Key words: complex structure, CR-structure, zero torsion, unitary matrice, Lie algebra, representation, holomorphic functions.

1. INTRODUCTION

The left invariant complex structures on the group $U(2)$ of unitary 2×2 matrices, i.e., complex structures on its Lie algebra $\mathfrak{u}(2)$, have been computed for the first time, up to equivalence, in [11] in the algebraic approach, that is by determining the complex Lie subalgebras $\mathfrak{m}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{u}(2)$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{m} \oplus \bar{\mathfrak{m}}$, bar denoting conjugation. More recently, and more generally, all left invariant maximal rank CR-structures on any finite dimensional compact Lie group have been classified up to equivalence in [2]. Independently, in the case of $SU(2) \times SU(2)$, the complex structures on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ have been computed in [4] by direct approach and computations.

In the present paper, we first compute the torsion-free linear maps $J : \mathfrak{su}(2) \to \mathfrak{su}(2)$. They appear to be maximal rank CR-structures, of the $CR0$-type in the classification of [2]. Then we show how to deduce, with the computer assisted methods of [6], a new description of the complex structures and their equivalence classes under the action of the automorphism group for the specific cases of $\mathfrak{u}(2)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, without resorting to the general results of [2]. Our method, which consists in growing dimensions starting with torsion-free linear maps of $\mathfrak{su}(2)$, is new and very different from that of [4] in the case $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

The cases $\mathfrak{u}(2) \oplus \mathfrak{u}(2)$, $\mathfrak{su}(2)^N$, $\mathfrak{u}(2)^N$ are considered as well. In these cases, the set of complex structures is a differentiable manifold, though we write down explicit proofs only in the cases of $\mathfrak{u}(2)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. We also examine the extension of complex structures from $\mathfrak{u}(2)$ to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, compute local complex charts for the complex manifolds associated to the complex structures, and determine what representations of $\mathfrak{u}(2)$ we can get from a substitute to the regular representation on a space of holomorphic functions for the complex structure.

2. PRELIMINARIES

Let G_0 be a connected finite dimensional real Lie group, with Lie algebra $\mathfrak{g}$. An almost complex structure on $\mathfrak{g}$ is a linear map $J : \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -1$. The almost complex structure J is said to be *integrable* if it satisfies the condition

$$(1) \qquad [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0, \quad \forall X, Y \in \mathfrak{g}.$$

From the Newlander-Nirenberg theorem [10], condition (1) means that G_0 can be given the structure of a complex manifold with the same underlying real structure and such that the canonical complex structure on G_0 is the left invariant almost complex structure $\widehat{J}$ associated to J. (For more details, see [6], [7].) By a complex structure on $\mathfrak{g}$, we will mean an *integrable* almost complex structure on $\mathfrak{g}$, that is one satisfying (1).

Let J a complex structure on $\mathfrak{g}$ and denote by $G = (G_0, J)$ the group G_0 endowed with the structure of complex manifold defined by $\widehat{J}$. The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ splits as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{(1,0)} \oplus \mathfrak{g}^{(0,1)}$ where $\mathfrak{g}^{(1,0)} = \{\widetilde{X} = X - \mathrm{i}JX;\ X \in \in \mathfrak{g}\}$, $\mathfrak{g}^{(0,1)} = \{\widetilde{X}^{-}X + \mathrm{i}JX;\ X \in \mathfrak{g}\}$. We will denote $\mathfrak{g}^{(1,0)}$ by $\mathfrak{m}$. The integrability of J amounts to $\mathfrak{m}$ being a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. In that way the set of complex structures on $\mathfrak{g}$ can be identified with the set of all complex subalgebras $\mathfrak{m}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{m} \oplus \bar{\mathfrak{m}}$, bar denoting conjugation in $\mathfrak{g}_{\mathbb{C}}$. In particular, J is said to be abelian if $\mathfrak{m}$ is. That is the algebraic approach. Our approach is more elementary. We fix a basis of $\mathfrak{g}$, write down the torsion equations $ij|k$ ($1 \le i, j, k \le n$) obtained by projecting on x_k the equation $[Jx_i, Jx_j] - [x_i, x_j] - J[Jx_i, x_j] - J[x_i, Jx_j] = 0$, where $(x_j)_{1 \le j \le n}$ is the basis of $\mathfrak{g}$ we use, and solve them in steps by specific programs with the computer algebra system *Reduce* by A. Hearn. These programs are downloadable in the electronic archive [3]. From now on, we will use the same notation J for J and $\widehat{J}$ as well. For any $x \in G_0$, the complexification $T_x(G_0)_{\mathbb{C}}$ of the tangent space also splits as the direct sum of the holomorphic vectors

$T_x(G_0)^{(1,0)} = \{\widetilde{X}X - \mathrm{i}JX;\ X \in T_x(G_0)\}$ and the antiholomorphic vectors $T_x(G_0)^{(0,1)} = \{\widetilde{X}^{-}X + \mathrm{i}JX;\ X \in T_x(G_0)\}$. For any open subset $V \subset G_0$, the space $H_{\mathbb{C}}(V)$ of complex valued holomorphic functions on V consists of all complex smooth functions f on V which are annihilated by any antiholomorphic vector field. This is equivalent to f being annihilated by all

$$\widetilde{X}_j^{-} = X_j + \mathrm{i}JX_j, \quad 1 \le j \le n$$

with $(X_j)_{1\le j\le n}$ the left invariant vector fields associated to the basis $(x_j)_{1\le j\le n}$ of $\mathfrak{g}$. Hence

$$H_{\mathbb{C}}(V) = \{f \in C^\infty(V);\ \widetilde{X}_j^{-} f = 0,\ \forall j,\ 1 \le j \le n\}.$$

Finally, the automorphism group $\operatorname{Aut}\mathfrak{g}$ of $\mathfrak{g}$ acts on the set $\mathfrak{X}_{\mathfrak{g}}$ of all complex structures on $\mathfrak{g}$ by $J \mapsto \Phi \circ J \circ \Phi^{-1}$ $\forall \Phi \in \operatorname{Aut}\mathfrak{g}$. Two complex structures J, J' on $\mathfrak{g}$ are said to be *equivalent* if they are on the same $\operatorname{Aut}\mathfrak{g}$ orbit. For simply connected G_0, this amounts to requiring the existence of an $f \in \operatorname{Aut} G_0$ such that $f : (G_0, J) \to (G_0, J')$ is biholomorphic.

3. $\boldsymbol{U(2)}$

Consider the Lie algebra $\mathfrak{su}(2)$ along with its basis $\{J_1, J_2, J_3\}$ defined by $J_1 = \frac{i}{2}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $J_2 = \frac{1}{2}\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, $J_3 = \frac{i}{2}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. One has

$$[J_1, J_2] = J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2 \tag{2}$$

and the corresponding one-parameter subgroups of $SU(2)$ are $e^{tJ_1}\begin{pmatrix}\cos\frac{t}{2} & \mathrm{i}\sin\frac{t}{2}\\ \mathrm{i}\sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix}$, $e^{tJ_2}\begin{pmatrix}\cos\frac{t}{2} & -\sin\frac{t}{2}\\ \sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix}$, $e^{tJ_3}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\frac{t}{2}} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\frac{t}{2}}\end{pmatrix}$. By means of the basis $\{J_1, J_2, J_3\}$, $\mathfrak{su}(2)$ can be identified to the euclidean vector space $\mathbb{R}^3$ the bracket being then identified to the vector product $\wedge$. Then $\operatorname{Aut}\mathfrak{su}(2)$ consists of the matrices $A = \operatorname{Mat}(\mathbf{a}, \mathbf{b}, \mathbf{a}\wedge\mathbf{b})$ with $\mathbf{a}, \mathbf{b}$ any two orthogonal normed vectors in $\mathbb{R}^3$, i.e., $\operatorname{Aut}\mathfrak{su}(2) \cong SO(3)$. Now, $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{c}$ where $\mathfrak{c} = \mathbb{R}\, J_4$ is the center of $\mathfrak{u}(2)$, $J_4 = \frac{i}{2}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. We use the basis (J_1, J_2, J_3, J_4) for $\mathfrak{u}(2)$. $\mathbb{R}^*$ stands for $\mathbb{R}\setminus\{0\}$.

Lemma 1. $\operatorname{Aut}\mathfrak{u}(2) \cong SO(3) \times \mathbb{R}^*$.

Proof. As the center is invariant, any $\Phi \in \operatorname{Aut}\mathfrak{u}(2)$ is of the form

$$\Phi = \begin{pmatrix} & & & 0\\ & A & & 0\\ & & & 0\\ b_1^4 & b_2^4 & b_3^4 & b_4^4\end{pmatrix}$$

with $A \in \operatorname{Aut}\mathfrak{su}(2) \cong SO(3)$ and $b_4^4 \in \mathbb{R}^*$. Necessarily, $b_1^4 = b_2^4 = b_3^4 = 0$, since $\Phi(J_k) \in [\mathfrak{u}(2), \mathfrak{u}(2)] = \mathfrak{su}(2)$, $1 \le k \le 3$. $\square$

LEMMA 2. *Let $J : \mathfrak{su}(2) \to \mathfrak{su}(2)$ linear. J has zero torsion, i.e., satisfies* (1), *if and only if there exists $R \in SO(3)$ such that*

$$R^{-1}JR \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi_3^3 \end{pmatrix}. \tag{3}$$

Proof. Let $J = (\xi_j^i)_{1\le i,j\le 3}$ in the basis (J_1, J_2, J_3). The 9 torsion equations are

$$\begin{array}{ll}
12|1 & \xi_3^1(\xi_2^2+\xi_1^1)+\xi_1^3(\xi_2^2-\xi_1^1)-\xi_2^3(\xi_1^2+\xi_2^1)=0,\\
12|2 & \xi_3^2(\xi_2^2+\xi_1^1)-\xi_2^3(\xi_2^2-\xi_1^1)-\xi_1^3(\xi_1^2+\xi_2^1)=0,\\
12|3 & \xi_2^1\xi_1^2-\xi_2^2\xi_1^1-(\xi_1^3)^2-(\xi_2^3)^2+\xi_3^3(\xi_2^2+\xi_1^1)+1=0,\\
13|1 & \xi_1^1(\xi_1^2-\xi_2^1)+\xi_3^2(\xi_3^1+\xi_1^3)-\xi_3^3(\xi_1^2+\xi_2^1)=0,\\
13|2 & \xi_3^1\xi_1^3+\xi_2^2\xi_1^1-(\xi_1^2)^2-(\xi_3^2)^2+\xi_3^3(\xi_2^2-\xi_1^1)+1=0,\\
13|3 & -\xi_1^1(\xi_3^2+\xi_2^3)+\xi_1^2(\xi_3^1+\xi_1^3)+\xi_3^3(\xi_3^2-\xi_2^3)=0,\\
23|1 & \xi_2^3\xi_3^2+\xi_2^2\xi_1^1-(\xi_3^1)^2-(\xi_2^1)^2-\xi_3^3(\xi_2^2-\xi_1^1)+1=0,\\
23|2 & \xi_2^2(\xi_1^2-\xi_2^1)-\xi_3^1(\xi_3^2+\xi_2^3)+\xi_3^3(\xi_1^2+\xi_2^1)=0,\\
23|3 & \xi_2^2(\xi_1^3+\xi_3^1)-\xi_2^1(\xi_3^2+\xi_2^3)+\xi_3^3(\xi_1^3-\xi_3^1)=0.
\end{array}$$

Again, we identify $\mathfrak{su}(2)$ to $\mathbb{R}^3$ with the vector product by means of the basis (J_1, J_2, J_3). J has at least one real eigenvalue λ. Let $\mathbf{f_3} \in \mathbb{R}^3$ some normed eigenvector associated to λ. Then there exist normed vectors $\mathbf{f_1}, \mathbf{f_2} \in \mathbb{R}^3$ such that $(\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3})$ is a direct orthonormal basis of $\mathbb{R}^3$. Hence there exists $R \in SO(3)$ such that

$$R^{-1}JR = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & \lambda \end{pmatrix}.$$

Hence we may suppose $\xi_3^1 = \xi_3^2 = 0$ in J. Now, the torsion equations 12|1 and 12|2 read respectively $\xi_1^3(\xi_2^2 - \xi_1^1) = \xi_2^3(\xi_1^2 + \xi_2^1)$, $\xi_1^3(\xi_1^2 + \xi_2^1) = \xi_2^3(\xi_1^1 - \xi_2^2)$ and imply the two equations $(\xi_1^3)^2(\xi_2^2 - \xi_1^1) = -(\xi_2^3)^2(\xi_2^2 - \xi_1^1)$, $(\xi_2^3)^2(\xi_1^2 + \xi_2^1) = -(\xi_1^3)^2(\xi_1^2 + \xi_2^1)$. Hence each one of the conditions $\xi_2^2 \neq \xi_1^1$ or $\xi_1^2 \neq -\xi_2^1$ implies $\xi_1^3 = \xi_2^3 = 0$. We now have two cases. Case 1: $\xi_1^3 = \xi_2^3 = 0$, Case 2: ξ_1^3, ξ_2^3 not both zero. In Case 2, one necessarily has $\xi_2^2 = \xi_1^1$ and $\xi_1^2 = -\xi_2^1$. Then equations 23|1 and 23|2 read $-(\xi_2^1)^2 + (\xi_1^1)^2 + 1 = 0$, $\xi_2^1\xi_1^1 = 0$ and give $\xi_1^1 = 0$, $\xi_2^1 = \pm 1$. Now equation 12|3 reads $(\xi_2^3)^2 + (\xi_1^3)^2 = 0$. Hence Case 2 doesn't occur, i.e., one may suppose $\xi_1^3 = \xi_2^3 = 0$. Then equations 13|1 and 23|2 read resp. $\xi_3^3(\xi_1^2 + \xi_2^1) = \xi_1^1(\xi_1^2 - \xi_2^1)$, $\xi_3^3(\xi_1^2 + \xi_2^1) = -\xi_2^2(\xi_1^2 - \xi_2^1)$, hence if $\xi_1^2 \neq \xi_2^1$, necessarily $\xi_2^2 = -\xi_1^1$. Now, $\xi_1^2 = \xi_2^1$ is impossible since it would imply either $\xi_3^3 = 0$ or $\xi_2^1 = 0$. In fact, first, if $\xi_2^1 = 0$, equations 12|3, 13|2 and 23|1 read resp. $\xi_3^3(\xi_2^2 + \xi_1^1) - \xi_2^2\xi_1^1 + 1 = 0$, $\xi_3^3(-\xi_2^2 + \xi_1^1) - \xi_2^2\xi_1^1 - 1 = 0$,

$\xi_3^3(-\xi_2^2+\xi_1^1)+\xi_2^2\xi_1^1+1=0$, so that $12|3+13|2$ gives $\xi_1^1(\xi_3^3-\xi_2^2)=0$ and $12|3+23|1$ gives $\xi_1^1\xi_3^3=-1$, hence $\xi_1^1\neq 0$ and $\xi_3^3=\xi_2^2$, which is impossible since then $12|3$ reads $(\xi_2^2)^2+1=0$. Second, if $\xi_3^3=0$, $12|3$, $13|2$ read resp. $-\xi_2^2\xi_1^1+(\xi_2^1)^2+1=0$, $-\xi_2^2\xi_1^1+(\xi_2^1)^2-1=0$, which is contradictory. Hence we get as asserted $\xi_1^2\neq\xi_2^1$ and $\xi_2^2=-\xi_1^1$. Now, we prove that $\xi_1^1=0$ and $\xi_2^1=\pm 1$. Since $\xi_2^2=-\xi_1^1$, equations $12|3$, $13|1$, $13|2$, $23|1$ read respectively

$$\begin{array}{ll}
12|3 & \xi_2^1\xi_1^2+(\xi_1^1)^2+1=0,\\
13|1 & \xi_3^3(\xi_1^2+\xi_2^1)-\xi_1^1(\xi_1^2-\xi_2^1)=0,\\
13|2 & 2\xi_3^3\xi_1^1+(\xi_1^2)^2+(\xi_1^1)^2-1=0,\\
23|1 & 2\xi_3^3\xi_1^1-(\xi_2^1)^2-(\xi_1^1)^2+1=0.
\end{array}$$

From $12|3$, $\xi_2^1\neq 0$ and $\xi_1^2=-\frac{1+(\xi_1^1)^2}{\xi_2^1}$. Then $13|1$, $13|2$, read respectively $Q=0$, $R=0$ with $Q=\xi_1^1((\xi_1^1)^2+(\xi_2^1)^2+1)-\xi_3^3((\xi_1^1)^2-(\xi_2^1)^2+1)$, $R=(\xi_2^1)^2(2\xi_3^3\xi_1^1+(\xi_1^1)^2-1)+((\xi_1^1)^2+1)^2$. Denote from $23|1$, $S=2\xi_3^3\xi_1^1-(\xi_2^1)^2-(\xi_1^1)^2+1$. Suppose $\xi_1^1\neq 0$. Then $N=\frac{R-S}{\xi_1^1}2\xi_3^3((\xi_2^1)^2-1)+\xi_1^1((\xi_1^1)^2+(\xi_2^1)^2+3)=0$ would give, for $\xi_2^1\neq\pm 1$, $\xi_3^3=-\frac{\xi_1^1((\xi_1^1)^2+(\xi_2^1)^2+3)}{2((\xi_2^1)^2-1)}$ and then $R-\frac{((\xi_2^1)^2-2\xi_2^1+(\xi_1^1)^2+1)((\xi_2^1)^2+2\xi_2^1+(\xi_1^1)^2+1)}{(\xi_2^1)^2-1}$ which is impossible since the polynomial $X^2\pm 2X+(\xi_1^1)^2+1$ has no real root. Hence $\xi_2^1=\pm 1$. Now, $S=0$ gives $\xi_3^3=\frac{\xi_1^1}{2}$ and then $R(\xi_1^1)^2((\xi_1^1)^2+4)\neq 0$. Hence $\xi_1^1=0$. Finally, that implies as asserted $\xi_2^1=\pm 1$, since $R=-(\xi_2^1)^2+1$. We conclude that $\xi_1^1=\xi_2^2=0, \xi_1^2=-\xi_2^1, \xi_2^1=\pm 1$. Changing if necessary Φ to $\Phi\,\mathrm{diag}(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right),-1))$, one may suppose $\xi_2^1=1$. $\square$

Remark 1. Recall that a rank r CR-structure on a real Lie algebra $\mathfrak{g}$ is a r-dimensional subalgebra $\mathfrak{m}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ such that $\mathfrak{m}\cap\bar{\mathfrak{m}}=\{0\}$. Then $\mathfrak{m}=\{X-\mathrm{i}J_{\mathfrak{p}}X;\ X\in\mathfrak{p}\}$ where $\mathfrak{p}$ (the real part of $\mathfrak{m}$) is a vector subspace of $\mathfrak{g}$ and $J_{\mathfrak{p}}:\mathfrak{p}\to\mathfrak{p}$ is a zero torsion linear map such that $J_{\mathfrak{p}}^2=-\mathrm{Id}_{\mathfrak{p}}$ and $[X,Y]-[J_{\mathfrak{p}}X,J_{\mathfrak{p}}Y]\in\mathfrak{p}$, $\forall X,Y\in\mathfrak{p}$. Alternatively, a CR-structure can be defined by such data $(\mathfrak{p},J_{\mathfrak{p}})$. For even-dimensional $\mathfrak{g}$, CR-structures of maximal rank $r=\frac{1}{2}\dim\mathfrak{g}$ are just complex structures on $\mathfrak{g}$. CR-structures of maximal rank on a real compact Lie algebra have been classified in [2]. For odd-dimensional $\mathfrak{g}$, they fall essentially into two classes: $CR0$ and (strict) CRI. For even-dimensional $\mathfrak{g}$ they are all $CR0$. From Lemma 2, any linear map $J:\mathfrak{su}(2)\to\mathfrak{su}(2)$ which has zero torsion is such that $\ker(J^2+\mathrm{Id})\neq\{0\}$, and hence defines a maximal rank CR-structure on $\mathfrak{su}(2)$. It is of type $CR0$. Let us elaborate on that point. $\mathfrak{a}_0=\mathbb{C}J_3$ is a Cartan subalgebra of $\mathfrak{su}(2)$. The complexification $\mathfrak{sl}(2)$ of $\mathfrak{su}(2)$ decomposes as $\mathfrak{sl}(2)=\mathbb{C}H_-\oplus\mathfrak{h}\oplus\mathbb{C}H_+$ with $H_\pm=\mathrm{i}J_1\mp J_2$, $H_3=\mathrm{i}J_3$, $\mathfrak{h}=\mathbb{C}H_3$. Any maximal rank CR-structure of $CR0$-type (respectively (strict) CRI-type) is equivalent to $\mathfrak{m} = \mathbb{C}H_+$ (respectively $\mathfrak{m} = \mathbb{C}(aJ_3 + H_+)$, $a \in \mathbb{R}^*$), and has real part $\mathfrak{p} = \mathbb{R}J_1 \oplus \mathbb{R}J_2$ (respectively $\mathfrak{p} = \mathbb{R}J_1 \oplus \mathbb{R}J_2'$, $J_2' = J_2 - aJ_3$). The corresponding endomorphism $J_\mathfrak{p}$ of $\mathfrak{p}$ has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the basis (J_1, J_2) (respectively (J_1, J_2')) and has zero torsion on $\mathfrak{p}$. Any extension of $J_\mathfrak{p}$ to $\mathfrak{su}(2)$ has matrix $\begin{pmatrix} 0 & 1 & \xi_3^1 \\ -1 & 0 & \xi_3^2 \\ 0 & 0 & \xi_3^3 \end{pmatrix}$ in the basis (J_1, J_2, J_3) (respectively (J_1, J_2', J_3)). In the $CR0$ case, it has zero torsion on the whole of $\mathfrak{su}(2)$ if and only if $\xi_3^1 = \xi_3^2 = 0$, i.e., is of the form (3). In the CRI case, it never has zero torsion on the whole of $\mathfrak{su}(2)$.

Lemma 3. *Let $\mathfrak{g} = \bigoplus_{j=1}^N \mathfrak{g}^{(j)}$, where $\mathfrak{g}^{(j)}$ are real Lie algebras with bases $\mathcal{B}_j = (X_k^{(j)})_{1 \le k \le n_j}$, and let $\pi^{(j)} : \mathfrak{g} \to \mathfrak{g}^{(j)}$ be the projections. Let $J : \mathfrak{g} \to \mathfrak{g}$ be a linear map, $\pi_j^i = \pi^{(i)} \circ J \circ \pi^{(j)}$, $\widetilde{\pi}_j^i = \pi^{(i)} \circ J \circ \pi^{(j)}|_{\mathfrak{g}^{(j)}}$. If J has zero torsion, then the two following conditions are satisfied:*

(i) *$\widetilde{\pi}_i^i$ has zero torsion for any i;*

(ii) *$[\pi_j^i X, \pi_j^i Y]\pi_j^i[JX, Y] + \pi_j^i[X, JY]$, $\forall X, Y \in \mathfrak{g}^{(j)}$ for any i, j such that $i \neq j$.*

Proof. For any i, j let $X, Y \in \mathfrak{g}$. Applying $\pi^{(i)}$ to the torsion equation (1) we get

$$[\pi^{(i)}JX, \pi^{(i)}JY] - [\pi^{(i)}X, \pi^{(i)}Y] - \pi^{(i)}J[JX, Y] - \pi^{(i)}J[X, JY] = 0. \tag{4}$$

Suppose first $i = j$ and $X, Y \in \mathfrak{g}^{(i)}$. Then $[JX, Y][\pi^{(i)}JX, Y] = \pi^{(i)}[\pi^{(i)}J\pi^{(i)}X, Y]$, and $[X, JY][X, \pi^{(i)}JY] = \pi^{(i)}[X, \pi^{(i)}J\pi^{(i)}Y]$, and moreover $[\pi^{(i)}X, \pi^{(i)}Y] = [X, Y]$, hence (4) gives $[\pi^{(i)}JX, \pi^{(i)}JY] - [X, Y] - \pi^{(i)}J\pi^{(i)}[\pi^{(i)}J\pi^{(i)}X, Y] - \pi^{(i)}J\pi^{(i)}[X, \pi^{(i)}J\pi^{(i)}Y] = 0$, i.e.,

$$[\widetilde{\pi}_i^i X, \widetilde{\pi}_i^i Y] - [X, Y] - \widetilde{\pi}_i^i[\widetilde{\pi}_i^i X, Y] - \widetilde{\pi}_i^i[X, \widetilde{\pi}_i^i Y] = 0,$$

that is $\widetilde{\pi}_i^i$ has no torsion. Suppose now $i \neq j$ and $X, Y \in \mathfrak{g}^{(j)}$. Then $[\pi^{(i)}X, \pi^{(i)}Y] = 0$ and (4) gives

$$[\pi^{(i)}JX, \pi^{(i)}JY] - \pi^{(i)}J\pi^{(j)}[JX, Y] - \pi^{(i)}J\pi^{(j)}[X, JY] = 0,$$

i.e.,

$$[\pi_j^i X, \pi_j^i Y] - \pi_j^i[JX, Y] - \pi_j^i[X, JY] = 0. \quad \square$$

Theorem 1. (i) *Let $J : \mathfrak{u}(2) \to \mathfrak{u}(2)$ linear. J has zero torsion, i.e., satisfies* (1), *if and only if there exists $\Phi \in SO(3) \times \mathbb{R}_+^*$ such that*

$$\Phi^{-1}J\Phi \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & \xi_4^3 \\ 0 & 0 & \xi_3^4 & \xi_4^4 \end{pmatrix}, \qquad \begin{pmatrix} \xi_3^3 & \xi_4^3 \\ \xi_3^4 & \xi_4^4 \end{pmatrix} \in \mathrm{gl}(2, \mathbb{R}).$$

(ii) *Any* $J\in\mathfrak{X}_{\mathfrak{u}(2)}$ *is equivalent to a unique*

$$J(\xi)\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & \xi & 1\\ 0 & 0 & -(1+\xi^2) & -\xi \end{pmatrix} \tag{5}$$

with $\xi\in\mathbb{R}$. $J(\xi)$ *and* $J(\xi')$ $(\xi,\xi'\in\mathbb{R})$ *are equivalent if and only if* $\xi=\xi'$.

Proof. (i) From Lemma 3,

$$J=\left(\begin{array}{ccc|c} & & & \xi^1_4\\ & J_1 & & \xi^2_4\\ & & & \xi^3_4\\ \xi^4_1 & \xi^4_2 & \xi^4_3 & \xi^4_4 \end{array}\right)$$

for some $J_1:\mathfrak{su}(2)\to\mathfrak{su}(2)$ with zero torsion. From Lemma 2, there exists $R\in SO(3)$ such that $R^{-1}J_1R\left(\begin{smallmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & \xi^3_3\end{smallmatrix}\right)$, whence

$$\Phi^{-1}J\Phi\begin{pmatrix} 0 & 1 & 0 & \xi^1_4\\ -1 & 0 & 0 & \xi^2_4\\ 0 & 0 & \xi^3_3 & \xi^3_4\\ \xi^4_1 & \xi^4_2 & \xi^4_3 & \xi^4_4 \end{pmatrix}$$

with $\Phi=\operatorname{diag}(R,1)$. Hence we may suppose $J_1\left(\begin{smallmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & \xi^3_3\end{smallmatrix}\right)$. Now the torsion equations $13|4, 23|4$ $14|3, 24|3$ give the two Cramer systems $\xi^4_2\xi^3_3+\xi^4_1=0$, $-\xi^4_2+\xi^3_3\xi^4_1=0$; $\xi^2_4\xi^3_3-\xi^1_4=0$, $\xi^2_4+\xi^3_3\xi^1_4=0$. Hence $\xi^4_1=\xi^4_2=\xi^1_4=\xi^2_4=0$. Then all torsion equations vanish, and (i) is proved ([3], `torsionu2.red`).

(ii) From (i), we may suppose

$$J=\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & \xi^3_3 & \xi^3_4\\ 0 & 0 & \xi^4_3 & \xi^4_4 \end{pmatrix}.$$

Now $J\in\mathfrak{X}_{\mathfrak{u}(2)}$ if and only if $\left(\begin{smallmatrix} \xi^3_3 & \xi^3_4\\ \xi^4_3 & \xi^4_4\end{smallmatrix}\right)^2=-I$, i.e.,

$$J=\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & \xi^3_3 & \xi^3_4\\ 0 & 0 & -\frac{1+(\xi^3_3)^2}{\xi^3_4} & -\xi^3_3 \end{pmatrix},\qquad \xi^3_4\neq 0.$$

Now, for any $\Phi = \operatorname{diag}(A, b) \in \operatorname{Aut}\mathfrak{u}(2)$ ($A \in SO(3), b \neq 0$),

$$\Phi J \Phi^{-1} \begin{pmatrix} AJ_1A^{-1} & b^{-1}A\begin{pmatrix}0\\0\\\xi_4^3\end{pmatrix} \\ b\begin{pmatrix}0 & 0 & -\frac{1+(\xi_3^3)^2}{\xi_4^3}\end{pmatrix}A^{-1} & -\xi_3^3 \end{pmatrix}. \tag{6}$$

Taking $A = I$, $b = \xi_4^3$, we get

$$\Phi J \Phi^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & 1 \\ 0 & 0 & -(1+(\xi_3^3)^2) & -\xi_3^3 \end{pmatrix}.$$

Hence J is equivalent to $J(\xi)$ in (5) with $\xi = \xi_3^3$. The last assertion of the theorem results from (6). $\square$

Remark 2. In [11], the equivalence classes of left invariant complex structures on $\mathfrak{u}(2)$ are shown to be parametrized by the complex subalgebras $\mathfrak{m}_d$ with basis $\{J_1 + \mathrm{i}J_2,\, 2\mathrm{i}J_3 + dJ_4\}$ with $d = -\frac{1+\mathrm{i}\xi}{1+\xi^2}$, $\xi \in \mathbb{R}$. The complex structure defined by $\mathfrak{m}_d$ has matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi & 2(1+\xi^2) \\ 0 & 0 & -\frac{1}{2} & -\xi \end{pmatrix} = \Phi J(\xi) \Phi^{-1},$$

with $\Phi = \operatorname{diag}\Big(1, 1, 1, \frac{1}{2(1+\xi^2)}\Big) \in \operatorname{Aut}\mathfrak{u}(2)$.

Remark 3. $\mathfrak{u}(2)$ has no abelian complex structures since, for $J(\xi)$, $\mathfrak{m} = \mathbb{C}\widetilde{J}_1 \oplus \mathbb{C}\widetilde{J}_3$ is the solvable Lie algebra $[\widetilde{J}_1, \widetilde{J}_3] = \mathrm{i}(1 - \mathrm{i}\xi)\widetilde{J}_1$.

Corollary 1. $\mathfrak{X}_{\mathfrak{u}(2)}$ *consists of the matrices*

$$\begin{pmatrix} (a_4^1)^2c^2\xi & (a_4^3 + a_4^2a_4^1c\xi)c & (a_4^3a_4^1c\xi - a_4^2)c & a_4^1 \\ -(a_4^3 - a_4^2a_4^1c\xi)c & (a_4^2)^2c^2\xi & (a_4^3a_4^2c\xi + a_4^1)c & a_4^2 \\ (a_4^3a_4^1c\xi + a_4^2)c & (a_4^3a_4^2c\xi - a_4^1)c & (a_4^3)^2c^2\xi & a_4^3 \\ -(\xi^2+1)c^2a_4^1 & -(\xi^2+1)c^2a_4^2 & -(\xi^2+1)c^2a_4^3 & -\xi \end{pmatrix}, \tag{7}$$

with the conditions

$$\xi \in \mathbb{R}, \quad \begin{pmatrix} a_4^1 \\ a_4^2 \\ a_4^3 \end{pmatrix} \in \mathbb{R}^3 \setminus \{0\}, \quad c = \pm\left((a_4^1)^2 + (a_4^2)^2 + (a_4^3)^2\right)^{-\frac{1}{2}}. \tag{8}$$

Proof. As is known, any $R \in SO(3)$ can be written

$$\begin{pmatrix} u^2-v^2-w^2+s^2 & -2(uv+ws) & 2(-uw+sv) \\ 2(-sw+uv) & u^2-v^2+w^2-s^2 & -2(su+vw) \\ 2(sv+uw) & 2(su-vw) & u^2+v^2-w^2-s^2 \end{pmatrix} \tag{9}$$

for $q=(u,v,w,s)\in\mathbb{S}^3$ (R can be written in exactly 2 ways by means of q and $-q$). Hence any $\Phi \in \operatorname{Aut}\mathfrak{u}(2)$ can be written

$$\Phi = \begin{pmatrix} & & & 0 \\ & R & & 0 \\ & & & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$$

with R as in (9) and $c\in\mathbb{R}^*$. Then we get for $\Phi J(\xi)\Phi^{-1}$ the matrix (7) with

$$a_4^1 = \frac{2}{c}(sv-uw), \tag{10}$$

$$a_4^2 = -\frac{2}{c}(su+vw), \tag{11}$$

$$a_4^3 = \frac{1}{c}(2u^2+2v^2-1). \tag{12}$$

From $u^2+v^2+w^2+s^2=1$, one gets $(a_4^1)^2+(a_4^2)^2+(a_4^3)^2=\frac{1}{c^2}$. Conversely, for any matrix J of the form (7) with conditions (8) there exist $\Phi\in\operatorname{Aut}\mathfrak{u}(2)$ and $\xi\in\mathbb{R}$ such that $J=\Phi J(\xi)\Phi^{-1}$. This amounts to the existence of $q=$ $=(u,v,w,s)\in\mathbb{S}^3$ such that equations (10), (11), (12) hold true, and follows from the fact that the map $\mathbb{S}^3\to\mathbb{S}^2$ $q\mapsto(ca_4^1, ca_4^2, ca_4^3)$ is the Hopf fibration. □

Corollary 2. *$\mathfrak{X}_{\mathfrak{u}(2)}$ is a closed 4-dimensional (smooth) submanifold of $\mathbb{R}^{16}$ with two connected components, each of them diffeomorphic to $\mathbb{R}\times\left(\mathbb{R}^3\setminus\{0\}\right)$.*

Proof. Denote $\mathfrak{X}^+_{\mathfrak{u}(2)}$ (respectively $\mathfrak{X}^-_{\mathfrak{u}(2)}$) the subset of those $J\in\mathfrak{X}_{\mathfrak{u}(2)}$ with $c>0$ (respectively $c<0$). As c is uniquely defined by the matrix $J=(a^i_j)\in\mathfrak{X}_{\mathfrak{u}(2)}$ by the formula

$$2c = \frac{a_4^3(a_2^1-a_1^2)+a_4^2(-a_3^1+a_1^3)+a_4^1(a_3^2-a_2^3)}{(a_4^1)^2+(a_4^2)^2+(a_4^3)^2},$$

one has $\mathfrak{X}_{\mathfrak{u}(2)}=\mathfrak{X}^+_{\mathfrak{u}(2)}\cup\mathfrak{X}^-_{\mathfrak{u}(2)}$ with disjoint union. $\mathfrak{X}^+_{\mathfrak{u}(2)}$ (respectively $\mathfrak{X}^-_{\mathfrak{u}(2)}$) is a closed subset of $\mathbb{R}^{16}$. It hence suffices to prove that $\mathfrak{X}^+_{\mathfrak{u}(2)}$ is a regular submanifold, the case of $\mathfrak{X}^-_{\mathfrak{u}(2)}$ being analogous. Let $F:\mathbb{R}\times\left(\mathbb{R}^3\setminus\{0\}\right)\to\mathfrak{X}^+_{\mathfrak{u}(2)}$ be the bijection defined by $F(\xi,(a_4^1,a_4^2,a_4^3))=J$ where J is the matrix (7) with $c=\left((a_4^1)^2+(a_4^2)^2+(a_4^3)^2\right)^{-\frac{1}{2}}$. We equip $\mathfrak{X}^+_{\mathfrak{u}(2)}$ with the differentiable structure transferred from $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \{0\}\right)$. The injection i from $\mathfrak{X}^+_{\mathfrak{u}(2)}$ into the open subset $X \subset \mathbb{R}^{16}$ defined by $(a^1_4)^2 + (a^2_4)^2 + (a^3_4)^2 \neq 0$ is smooth. Now, there is a smooth retraction $r : X \mapsto \mathfrak{X}^+_{\mathfrak{u}(2)}$ defined by $r(A) = F(-a^4_4, (a^1_4, a^2_4, a^3_4))$ for $A = (a^i_j) \in X$. Hence i is an immersion and the topology of $\mathfrak{X}^+_{\mathfrak{u}(2)}$ is the induced topology of $\mathbb{R}^{16}$. $\square$

4. $\boldsymbol{SU(2) \times SU(2)}$

Lemma 4. Aut $(\mathfrak{su}(2) \oplus \mathfrak{su}(2))$ $(SO(3) \times SO(3)) \cup \tau\,(SO(3) \times SO(3))$ *where* $\tau = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ *is the switch between the two factors of* $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

Proof. Let $J^{(1)}_k$, $1 \leq k \leq 3$, (respectively $J^{(2)}_\ell$, $1 \leq \ell \leq 3$) be the basis for the first (respectively the second) factor $\mathfrak{su}(2)^{(1)}$ (respectively $\mathfrak{su}(2)^{(2)}$) of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ with relations (2), and $\pi^{(1)}$ (respectively $\pi^{(2)}$) the corresponding projections. Let $\Phi = \left(\begin{smallmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{smallmatrix}\right) \in \operatorname{Aut}(\mathfrak{su}(2) \oplus \mathfrak{su}(2))$, each Φ_j being a 3×3 matrix. $\Phi_1 = \left(\pi^{(1)} \circ \Phi\right)_{|\mathfrak{su}(2)^{(1)}}$ is an homomorphism of $\mathfrak{su}(2)^{(1)}$ into itself. Hence the three columns of Φ_1 are two-by-two orthogonal vectors in $\mathbb{R}^3$ and if one of them is zero, then the three of them are zero. In particular, if $\Phi_1 \neq 0$, then $\Phi_1 \in SO(3)$. With the same reasoning, the same property holds true for Φ_2, Φ_3, Φ_4. Suppose first $\Phi_1 \neq 0$. For $k, \ell = 1, 2, 3$, $[\pi^{(1)}(\Phi(J^{(1)}_k)), \pi^{(1)}(\Phi(J^{(2)}_\ell))] = \pi^{(1)}(\Phi([J^{(1)}_k, J^{(2)}_\ell])) = 0$. That implies that any column of Φ_1 is collinear with any column of Φ_2, hence $\Phi_2 = 0$ since the columns of Φ_1 are linearly independent. Then $\det \Phi_4 \neq 0$, whence $\Phi_4 \in SO(3)$ and finally $\Phi_3 = 0$ by the above reasoning. Hence $\Phi = \left(\begin{smallmatrix} \Phi_1 & 0 \\ 0 & \Phi_4 \end{smallmatrix}\right) \in SO(3) \times SO(3)$. Suppose now $\Phi_1 = 0$. Then $\det \Phi_2 \neq 0$, whence $\Phi_2 \in SO(3)$, and $\det \Phi_3 \neq 0$, whence $\Phi_3 \in SO(3)$. By the same argument as before, $\Phi_4 = 0$. Hence $\Phi = \left(\begin{smallmatrix} 0 & \Phi_2 \\ \Phi_3 & 0 \end{smallmatrix}\right) \tau \left(\begin{smallmatrix} \Phi_3 & 0 \\ 0 & \Phi_2 \end{smallmatrix}\right) \in \tau\,(SO(3) \times SO(3))$. $\square$

Theorem 2. *Let* $J : \mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ *linear.* J *has zero torsion, i.e., satisfies* (1), *if and only if there exists* $\Phi \in SO(3) \times SO(3)$ *such that*

$$\Phi^{-1} J \Phi \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^3_3 & 0 & 0 & \xi^3_6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \xi^6_3 & 0 & 0 & \xi^6_6 \end{pmatrix}. \tag{13}$$

Proof. From Lemmas 3 and 2, there exists $\Phi \in SO(3)\times SO(3)$ such that

$$\Phi^{-1}J\Phi \begin{pmatrix} 0 & 1 & 0 & \xi_4^1 & \xi_5^1 & \xi_6^1 \\ -1 & 0 & 0 & \xi_4^2 & \xi_5^2 & \xi_6^2 \\ 0 & 0 & \xi_3^3 & \xi_4^3 & \xi_5^3 & \xi_6^3 \\ \xi_1^4 & \xi_2^4 & \xi_3^4 & 0 & 1 & 0 \\ \xi_1^5 & \xi_2^5 & \xi_3^5 & -1 & 0 & 0 \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & 0 & 0 & \xi_6^6 \end{pmatrix}. \tag{14}$$

Hence we may suppose J of the form (14). The matrix $(\xi_j^i)_{1\le i\le 3, 4\le j\le 6}$ (respectively $(\xi_j^i)_{4\le i\le 6, 1\le j\le 3}$) is the matrix of π_2^1 (respectively π_1^2) of Lemma 3. Consider the vectors $\mathbf{u} = \pi_2^1 J_1^{(2)}, \mathbf{v} = \pi_2^1 J_2^{(2)}, \mathbf{w} = \pi_2^1 J_3^{(2)}$. From Lemma 2 (ii) one has

$$[\pi_2^1 J_1^{(2)}, \pi_2^1 J_2^{(2)}]\pi_2^1 [JJ_1^{(2)}, J_2^{(2)}] + \pi_2^1 [J_1^{(2)}, JJ_2^{(2)}] =$$
$$= \pi_2^1 [-J_2^{(2)}, J_2^{(2)}] + \pi_2^1 [J_1^{(2)}, J_1^{(2)}] = 0,$$

$$[\pi_2^1 J_2^{(2)}, \pi_2^1 J_3^{(2)}]\pi_2^1 [JJ_2^{(2)}, J_3^{(2)}] + \pi_2^1 [J_2^{(2)}, JJ_3^{(2)}] =$$
$$= \pi_2^1 [J_1^{(2)}, J_3^{(2)}] + \pi_2^1 [J_2^{(2)}, \xi_6^6 J_3^{(2)}] = -\pi_2^1 J_2^{(2)} + \xi_6^6 \pi_2^1 J_1^{(2)},$$

$$[\pi_2^1 J_1^{(2)}, \pi_2^1 J_3^{(2)}]\pi_2^1 [JJ_1^{(2)}, J_3^{(2)}] + \pi_2^1 [J_1^{(2)}, JJ_3^{(2)}] =$$
$$= \pi_2^1 [-J_2^{(2)}, J_3^{(2)}] + \pi_2^1 [J_1^{(2)}, \xi_6^6 J_3^{(2)}] = -\pi_2^1 J_1^{(2)} - \xi_6^6 \pi_2^1 J_2^{(2)}.$$

That is,

$$\mathbf{u}\wedge\mathbf{v} = 0, \quad \mathbf{v}\wedge\mathbf{w} = -\mathbf{v} + \xi_6^6\mathbf{u}, \quad \mathbf{u}\wedge\mathbf{w} = -\mathbf{u} - \xi_6^6\mathbf{v},$$

which implies $\mathbf{u} = \mathbf{v} = 0$. With the same reasoning for π_1^2, we get

$$J \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \xi_6^1 \\ -1 & 0 & 0 & 0 & 0 & \xi_6^2 \\ 0 & 0 & \xi_3^3 & 0 & 0 & \xi_6^3 \\ 0 & 0 & \xi_3^4 & 0 & 1 & 0 \\ 0 & 0 & \xi_3^5 & -1 & 0 & 0 \\ 0 & 0 & \xi_3^6 & 0 & 0 & \xi_6^6 \end{pmatrix}. \tag{15}$$

Now, the torsion equations $16|3, 26|3\ 36|4, 36|5$ give the 2 Cramer systems $\xi_6^2\xi_3^3 - \xi_6^1 = 0$, $\xi_6^2 + \xi_3^3\xi_6^1 = 0$; $\xi_3^5\xi_6^6 + \xi_3^4 = 0$, $-\xi_3^5 + \xi_6^6\xi_3^4 = 0$. Hence $\xi_6^1 = \xi_6^2 = \xi_3^4 = \xi_3^5 = 0$. Then all torsion equations vanish, and the theorem is proved ([3]). $\square$

Corollary 3. *Any* $J \in \mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ *is equivalent under some member of* $SO(3)\times SO(3)$ *to*

$$J(\xi,\eta)\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \xi & 0 & 0 & \eta\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & -\frac{1+\xi^2}{\eta} & 0 & 0 & -\xi \end{pmatrix} \tag{16}$$

with $\xi,\eta\in\mathbb{R}$, $\eta\neq 0$. $J(\xi,\eta)$ *and* $J(\xi',\eta')$ *are equivalent under some member of* $SO(3)\times SO(3)$ (*respectively* $\tau\,(SO(3)\times SO(3))$) *if and only if* $\xi'=\xi$ *and* $\eta'=\eta$ (*respectively* $\xi'=-\xi$ *and* $\eta'=-\frac{1+\xi^2}{\eta}$).

Proof. J in (13) satisfies $J^2=-I$ if and only if $\xi_6^3\neq 0$ and $\xi_3^6=-\frac{1+(\xi_3^3)^2}{\xi_6^3}$, $\xi_6^6=-\xi_3^3$, leading to $J(\xi,\eta)$ in (16) with $\xi=\xi_3^3,\eta=\xi_6^3$.

Suppose $J(\xi',\eta')\Phi J(\xi,\eta)\Phi^{-1}$ with $\Phi\left(\begin{smallmatrix}\Phi_1 & 0\\ 0 & \Phi_2\end{smallmatrix}\right)\in SO(3)\times SO(3)$. Then $\left(\begin{smallmatrix}0&1&0\\-1&0&0\\0&0&\xi'\end{smallmatrix}\right)\Phi_1\left(\begin{smallmatrix}0&1&0\\-1&0&0\\0&0&\xi\end{smallmatrix}\right)\Phi_1^{-1}$ and $\left(\begin{smallmatrix}0&1&0\\-1&0&0\\0&0&-\xi'\end{smallmatrix}\right)\Phi_2\left(\begin{smallmatrix}0&1&0\\-1&0&0\\0&0&-\xi\end{smallmatrix}\right)\Phi_2^{-1}$, which imply first $\xi'=\xi$ and second $\Phi_1=\mathrm{diag}(R_1,1)$, $\Phi_2=\mathrm{diag}(R_2,1)$ with $R_1,R_2\in SO(2)$. Then $\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&\eta'\end{smallmatrix}\right)\Phi_1\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&\eta\end{smallmatrix}\right)\Phi_2^{-1}$ implies $\eta'=\eta$.

Now, suppose $J(\xi',\eta')=\Psi J(\xi,\eta)\Psi^{-1}$ with $\Psi=\tau\Phi\in\tau\,(SO(3)\times SO(3))$, $\Phi\left(\begin{smallmatrix}\Phi_1 & 0\\ 0 & \Phi_2\end{smallmatrix}\right)\in SO(3)\times SO(3)$. Then $\Phi J(\xi,\eta)\Phi^{-1}\tau J(\xi',\eta')\tau=J(-\xi',-\frac{1+\xi'^2}{\eta'})$. Hence $\xi=-\xi'$ and $\eta=-\frac{1+\xi'^2}{\eta'}$, i.e., $\eta'=-\frac{1+\xi^2}{\eta}$. $\square$

Remark 4. Lemma 1 in [4] states that a left invariant almost complex structure on $SU(2)\times SU(2)$ is integrable if and only if it has the form $AI_{a,c}A^{-1}$ with $A\in SO(3)\times SO(3)$, $a\in\mathbb{R},c\in\mathbb{R}^*$, and

$$I_{a,c}\begin{pmatrix} \frac{a}{c} & 0 & 0 & -\frac{a^2+c^2}{c} & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ \frac{1}{c} & 0 & 0 & -\frac{a}{c} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

One has $\Phi^{-1}I_{a,c}\Phi=J(\frac{a}{c},-\frac{a^2+c^2}{c})$ with $\Phi=\mathrm{diag}\left(\left(\begin{smallmatrix}0&0&-1\\0&1&0\\1&0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&-1\\0&1&0\\1&0&0\end{smallmatrix}\right)\right)\in$ $\in SO(3)\times SO(3)$.

Remark 5. $\mathfrak{su}(2)\times\mathfrak{su}(2)$ has no abelian complex structures since, for $J(\xi,\eta)$, $\mathfrak{m}=\mathbb{C}\widetilde{J}_1^{(1)}\oplus\mathbb{C}\widetilde{J}_3^{(1)}\oplus\mathbb{C}\widetilde{J}_1^{(2)}$ is the solvable Lie algebra $[\widetilde{J}_1^{(1)},\widetilde{J}_3^{(1)}]=\mathrm{i}(1-\mathrm{i}\xi)\widetilde{J}_1^{(1)}$, $[\widetilde{J}_3^{(1)},\widetilde{J}_1^{(2)}]=\frac{1+\xi^2}{\eta}\widetilde{J}_1^{(2)}$.

Corollary 4. *$\mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ consists of the matrices*

(17)
$$\begin{pmatrix}
\lambda_1^2\xi & -\lambda_1\mu_1\xi+\nu_1 & \lambda_1\nu_1\xi+\mu_1 & \eta\lambda_1\lambda_2 & -\eta\lambda_1\mu_2 & \eta\lambda_1\nu_2\\
-\lambda_1\mu_1\xi-\nu_1 & \mu_1^2\xi & \lambda_1-\mu_1\nu_1\xi & -\eta\mu_1\lambda_2 & \eta\mu_1\mu_2 & -\eta\mu_1\nu_2\\
\lambda_1\nu_1\xi-\mu_1 & -\lambda_1-\mu_1\nu_1\xi & \nu_1^2\xi & \eta\nu_1\lambda_2 & -\eta\nu_1\mu_2 & \eta\nu_1\nu_2\\
-\frac{\xi^2+1}{\eta}\lambda_1\lambda_2 & \frac{\xi^2+1}{\eta}\mu_1\lambda_2 & -\frac{\xi^2+1}{\eta}\nu_1\lambda_2 & -\lambda_2^2\xi & \lambda_2\mu_2\xi+\nu_2 & -\lambda_2\nu_2\xi+\mu_2\\
\frac{\xi^2+1}{\eta}\lambda_1\mu_2 & -\frac{\xi^2+1}{\eta}\mu_1\mu_2 & \frac{\xi^2+1}{\eta}\nu_1\mu_2 & \lambda_2\mu_2\xi-\nu_2 & -\mu_2^2\xi & \lambda_2+\mu_2\nu_2\xi\\
-\frac{\xi^2+1}{\eta}\lambda_1\nu_2 & \frac{\xi^2+1}{\eta}\mu_1\nu_2 & -\frac{\xi^2+1}{\eta}\nu_1\nu_2 & -\lambda_2\nu_2\xi-\mu_2 & -\lambda_2+\mu_2\nu_2\xi & -\nu_2^2\xi
\end{pmatrix}$$

with

(18) $$(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^*,\quad \begin{pmatrix}\lambda_i\\ \mu_i\\ \nu_i\end{pmatrix}\in\mathbb{S}^2,\quad i=1,2.$$

Proof. $\mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ consists of the matrices $\Phi J(\xi,\eta)\Phi^{-1}$, $(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^*$, $\Phi\in SO(3)\times SO(3)$. Let $\Phi=\begin{pmatrix}\Phi_1 & 0\\ 0 & \Phi_2\end{pmatrix}\in SO(3)\times SO(3)$. Φ_1,Φ_2 can be written in the form (9) for respectively $q_1=(u_1,v_1,w_1,s_1)$, $q_2=(u_2,v_2,w_2,s_2)$ $\in\mathbb{S}^3$. Then $\Phi J(\xi,\eta)\Phi^{-1}$ is the matrix (17) with, for $i=1,2$,

(19) $$\lambda_i = 2(s_iv_i-u_iw_i),$$

(20) $$\mu_i = 2(s_iu_i+v_iw_i),$$

(21) $$\nu_i = 2u_i^2+2v_i^2-1.$$

One has $\lambda_i^2+\mu_i^2+\nu_i^2=1$. Conversely, for any matrix J of the form (17) with condition (18) there exist $\Phi\in SO(3)\times SO(3)$ and $(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^*$ such that $J=\Phi J(\xi,\eta)\Phi^{-1}$. This amounts to the existence for $i=1,2$ of $q_i=(u_i,v_i,w_i,s_i)\in\mathbb{S}^3$ such that equations (19), (20), (21) hold true, which again follows from the Hopf fibration. $\square$

Corollary 5. *$\mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ is a closed 6-dimensional (smooth) submanifold of $\mathbb{R}^{36}$ diffeomorphic to $\mathbb{R}\times\mathbb{R}^*\times(\mathbb{S}^2)^2$.*

Proof. Let X the open subset of $\mathbb{R}^{36}$ of those matrices $\left(a_j^i\right)_{1\le i,j\le 6}$ such that $H^2N_1N_2\neq 0$, where $H^2=\sum_{i=1}^3\sum_{j=4}^6(a_j^i)^2$, $N_1=(a_3^2-a_2^3)^2+(a_3^1-a_1^3)^2+(a_2^1-a_1^2)^2$, $N_2=(a_6^5-a_5^6)^2+(a_6^4-a_4^6)^2+(a_5^4-a_4^5)^2$ and consider $F:\mathbb{R}\times\mathbb{R}^*\times\left(\mathbb{S}^2\right)^2\to X$ defined by $F(\xi,\eta,(\lambda_1,\mu_1,\nu_1),(\lambda_2,\mu_2,\nu_2))=J$, where J is the matrix (17).

Observe first that F is injective. In fact, $\xi,(\lambda_1,\mu_1,\nu_1),(\lambda_2,\mu_2,\nu_2)$ can be retrieved from $(a_j^i)F(\xi,\eta,(\lambda_1,\mu_1,\nu_1),(\lambda_2,\mu_2,\nu_2))$ by the formulas $\xi=a_1^1+a_2^2+$

a^3_3, $(\lambda_1,\mu_1,\nu_1)\left(\frac{a^2_3-a^3_2}{\sqrt{N_1}},\frac{a^1_3-a^3_1}{\sqrt{N_1}},\frac{a^1_2-a^2_1}{\sqrt{N_1}}\right)$, $(\lambda_2,\mu_2,\nu_2)\left(\frac{a^5_6-a^6_5}{\sqrt{N_2}},\frac{a^4_6-a^6_4}{\sqrt{N_2}},\frac{a^4_5-a^5_4}{\sqrt{N_2}}\right)$; hence $F(\xi,\eta,(\lambda_1,\mu_1,\nu_1),(\lambda_2,\mu_2,\nu_2))F(\xi',\eta',(\lambda'_1,\mu'_1,\nu'_1),(\lambda'_2,\mu'_2,\nu'_2))$ implies $\xi=\xi'$, $(\lambda'_1,\mu'_1,\nu'_1)=(\lambda_1,\mu_1,\nu_1),(\lambda'_2,\mu'_2,\nu'_2)(\lambda_2,\mu_2,\nu_2)$, and then $\eta=\eta'$ since

$$\begin{pmatrix} \lambda_1\lambda_2 & -\lambda_1\mu_2 & \lambda_1\nu_2 \\ -\mu_1\lambda_2 & \mu_1\mu_2 & -\mu_1\nu_2 \\ \nu_1\lambda_2 & -\nu_1\mu_2 & \nu_1\nu_2 \end{pmatrix} \neq 0.$$

From the injectivity of F, $\mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}\mathfrak{X}^{+}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}\cup\mathfrak{X}^{-}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ with disjoint union, where $\mathfrak{X}^{\epsilon}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ denotes the set of those Js having η the sign of ϵ ($\epsilon=\pm$). Now, the map $G_\epsilon : X\to\mathbb{R}\times\mathbb{R}^*_\epsilon\times\left(\mathbb{S}^2\right)^2$ defined by

$$G_\epsilon((a^i_j))\left(a^1_1+a^2_2+a^3_3,\ \epsilon\sqrt{\sum_{i=1}^{3}\sum_{j=4}^{6}\left(a^i_j\right)^2},\right.$$
$$\left.\left(\frac{a^2_3-a^3_2}{\sqrt{N_1}},\frac{a^1_3-a^3_1}{\sqrt{N_1}},\frac{a^1_2-a^2_1}{\sqrt{N_1}}\right),\left(\frac{a^5_6-a^6_5}{\sqrt{N_2}},\frac{a^4_6-a^6_4}{\sqrt{N_2}},\frac{a^4_5-a^5_4}{\sqrt{N_2}}\right)\right)$$

is a smooth retraction for the restriction F_ϵ of F to $\mathbb{R}\times\mathbb{R}^*_\epsilon\times\left(\mathbb{S}^2\right)^2$. Hence F_ϵ is an immersion and the topology of $\mathfrak{X}^{\epsilon}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ is the induced topology from X. The corollary follows. □

Remark 6. We may consider $\mathfrak{u}(2)$ as a subalgebra of $\mathfrak{su}(2)\oplus\ \mathfrak{su}(2)$ by identifying J_1,J_2,J_3,J_4 to $J^{(1)}_1,J^{(1)}_2,J^{(1)}_3,J^{(2)}_3$ respectively. Then the complex structure J in (17) leaves $\mathfrak{u}(2)$ invariant if and only if $\lambda_2=\mu_2=0,\nu_2=\pm1$. For the restriction of J to $\mathfrak{u}(2)$ to be (7), one must take $\lambda_1=\frac{a^1_4}{\eta\nu_2},\mu_1=-\frac{a^2_4}{\eta\nu_2},\nu_1=\frac{a^3_4}{\eta\nu_2}$ with $c=\frac{\nu_2}{\eta}$. Then

$$J\begin{pmatrix} (a^1_4)^2c^2\xi & (a^3_4+a^2_4a^1_4c\xi)c & (a^3_4a^1_4c\xi-a^2_4)c & 0 & 0 & a^1_4 \\ -(a^3_4-a^2_4a^1_4c\xi)c & (a^2_4)^2c^2\xi & (a^3_4a^2_4c\xi+a^1_4)c & 0 & 0 & a^2_4 \\ (a^3_4a^1_4c\xi+a^2_4)c & (a^3_4a^2_4c\xi-a^1_4)c & (a^3_4)^2c^2\xi & 0 & 0 & a^3_4 \\ 0 & 0 & 0 & 0 & \nu_2 & 0 \\ 0 & 0 & 0 & -\nu_2 & 0 & 0 \\ -(\xi^2+1)c^2a^1_4 & -(\xi^2+1)c^2a^2_4 & -(\xi^2+1)c^2a^3_4 & 0 & 0 & -\xi \end{pmatrix}.$$

Hence any complex structure on $\mathfrak{u}(2)$ can be extended in 2 (in general non equivalent) ways to a complex structure on $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. For example, $J(\xi)$ can be extended (here $a^1_4=a^2_4=0$, $a^3_4=1$, $c=1$) with $\nu_2=1$ to $J(\xi,1)$ or with $\nu_2=-1$ to $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \xi & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -(1+\xi^2) & 0 & 0 & -\xi \end{pmatrix}$ which is equivalent to $J(\xi,-1)$. Now, $J(\xi,-1)\cong J(\xi,1)\Leftrightarrow \xi=0$.

5. $\boldsymbol{SU(2)^N}$

The results of Lemma 4, Theorem 2, and Corollary 3 easily generalize in the following way.

Lemma 5. *For any $N\in\mathbb{N}^*$,*

$$\mathrm{Aut}\,(\mathfrak{su}(2))^{\mathrm{N}} SO(3)^N \cup \left(\bigcup_{\sigma\in\Sigma}\tau_\sigma\left(SO(3)^N\right)\right) \qquad (\textit{disjoint reunion}),$$

where Σ is the set of circular permutations of $\{1,\dots,N\}$ having no fixed point, and $\tau_\sigma=(T^i_j)_{1\le i,j\le N}$ with the T^i_js the 3×3 blocks $T^i_j=\delta_{i,\sigma(j)}\,I$ (I the 3×3 identity and $\delta_{k,\ell}$ the Kronecker symbol).

Theorem 3. *Let $J:\mathfrak{su}(2)^N\to\mathfrak{su}(2)^N$ linear. J has zero torsion if and only if there exist $\Phi\in SO(3)^N$ and $M=(\xi^{3i}_{3j})_{1\le i,j\le N}\in \mathrm{gl}(N,\mathbb{R})$ such that $\Phi^{-1}J\Phi=J(M)$ with $J(M)=(J^i_j(M))_{1\le i,j\le N}$ and the $J^i_j(M)$s the following 3×3 blocks*

$$J^i_i(M)\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & \xi^{3i}_{3i}\end{pmatrix},\quad 1\le i\le N, \tag{22}$$

$$J^i_j(M)\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \xi^{3i}_{3j}\end{pmatrix},\quad 1\le i,j\le N,\ i\ne j.$$

(Here we used that the analogs of $16|3, 26|3\ 36|4, 36|5$ at the end of the proof of Theorem 2 are respectively, with $i<j$,

$$\begin{aligned}
3i-2,3j|3i:&\quad \xi^{3i-2}_{3j}-\xi^{3i}_{3i}\xi^{3i-1}_{3j}=0\\
3i-1,3j|3i:&\quad \xi^{3i-1}_{3j}+\xi^{3i}_{3i}\xi^{3i-2}_{3j}=0\\
3i,3j|3j-2:&\quad \xi^{3j-2}_{3i}+\xi^{3j}_{3j}\xi^{3j-1}_{3i}=0\\
3i,3j|3j-1:&\quad -\xi^{3j-1}_{3i}+\xi^{3j}_{3j}\xi^{3j-2}_{3i}=0
\end{aligned}$$

and give $\xi^{3i-2}_{3j}=\xi^{3i-1}_{3j}\xi^{3j-2}_{3i}=\xi^{3j-1}_{3i}=0$. Then all torsion equations vanish.)

Example 1. For $N = 4$,

$$
J(M) = \left(\begin{array}{ccc|ccc||ccc|ccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boxed{\xi_3^3} & 0 & 0 & \boxed{\xi_6^3} & 0 & 0 & \boxed{\xi_9^3} & 0 & 0 & \boxed{\xi_{12}^3} \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boxed{\xi_3^6} & 0 & 0 & \boxed{\xi_6^6} & 0 & 0 & \boxed{\xi_9^6} & 0 & 0 & \boxed{\xi_{12}^6} \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boxed{\xi_3^9} & 0 & 0 & \boxed{\xi_6^9} & 0 & 0 & \boxed{\xi_9^9} & 0 & 0 & \boxed{\xi_{12}^9} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & \boxed{\xi_3^{12}} & 0 & 0 & \boxed{\xi_6^{12}} & 0 & 0 & \boxed{\xi_9^{12}} & 0 & 0 & \boxed{\xi_{12}^{12}}
\end{array}\right),
$$

$$
M \begin{pmatrix}
\xi_3^3 & \xi_6^3 & \xi_9^3 & \xi_{12}^3 \\
\xi_3^6 & \xi_6^6 & \xi_9^6 & \xi_{12}^6 \\
\xi_3^9 & \xi_6^9 & \xi_9^9 & \xi_{12}^9 \\
\xi_3^{12} & \xi_6^{12} & \xi_9^{12} & \xi_{12}^{12}
\end{pmatrix}.
$$

COROLLARY 6. *For even N, any $J \in \mathfrak{X}_{\mathfrak{su}(2)^N}$ is equivalent under some member of $SO(3)^N$ to some $J(M) = (J_j^i(M))_{1\le i,j\le N}$ with $M = (\xi_{3j}^{3i})_{1\le i,j\le N}$ such that $M^2 = -I$ and $J_j^i(M)$ defined in (22). $J(M)$ and $J(M')$ are equivalent under some member of $SO(3)^N$ (respectively $\tau_\sigma\left(SO(3)^N\right)$, $\sigma \in \Sigma$) if and only if $M' = M$ (respectively $M' = M^{\sigma^{-1}}$, $M^{\sigma^{-1}} = \left(\xi_{3\,\sigma^{-1}(j)}^{3\,\sigma^{-1}(i)}\right)_{1\le i,j\le N}$). Here we make use of $(\tau_\sigma)^{-1} = \tau_{\sigma^{-1}}$ and $\tau_\sigma J(M')(\tau_\sigma)^{-1} = \left(J_{\sigma^{-1}(j)}^{\sigma^{-1}(i)}(M')\right)_{1\le i,j\le N}$.*

Example 2. For $N = 2$, Σ consists only of the transposition $(1,2)$;

$$
M = \begin{pmatrix} \xi_3^3 & \xi_6^3 \\ -\frac{1+(\xi_3^3)^2}{\xi_6^3} & -\xi_3^3 \end{pmatrix}, \quad M' = \begin{pmatrix} {\xi'}_3^3 & {\xi'}_6^3 \\ -\frac{1+({\xi'}_3^3)^2}{{\xi'}_6^3} & -{\xi'}_3^3 \end{pmatrix}.
$$

For $\sigma = (1,2)$, the condition $M' = M^{\sigma^{-1}}$ reads ${\xi'}_3^3 = -\xi_3^3$, ${\xi'}_6^3 - \frac{1+(\xi_3^3)^2}{\xi_6^3}$ and is that of Corollary 3.

6. $\boldsymbol{U(2)\times U(2)}$

LEMMA 6. Aut $(\mathfrak{u}(2)\oplus\mathfrak{u}(2))H\cup\tau H$, *where* $\tau=\left(\begin{array}{c|c}0&I\\\hline I&0\end{array}\right)$ *is the switch between the two factors of* $\mathfrak{u}(2)\oplus\mathfrak{u}(2)$,

$$H\left\{\left(\begin{array}{cc|cc}\Phi_1&0&0&0\\0&b_4^4&0&b_8^4\\\hline 0&0&\Phi_4&0\\0&b_4^8&0&b_8^8\end{array}\right),\ \Phi_1,\Phi_4\in SO(3),\ b_4^4b_8^8-b_8^4b_4^8\neq 0\right\},$$

$$\tau H\left\{\left(\begin{array}{cc|cc}0&0&\Phi_2&0\\0&b_4^4&0&b_8^4\\\hline \Phi_3&0&0&0\\0&b_4^8&0&b_8^8\end{array}\right),\ \Phi_2,\Phi_3\in SO(3),\ b_4^4b_8^8-b_8^4b_4^8\neq 0\right\}.$$

Proof. Analogous to that of Lemma 4. $\square$

THEOREM 4. (i) *Let* $J:\mathfrak{u}(2)\oplus\mathfrak{u}(2)\to\mathfrak{u}(2)\oplus\mathfrak{u}(2)$ *linear.* J *has zero torsion if and only if there exists* $\Phi\in\left(SO(3)\times\mathbb{R}_+^*\right)^2\subset H$ *and* $M\in\mathrm{gl}(4,\mathbb{R})$ *such that* $\Phi^{-1}J\Phi=K(M)$, *where*

$$(23)\quad K(M)\left(\begin{array}{cccc|cccc}0&1&0&0&0&0&0&0\\-1&0&0&0&0&0&0&0\\0&0&\xi_3^3&\xi_4^3&0&0&\xi_7^3&\xi_8^3\\0&0&\xi_3^4&\xi_4^4&0&0&\xi_7^4&\xi_8^4\\\hline 0&0&0&0&0&1&0&0\\0&0&0&0&-1&0&0&0\\0&0&\xi_3^7&\xi_4^7&0&0&\xi_7^7&\xi_8^7\\0&0&\xi_3^8&\xi_4^8&0&0&\xi_7^8&\xi_8^8\end{array}\right),\quad M\begin{pmatrix}\xi_3^3&\xi_4^3&\xi_7^3&\xi_8^3\\\xi_3^4&\xi_4^4&\xi_7^4&\xi_8^4\\\xi_3^7&\xi_4^7&\xi_7^7&\xi_8^7\\\xi_3^8&\xi_4^8&\xi_7^8&\xi_8^8\end{pmatrix}.$$

(ii) *For* $M,M'\in\mathrm{gl}(4,\mathbb{R})$, *there exists some* $\Phi\in H$ *such that* $K(M')=$ $=\Phi K(M)\Phi^{-1}$ *if and only if there exists* $\begin{pmatrix}b_4^4&b_8^4\\b_4^8&b_8^8\end{pmatrix}\in GL(2,\mathbb{R})$ *such that* $M'=$ $=GMG^{-1}$, *with*

$$(24)\qquad G\begin{pmatrix}1&0&0&0\\0&b_4^4&0&b_8^4\\0&0&1&0\\0&b_4^8&0&b_8^8\end{pmatrix}\in GL(4,\mathbb{R}).$$

(iii) *For* $M,M'\in\mathrm{gl}(4,\mathbb{R})$, *there exists* $\Psi\in\tau H$ *such that* $K(M')=$ $=\Psi K(M)\Psi^{-1}$ *if and only if there exists* $\Phi\in H$ *such that* $K(M')=$ $=\Phi K(M)\Phi^{-1}$.

Proof. (i) From Lemma 3 and Theorem 1(i), there exists $\Phi \in (SO(3)\times \times\mathbb{R}_+^*)^2 \subset H$ such that

$$\Phi^{-1}J\Phi \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & \xi^1_5 & \xi^1_6 & \xi^1_7 & \xi^1_8 \\ -1 & 0 & 0 & 0 & \xi^2_5 & \xi^2_6 & \xi^2_7 & \xi^2_8 \\ 0 & 0 & \xi^3_3 & \xi^3_4 & \xi^3_5 & \xi^3_6 & \xi^3_7 & \xi^3_8 \\ 0 & 0 & \xi^4_3 & \xi^4_4 & \xi^3_5 & \xi^3_6 & \xi^4_7 & \xi^4_8 \\ \hline \xi^5_1 & \xi^5_2 & \xi^5_3 & \xi^5_4 & 0 & 1 & 0 & 0 \\ \xi^6_1 & \xi^6_2 & \xi^6_3 & \xi^6_4 & -1 & 0 & 0 & 0 \\ \xi^7_1 & \xi^7_2 & \xi^7_3 & \xi^7_4 & 0 & 0 & \xi^7_7 & \xi^7_8 \\ \xi^8_1 & \xi^8_2 & \xi^8_3 & \xi^8_4 & 0 & 0 & \xi^8_7 & \xi^8_8 \end{array}\right). \tag{25}$$

Hence we may suppose J of the form (25). The matrix $(\xi^i_j)_{1\le i\le 4, 5\le j\le 8}$ (respectively $(\xi^i_j)_{5\le i\le 8, 1\le j\le 4}$) is the matrix of π^1_2 (respectively π^2_1) of Lemma 3. Consider $\mathbf{u} = \pi^1_2 J^{(2)}_1$, $\mathbf{v} = \pi^1_2 J^{(2)}_2$, $\mathbf{w} = \pi^1_2 J^{(2)}_3$, $\mathbf{z} = \pi^1_2 J^{(2)}_4$. From Lemma 2 (ii) one has

$$[\mathbf{u},\mathbf{v}] = 0, \quad [\mathbf{v},\mathbf{w}] = -\mathbf{v} + \xi^7_7\mathbf{u}, \quad [\mathbf{u},\mathbf{w}] = -\mathbf{u} - \xi^7_7\mathbf{v},$$
$$[\mathbf{u},\mathbf{z}] = -\xi^7_8\mathbf{v}, \quad [\mathbf{v},\mathbf{z}] = \xi^7_8\mathbf{u}, \quad [\mathbf{w},\mathbf{z}] = 0,$$

which implies $\mathbf{u} = \mathbf{v} = 0$. With the same reasoning for π^2_1, we get

$$J \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & \xi^1_7 & \xi^1_8 \\ -1 & 0 & 0 & 0 & 0 & 0 & \xi^2_7 & \xi^2_8 \\ 0 & 0 & \xi^3_3 & \xi^3_4 & 0 & 0 & \xi^3_7 & \xi^3_8 \\ 0 & 0 & \xi^4_3 & \xi^4_4 & 0 & 0 & \xi^4_7 & \xi^4_8 \\ \hline 0 & 0 & \xi^5_3 & \xi^5_4 & 0 & 1 & 0 & 0 \\ 0 & 0 & \xi^6_3 & \xi^6_4 & -1 & 0 & 0 & 0 \\ 0 & 0 & \xi^7_3 & \xi^7_4 & 0 & 0 & \xi^7_7 & \xi^7_8 \\ 0 & 0 & \xi^8_3 & \xi^8_4 & 0 & 0 & \xi^8_7 & \xi^8_8 \end{array}\right).$$

Now the torsion equations $17|3, 27|3$, $18|3, 28|3$, $35|7, 36|7$, $45|7, 46|7$, give the four Cramer systems $\xi^2_7\xi^3_3 - \xi^1_7 = 0$, $\xi^2_7 + \xi^3_3\xi^1_7 = 0$; $\xi^2_8\xi^3_3 - \xi^1_8 = 0$, $\xi^2_8 + \xi^3_3\xi^1_8 = 0$; $\xi^6_3\xi^7_7 - \xi^5_3 = 0$, $\xi^6_3 + \xi^7_7\xi^5_3 = 0$, $\xi^6_4\xi^7_7 - \xi^5_4 = 0$, $\xi^6_4 + \xi^7_7\xi^5_4 = 0$.

Hence $\xi^1_7 = \xi^2_7 = \xi^1_8 = \xi^2_8 = 0\xi^5_3 = \xi^6_3 = \xi^5_4 = \xi^6_4 = 0$. Then all torsion equations vanish ([3]).

(ii) Suppose there exists

$$\Phi = \left(\begin{array}{cc|cc} \Phi_1 & 0 & 0 & 0 \\ 0 & b^4_4 & 0 & b^4_8 \\ \hline 0 & 0 & \Phi_4 & 0 \\ 0 & b^8_4 & 0 & b^8_8 \end{array}\right) \in H, \quad \Phi_1, \Phi_4 \in SO(3),\ b^4_4 b^8_8 - b^4_8 b^8_4 \neq 0,$$

such that $\Phi K(M) = K(M')\Phi$, with $M = \begin{pmatrix} \xi^3_3 & \xi^3_4 & \xi^3_7 & \xi^3_8 \\ \xi^4_3 & \xi^4_4 & \xi^4_7 & \xi^4_8 \\ \xi^7_3 & \xi^7_4 & \xi^7_7 & \xi^7_8 \\ \xi^8_3 & \xi^8_4 & \xi^8_7 & \xi^8_8 \end{pmatrix}$, $M' = \begin{pmatrix} \xi'^3_3 & \xi'^3_4 & \xi'^3_7 & \xi'^3_8 \\ \xi'^4_3 & \xi'^4_4 & \xi'^4_7 & \xi'^4_8 \\ \xi'^7_3 & \xi'^7_4 & \xi'^7_7 & \xi'^7_8 \\ \xi'^8_3 & \xi'^8_4 & \xi'^8_7 & \xi'^8_8 \end{pmatrix}$. One has $\Phi K(M) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A = \left(\begin{array}{ccc|c} & A_1 & & * \\ \hline * & * & * & * \end{array}\right)$, $D = \left(\begin{array}{ccc|c} & D_1 & & * \\ \hline * & * & * & * \end{array}\right)$, $A_1 = \Phi_1 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi^3_3 \end{pmatrix}$, $D_1 = \Phi_4 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi^7_7 \end{pmatrix}$, and $K(M')\Phi = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ with $A' = \left(\begin{array}{ccc|c} & A'_1 & & * \\ \hline * & * & * & * \end{array}\right)$, $D' = \left(\begin{array}{ccc|c} & D'_1 & & * \\ \hline * & * & * & * \end{array}\right)$, $A'_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi'^3_3 \end{pmatrix} \Phi_1$, $D'_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi'^7_7 \end{pmatrix} \Phi_4$. Hence $\xi'^3_3 = \xi^3_3$, $\xi'^7_7 = \xi^7_7$, and $\Phi_1 = \operatorname{diag}(R_1, 1)$, $\Phi_4 = \operatorname{diag}(R_4, 1)$, $R_1, R_4 \in SO(2)$. Now, since

$$\Phi = \begin{pmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^4_4 & 0 & 0 & b^4_8 \\ 0 & 0 & 0 & R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & b^8_4 & 0 & 0 & b^8_8 \end{pmatrix},$$

we reindex the basis as the new basis $(J^{(1)}_1, J^{(1)}_2, J^{(2)}_1, J^{(2)}_2, J^{(1)}_3, J^{(1)}_4, J^{(2)}_3, J^{(2)}_4)$, so that Φ, $K(M)$, $K(M')$ have respective matrices in the new basis

$$\Phi = \left(\begin{array}{cc|cccc} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b^4_4 & 0 & b^4_8 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b^8_4 & 0 & b^8_8 \end{array}\right),$$

$$K(M) = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & M \end{array}\right), \quad K(M') = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & M' \end{array}\right).$$

The conclusion follows.

(iii) One has $\tau K(M')\tau = K(\tau_1 M' \tau_1)$ with τ_1 the same as τ yet with 2×2 blocks: $\tau_1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Now, let $\Psi = \tau\Phi \in \tau H$, $\Phi \in H$. Then $K(M') = \Psi K(M) \Psi^{-1}$ if and only if $\Phi K(M) \Phi^{-1} = \tau K(M') \tau K(\tau_1 M' \tau_1)$, i.e., there exists $\begin{pmatrix} b^4_4 & b^4_8 \\ b^8_4 & b^8_8 \end{pmatrix} \in GL(2, \mathbb{R})$ such that $\tau_1 M' \tau_1 = GMG^{-1}$ with G as in (24), i.e.,

$M' = (\tau_1 G) M (\tau_1 G)^{-1} = G_1 M G_1^{-1}$ with $G_1 = \tau_1 G \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_8^8 & 0 & b_4^8 \\ 0 & 0 & 1 & 0 \\ 0 & b_8^4 & 0 & b_4^4 \end{pmatrix}$ which is simply the matrix corresponding to $\begin{pmatrix} b_8^8 & b_4^8 \\ b_8^4 & b_4^4 \end{pmatrix} \in GL(2,\mathbb{R})$ in the formula (24). $\square$

Corollary 7. *Any $J \in \mathfrak{X}_{\mathfrak{u}(2)\oplus\mathfrak{u}(2)}$ is equivalent under some member of $\left(SO(3) \times \mathbb{R}_+^*\right)^2$ to $K(M)$ in (23) with $M \in GL(4,\mathbb{R})$, $M^2 = -I$. $K(M)$, $K(M')$ are equivalent if and only if there exists some $\begin{pmatrix} b_4^4 & b_8^4 \\ b_4^8 & b_8^8 \end{pmatrix} \in GL(2,\mathbb{R})$ such that $M' = GMG^{-1}$ with G as in (24).*

Proof. Follows readily from Theorem 4. $\square$

7. $U(2)^N$

The results of Lemma 6, Theorem 4 and Corollary 7 generalize in the following way.

Lemma 7. $\forall\, N \in \mathbb{N}^*$, $\mathrm{Aut}\,(\mathfrak{u}(2))^{\mathrm{N}} H_N \cup \left(\bigcup_{\sigma\in\Sigma} \tau_\sigma\, H_N\right)$ *(disjoint reunion) where*

- $H_N = \Big\{ (U_j^i)_{1\le i,j\le N};\ U_i^i \begin{pmatrix} \Phi_i & 0 \\ 0 & b_i^i \end{pmatrix},\ U_j^i \begin{pmatrix} 0 & 0 \\ 0 & b_i^j \end{pmatrix}\ (i \neq j),\ \Phi_i \in SO(3),$ $\det(b_i^j) \neq 0 \Big\}$;
- *Σ is the set of circular permutations of $\{1,\dots,N\}$ having no fixed point, and $\tau_\sigma = (T_j^i)_{1\le i,j\le N}$ with the T_j^is the 4×4 blocks $T_j^i = \delta_{i,\sigma(j)}\, I$ (I the 4×4 identity and $\delta_{k,\ell}$ the Kronecker symbol).*

Theorem 5. *Let $J : \mathfrak{u}(2)^N \to \mathfrak{u}(2)^N$. J has zero torsion if and only if there exists $\Phi \in \left(SO(3)\times\mathbb{R}_+^*\right)^N \subset H_N$ and $M = (M_j^i)_{1\le i,j\le N} \in \mathrm{gl}(2N,\mathbb{R})$, $M_j^i \begin{pmatrix} \xi_{4j-1}^{4i-1} & \xi_{4j}^{4i-1} \\ \xi_{4j-1}^{4i} & \xi_{4j}^{4i} \end{pmatrix}$, such that $\Phi^{-1} J \Phi = K(M)$ with*

$$K(M) = (K_j^i(M))_{1\le i,j\le N} \tag{26}$$

and the $K_j^i(M)$s the 4×4 blocks

$$K_i^i(M) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & M_i^i \end{array}\right), \quad 1 \le i \le N,$$

$$K_j^i(M) = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & M_j^i \end{array}\right), \quad 1 \le i,j \le N,\ i \neq j.$$

(Here we used that the analogs of $17|3, 27|3$ $18|3, 28|3$ $35|7, 36|7$ $45|7, 46|7$ at the end of (i) in the proof of Theorem 4 are respectively, with $i<j$,

$$\begin{aligned}
4i-3, 4j-1|4i-1:&\quad \xi^{4i-3}_{4j-1} - \xi^{4i-1}_{4i-1}\xi^{4i-2}_{4j-1} = 0\\
4i-2, 4j-1|4i-1:&\quad \xi^{4i-2}_{4j-1} + \xi^{4i-1}_{4i-1}\xi^{4i-3}_{4j-1} = 0\\
4i-3, 4j|4i-1:&\quad \xi^{4i-3}_{4j} - \xi^{4i-1}_{4i-1}\xi^{4i-2}_{4j} = 0\\
4i-2, 4j|4i-1:&\quad \xi^{4i-2}_{4j} + \xi^{4i-1}_{4i-1}\xi^{4i-3}_{4j} = 0\\
4i-1, 4j-3|4j-1:&\quad \xi^{4j-3}_{4i-1} - \xi^{4j-1}_{4j-1}\xi^{4j-2}_{4i-1} = 0\\
4i-1, 4j-2|4j-1:&\quad \xi^{4j-2}_{4i-1} + \xi^{4j-1}_{4j-1}\xi^{4j-3}_{4i-1} = 0\\
4i, 4j-3|4j-1:&\quad \xi^{4j-3}_{4i} - \xi^{4j-1}_{4j-1}\xi^{4j-2}_{4i} = 0\\
4i, 4j-2|4j-1:&\quad \xi^{4j-2}_{4i} + \xi^{4j-1}_{4j-1}\xi^{4j-3}_{4i} = 0
\end{aligned}$$

and give $\xi^{4i-3}_{4j-1} = \xi^{4i-2}_{4j-1}\xi^{4i-3}_{4j} = \xi^{4i-2}_{4j}\xi^{4j-3}_{4i-1} = \xi^{4j-2}_{4i-1}\xi^{4j-3}_{4i} = \xi^{4j-2}_{4i}0$. Then all torsion equations vanish.)

Corollary 8. *Any* $J \in \mathfrak{X}_{\mathfrak{u}(2)^N}$ *is equivalent under some member of* $(SO(3)\times\mathbb{R}^*_+)^N$ *to* $K(M)$ *in* (26) *with* $M \in GL(2N,\mathbb{R})$, $M^2=-I$. $K(M)$, $K(M')$ *are equivalent if and only if there exists some* $\left(b^{4i}_{4j}\right)_{1\le i,j\le N} \in GL(N,\mathbb{R})$ *such that* $M' = GMG^{-1}$ *with* $G = (G^i_j(M))_{1\le i,j\le N}$, $G^i_i \begin{pmatrix}1 & 0\\ 0 & b^{4i}_{4i}\end{pmatrix}$, $G^i_j \begin{pmatrix}0 & 0\\ 0 & b^{4i}_{4j}\end{pmatrix}$, $i\neq j$.

Remark 7. The closed set $\mathcal{R} = \{M \in GL(2N,\mathbb{R});\ M^2=-I\}$ consists of the conjugates of $\mathcal{T} = \begin{pmatrix}0 & -I_N\\ I_N & 0\end{pmatrix}$ (I_N the $N\times N$ identity) under the action of $GL(2N,\mathbb{R})$. Hence it is a $2N^2$-dimensional submanifold of $\mathbb{R}^{4N^2}$ with a diffeomorphism

$$\chi : GL(2N,\mathbb{R})\,/\,\mathcal{S} \to \mathcal{R},$$

$\mathcal{S} = \left\{Q = \begin{pmatrix}R & -S\\ S & R\end{pmatrix};\ R,S \in GL(N,\mathbb{R}),\ \det Q \neq 0\right\}$ the stabilizer of $\mathcal{T}$, and χ defined by $\chi[P] = P\mathcal{T}P^{-1}$ for $[P]$ the class mod $\mathcal{S}$ of $P\in GL(2N,\mathbb{R})$. For $N=2$, $\chi\left[\begin{pmatrix}-\eta & 0\\ \xi & 1\end{pmatrix}\right] = \begin{pmatrix}\xi & \eta\\ -\frac{1+\xi^2}{\eta} & -\xi\end{pmatrix}$, $(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^*$. For general N and for $G\in GL(2N,\mathbb{R})$, $M=\chi[P]$, $M'\chi[P']\in\mathcal{R}$, $GMG^{-1}\chi[GP]$ and the condition $M' = GMG^{-1}$ reads $[P'][GP]$.

8. LOCAL CHART AND A REPRESENTATION FOR $(\boldsymbol{U(2)}, \boldsymbol{J}(\xi))$

8.1. Local chart

For any fixed $\xi \in \mathbb{R}$, denote simply J the complex structure $J(\xi)$ on $\mathfrak{u}(2)$ and by G the group $U(2)$ endowed with the left invariant structure of complex manifold defined by J. For any open subset $V \subset U(2)$, the space $H_{\mathbb{C}}(V)$ of complex valued holomorphic functions on V (considered here as a subset of G) consists of all complex smooth functions f on V which are annihilated by all $\widetilde{X}_j^- = X_j + \mathrm{i}JX_j$, $1 \le j \le 4$, with $(X_j)_{1 \le j \le 4}$ (respectively (JX_j)) the left invariant vector fields associated to the basis $(J_j)_{1 \le j \le 4}$ of $\mathfrak{u}(2)$ (respectively to $(J\,J_j)$). One has $\widetilde{X}_1^- = X_1 - \mathrm{i}X_2$, $\widetilde{X}_2^- = \mathrm{i}\widetilde{X}_1^-$, $\widetilde{X}_4^- = \mathrm{i}X_3 + (1-\mathrm{i}\xi)X_4$, $\widetilde{X}_3^- = -\mathrm{i}(1+\mathrm{i}\xi)\widetilde{X}_4^-$, hence

$$H_{\mathbb{C}}(V) = \{f \in C^\infty(V);\ \widetilde{X}_1^- f \widetilde{X}_4^- f = 0\}. \tag{27}$$

As is known, the map $\mathbb{S}^1 \times SU(2) \to U(2)$ defined by

$$(\zeta, A) \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} A$$

is a diffeomorphism of manifolds (not of groups). Introducing Euler angles as coordinates in the open subset $\Omega = SU(2) \setminus \left(\mathrm{e}^{\mathbb{R}J_1} \cup \mathrm{e}^{\pi J_3}\mathrm{e}^{\mathbb{R}J_1}\right)$ of $SU(2)$, one gets the coordinates $(s, \theta, \varphi, \psi)$ in the open subset

$$V = \left(\mathbb{S}^1 \setminus \{-1\}\right) \times \Omega \tag{28}$$

such that u defined by

$$u(s,\theta,\varphi,\psi) \begin{pmatrix} \mathrm{e}^{\mathrm{i}s} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{e}^{\varphi J_3}\mathrm{e}^{\theta J_1}\mathrm{e}^{\psi J_3} \begin{pmatrix} \mathrm{e}^{\mathrm{i}s}\mathrm{e}^{\mathrm{i}\frac{\varphi+\psi}{2}}\cos\frac{\theta}{2} & \mathrm{i}\mathrm{e}^{\mathrm{i}s}\mathrm{e}^{\mathrm{i}\frac{\varphi-\psi}{2}}\sin\frac{\theta}{2} \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\varphi-\psi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{-\mathrm{i}\frac{\varphi+\psi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \tag{29}$$

is a diffeomorphism of $]-\pi,\pi[\, \times\,]0,\pi[\, \times\,]0,2\pi[\, \times\,]-2\pi,2\pi[$ on V. Then one gets on V (see e.g. [12], p. 141)

$$\begin{aligned}
X_1 &= \cos\psi\,\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\,\frac{\partial}{\partial\varphi} - \cot\theta\sin\psi\,\frac{\partial}{\partial\psi}, \\
X_2 &= -\sin\psi\,\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\,\frac{\partial}{\partial\varphi} - \cot\theta\cos\psi\,\frac{\partial}{\partial\psi}, \\
X_3 &= \frac{\partial}{\partial\psi}, \\
X_4 &= \frac{\partial}{\partial s} - \frac{\partial}{\partial\varphi}.
\end{aligned}$$

Hence $f\in C^\infty(V)$ is in $H_{\mathbb{C}}(V)$ if and only if it satisfies the two equations

$$\mathrm{i}\sin\theta\,\frac{\partial f}{\partial\theta}+\frac{\partial f}{\partial\varphi}-\cos\theta\,\frac{\partial f}{\partial\psi}=0,$$
$$\mathrm{i}\frac{\partial f}{\partial\psi}+(1-\mathrm{i}\xi)\left(\frac{\partial f}{\partial s}-\frac{\partial f}{\partial\varphi}\right)=0.$$

The two functions

$$w^1=\mathrm{e}^{\mathrm{i}(s+\varphi)}\cot\frac{\theta}{2}, \tag{30}$$
$$w^2=\mathrm{e}^{(1+\mathrm{i}\xi)\frac{s}{2(1+\xi^2)}}\,\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}\sqrt{\sin\theta} \tag{31}$$

are holomorphic on V. Let $F:V\to\mathbb{C}^2$ defined by $F=(w^1,w^2)$. It is easily seen that F is injective, with jacobian $-\frac{1}{4(1+\xi^2)}\,\mathrm{e}^{\frac{s}{1+\xi^2}}\,(\cot\frac{\theta}{2})^2\neq 0$, hence F is a biholomorphic bijection of V onto an open subset $F(V)$ of $\mathbb{C}^2$, i.e., (V,F) is a chart of G. $F(V)$ is the set of those $(w^1,w^2)\in\mathbb{C}^2$ satisfying the following conditions, where $r_1=|w^1|$, $r_2=|w^2|$ and $\omega(r_1,r_2)=\log r_2-\frac{1}{2}\log\frac{2r_1}{1+r_1^2}$: $r_1r_2\neq 0$, $\sqrt{\frac{2r_1}{1+r_1^2}}\,\mathrm{e}^{-\frac{\pi}{2(1+\xi^2)}}<r_2<\sqrt{\frac{2r_1}{1+r_1^2}}\,\mathrm{e}^{\frac{\pi}{2(1+\xi^2)}}$, $\arg w_1\not\equiv 2(1+\xi^2)\omega(r_1,r_2)$ mod 2π, $\arg w_2\not\equiv\xi\omega(r_1,r_2)+\pi$ mod 2π. For example, if $\xi=0$,

$$V\bigcup_{r_1>0}\ \bigcup_{\mathrm{e}^{-\frac{\pi}{2}}y(r_1)<r_2<\mathrm{e}^{\frac{\pi}{2}}y(r_1)}\left(\left(\mathcal{C}^{(1)}_{r_1}\setminus\{\arg\equiv 2\omega(r_1,r_2)\}\right)\times\left(\mathcal{C}^{(2)}_{r_2}\setminus\{\arg\equiv\pi\}\right)\right),$$

where $\mathcal{C}^j_{r_j}$, $j=1,2$ is the circle with radius r_j in the w^j-plane and $y(x)= =\sqrt{\frac{2x}{1+x^2}}$, $x>0$.

8.2. A representation on a space of holomorphic functions

As $U(2)$ is compact, there are no nonconstant holomorphic functions on the whole of $U(2)$. Instead, we consider the space $H_{\mathbb{C}}(V)$ of holomorphic functions on the open subset V (28), and we compute (as kind of substitute for the regular representation) the representation λ of the Lie algebra $\mathfrak{u}(2)$ we get by Lie derivatives on $H_{\mathbb{C}}(V)$. First, note that for any $x=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in V$ as in (29), the complex coordinates w^1, w^2 of x (30), (31) satisfy

$$w^1=-\mathrm{i}\mathrm{e}^{-\mathrm{i}s}\frac{a}{\bar b},$$
$$\left(w^2\right)^2=2\mathrm{i}a\bar b\,\mathrm{e}^{s\frac{1+\mathrm{i}\xi}{1+\xi^2}}.$$

Then, one gets for the complex coordinates $w^1_{\mathrm{e}^{-tJ_1}x}, w^2_{\mathrm{e}^{-tJ_1}x}$ of $\mathrm{e}^{-tJ_1}x$ ($x \in V$, $t \in \mathbb{R}$ sufficiently small)

$$w^1_{\mathrm{e}^{-tJ_1}x} = \frac{1 + w^1 \cot \frac{t}{2}}{\cot \frac{t}{2} - w^1},$$
$$(w^2_{\mathrm{e}^{-tJ_1}x})^2 = \left(w^2\right)^2 \left(\cos t + \frac{\sin t}{2} \frac{1 - \left(w^1\right)^2}{w^1}\right).$$

Whence for any $f \in H_{\mathbb{C}}(V)$, denoting $J_1 f$ instead of $\lambda(J_1)f$,

$$(J_1 f)\,(w^1, w^2) = \left[\frac{\mathrm{d}}{\mathrm{d}t} f(w^1_{\mathrm{e}^{-tJ_1}x}, w^2_{\mathrm{e}^{-tJ_1}x})\right]_{t=0} \frac{1 + (w^1)^2}{2} \frac{\partial f}{\partial w^1} + \\ + \frac{w^2\left(1 - (w^1)^2\right)}{4w^1} \frac{\partial f}{\partial w^2}.$$

In the same way,

$$w^1_{\mathrm{e}^{-tJ_2}x} = -\mathrm{i}\,\frac{\mathrm{i}\sin \frac{t}{2} + w^1 \cos \frac{t}{2}}{-\mathrm{i}\cos \frac{t}{2} + w^1 \sin \frac{t}{2}},$$
$$(w^2_{\mathrm{e}^{-tJ_2}x})^2 = \left(w^2\right)^2 \left(\cos t + \mathrm{i}\,\frac{\sin t}{2} \frac{1 + \left(w^1\right)^2}{w^1}\right),$$
$$(J_2 f)\,(w^1, w^2) = \frac{\mathrm{i}(1 - (w^1)^2)}{2} \frac{\partial f}{\partial w^1} + \frac{\mathrm{i}w^2\left(1 + (w^1)^2\right)}{4w^1} \frac{\partial f}{\partial w^2},$$
$$w^1_{\mathrm{e}^{-tJ_3}x} = \mathrm{e}^{-\mathrm{i}t} w^1, \quad (w^2_{\mathrm{e}^{-tJ_3}x})^2 = \left(w^2\right)^2,$$
$$(J_3 f)\,(w^1, w^2) = -\mathrm{i}w^1 \frac{\partial f}{\partial w^1}.$$

Finally,

$$w^1_{\mathrm{e}^{-tJ_4}x} = w^1, \quad (w^2_{\mathrm{e}^{-tJ_4}x})^2 = \left(w^2\right)^2 \mathrm{e}^{-t\frac{1+\mathrm{i}\xi}{1+\xi^2}},$$
$$(J_4 f)\,(w^1, w^2) = -\frac{1 + \mathrm{i}\xi}{1 + \xi^2}\, w^2 \frac{\partial f}{\partial w^2}.$$

In the complexification $\mathfrak{sl}(2) \oplus \mathbb{C}J_4$ of $\mathfrak{u}(2)$, introduce as usual

$$H_\pm = \mathrm{i}J_1 \mp J_2, \quad H_3 = \mathrm{i}J_3,$$

so that

$$[H_3, H_\pm] = \pm H_\pm, \quad [H_+, H_-] = 2H_3.$$

Then, extending the representation λ to $\mathfrak{sl}(2)\oplus\mathbb{C}J_4$, one has, with $H_4 = -(1-\mathrm{i}\xi)J_4$,

$$(H_+f)\,(w^1,w^2) = \mathrm{i}\left((w^1)^2\frac{\partial f}{\partial w^1} - \frac{1}{2}\,w^1w^2\,\frac{\partial f}{\partial w^2}\right),$$
$$(H_-f)\,(w^1,w^2) = \mathrm{i}\left(\frac{\partial f}{\partial w^1} + \frac{w^2}{2w^1}\,\frac{\partial f}{\partial w^2}\right),$$
$$(H_3f)\,(w^1,w^2) = w^1\frac{\partial f}{\partial w^1},\quad (H_4f)\,(w^1,w^2) = w^2\,\frac{\partial f}{\partial w^2}.$$

8.3. **A subrepresentation**

We restrict λ to $H(\mathbb{C}^*\times\mathbb{C}^*)$, $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, and denote $\varphi_{p,q}$ the function $\varphi_{p,q}(w^1,w^2) = (w^1)^p(w^2)^q$ for $p,q\in\mathbb{Z}$. The system $(\varphi_{p,q})_{p,q\in\mathbb{Z}}$ is total in $H(\mathbb{C}^*\times\mathbb{C}^*)$, and one has

$$H_+\,\varphi_{p,q} = \mathrm{i}\left(p-\frac{q}{2}\right)\varphi_{p+1,q}, \tag{32}$$
$$H_-\,\varphi_{p,q} = \mathrm{i}\left(p+\frac{q}{2}\right)\varphi_{p-1,q}, \tag{33}$$
$$H_3\,\varphi_{p,q} = p\,\varphi_{p,q}, \tag{34}$$
$$H_4\,\varphi_{p,q} = q\,\varphi_{p,q}. \tag{35}$$

For any $q\in\mathbb{Z}$, the subspace $\mathcal{H}_q$ of functions of the form $(w^2)^q g(w^1)$, $g\in H(\mathbb{C}^*)$, is a closed invariant subspace of $H(\mathbb{C}^*\times\mathbb{C}^*)$, and $H(\mathbb{C}^*\times\mathbb{C}^*)$ is the closure of $\bigoplus_{q\in\mathbb{Z}}\mathcal{H}_q$.

8.4. **A lemma**

Lemma 8. *Let $\mathcal{E}=H(\mathbb{C}^*)$ the Fréchet space of holomorphic functions of the complex variable z on $\mathbb{C}^*$. Let $\mathcal{F}$ be any closed vector subspace of $\mathcal{E}$ that is invariant by the operator $z\frac{\mathrm{d}}{\mathrm{d}z}$. Let $f\in\mathcal{F}$ and $f(z)=\sum_{p=-\infty}^{+\infty} c_pz^p$ its Laurent expansion in $\mathbb{C}^*$. If for some $p\in\mathbb{Z}$, $c_p\neq 0$, then the function $z\mapsto z^p$ belongs to $\mathcal{F}$.*

Proof. We show first that the function $z\mapsto f(\mathrm{e}^{\mathrm{i}\theta}z)$ belongs to $\mathcal{F}$, $\forall\theta\in\mathbb{R}$, $\forall f\in\mathcal{F}$. Let $f\in\mathcal{F}$ and $f(z)=\sum_{p=-\infty}^{+\infty} c_pz^p$ its Laurent expansion in $\mathbb{C}^*$. Since it is uniformly and absolutely convergent on compact subsets of $\mathbb{C}^*$, and since the operator $H=z\frac{\mathrm{d}}{\mathrm{d}z}$ is continuous on $\mathcal{E}$,

$$\frac{(\mathrm{i}\theta)^k}{k!}\,(H^kf)(z)\sum_{p=-\infty}^{\infty} c_p\frac{(\mathrm{i}\theta p)^k}{k!}\,z^p,\quad \forall k\in\mathbb{N},\,\forall\theta\in\mathbb{R},\,\forall z\in\mathbb{C}^*.$$

On the other hand, for any fixed $\theta \in \mathbb{R}$, the double series

$$\sum_{k=0}^{+\infty} \sum_{p=-\infty}^{+\infty} c_p \frac{(\mathrm{i}\theta p)^k}{k!} z^p$$

is absolutely and uniformly summable in the annulus $A(r,R)$ for any $0 < r < R < +\infty$ since

$$\sum_{k=0}^{+\infty} \sum_{p=-\infty}^{+\infty} |c_p| \frac{(|\theta|\,|p|)^k}{k!} |z|^p \leq \sum_{p<0} |c_p| \big(\mathrm{e}^{-|\theta|} r\big)^p + \sum_{p\geqslant 0} |c_p| \big(\mathrm{e}^{|\theta|} R\big)^p < +\infty.$$

From the associativity theorem for summable families, we have

$$f(\mathrm{e}^{\mathrm{i}\theta} z) \sum_{k=0}^{+\infty} \frac{(\mathrm{i}\theta)^k}{k!} (H^k f)(z)$$

with the series uniformly and absolutely convergent on compact subsets of $\mathbb{C}^*$. The conclusion follows, since $H^k f \in \mathcal{F}$, $\forall k$. Now we use the same trick as in [5], p. 14. For any $z \in \mathbb{C}^*$, denote f_z the periodic function on $\mathbb{R} : \theta \mapsto f(\mathrm{e}^{\mathrm{i}\theta} z)$. Its Fourier expansion is $f(\mathrm{e}^{\mathrm{i}\theta} z) = \sum_{p=-\infty}^{+\infty} \widetilde{c}_p(z) \mathrm{e}^{\mathrm{i}p\theta}$ where

$$\widetilde{c}_p(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta} z) \mathrm{e}^{-\mathrm{i}p\theta} \mathrm{d}\theta.$$

The function $z \mapsto \widetilde{c}_p(z)$ belongs to $\mathcal{F}$ as the right-hand side is a limit in $\mathcal{E}$ of linear combinations of functions $z \mapsto f(\mathrm{e}^{\mathrm{i}\theta} z)$. But with the Laurent expansion of f one gets $f(\mathrm{e}^{\mathrm{i}\theta} z) = \sum_{p=-\infty}^{+\infty} c_p z^p \mathrm{e}^{\mathrm{i}p\theta}$. For any z, that series is a trigonometric series that converges uniformly on $\mathbb{R}$, hence it coincides with the Fourier series of f_z and $\widetilde{c}_p(z) = c_p z^p$, $\forall p \in \mathbb{Z}$. Hence if for some $p \in \mathbb{Z}$, $c_p \neq 0$, then the function $z \mapsto z^p$ belongs to $\mathcal{F}$. $\square$

8.5. **A closer look at the subrepresentation**

Introduce the Casimir $C = H_+H_- + (H_3)^2 - H_3$. On $\mathcal{H}_q$, $C = u(u+1)$ with $u = \frac{q}{2}$. Now, we distinguish three cases. We use both the notation $\uparrow_u^q$, $D^q(2k)$, etc., of [8], Theorem 2.3, for representations of $\mathfrak{u}(2)$ and the usual notation of, e.g., [9], 7.3, $\uparrow_u$, $D^{(k)}$, etc. for representations of $\mathfrak{sl}(2)$. One has $\uparrow_u^q = \uparrow_u \otimes q$, $D^q(2k) = D^{(k)} \otimes q$ etc.

Case 1: $q = -2k$, $k \in \mathbb{N} \setminus \{0\}$. Then from (32), (33), the closed subspace $\mathcal{H}_q^\uparrow$ (respectively $\mathcal{H}_q^\downarrow$) generated by $\{\varphi_{k+n,q},\ n \in \mathbb{N}\}$ (respectively $\{\varphi_{-k-n,q},\ n \in \mathbb{N}\}$) which consists of the functions $(w^2)^{-2k}(w^1)^k g(w^1)$ (respectively $(w^2)^{-2k}\ (w^1)^{-k} g(\frac{1}{w^1})$), $g \in H(\mathbb{C})$, is invariant and topologically irreducible from Lemma 8. $\mathcal{H}_q^\uparrow = \uparrow_{-k}^q = \uparrow_{-k} \otimes q$, $\mathcal{H}_q^\downarrow = \downarrow_{-k}^q = \downarrow_{-k} \otimes q$. $\mathcal{H}_q$ is indecomposable and $\mathcal{H}_q/\big(\mathcal{H}_q^{\uparrow}\oplus\mathcal{H}_q^{\downarrow}\big)$ is topologically irreducible and equal to $D^q(2(k-1)) = D^{(k-1)}\otimes q$, i.e., $\mathcal{H}_q$ is a nontrivial extension of $D^q(2(k-1))$ by $\uparrow^q_{-k}\oplus\downarrow^q_{-k}$.

Case 2: $q=2k$, $k\in\mathbb{N}$. The closed subspace $\mathcal{H}_q^D$ generated by $\{\varphi_{-k+n,q}$, $n\in\mathbb{N}$, $0\le n\le 2k\}$, which consists of the functions $(w^2)^{2k}(w^1)^{-k}P(w^1)$, $P\in\mathbb{C}[w^1]$, $\deg P\le 2k$, is invariant and topologically irreducible from Lemma 8, and $\mathcal{H}_q^D = D^q(2k) = D^{(k)}\otimes q$. There are exactly 2 closed invariant (nontrivial) subspaces containing $\mathcal{H}_q^D$. Each one is indecomposable, with topologically irreducible quotient by $\mathcal{H}_q^D$ equal respectively to $\uparrow^q_{-k-1}$ or $\downarrow^q_{-k-1}$.

Case 3: $q\notin 2\mathbb{Z}$. In that case $\mathcal{H}_q = D^q(u,0)$ is topologically irreducible.

We see that λ is quite different from the regular representation, since the differentials of the representation in the unitary dual of $U(2)$ are $D^{(\ell)}\otimes m$, $2\ell\in\mathbb{N}$, $m\in\mathbb{Z}$, with $2\ell+m$ even ([1], p. 87).

9. CHART FOR $(\boldsymbol{SU}(\mathbf{2})\times\boldsymbol{SU}(\mathbf{2}),\boldsymbol{J}(\xi,\eta))$

In this last section, we compute an holomorphic chart for $J(\xi,\eta)$, $(\xi,\eta)\in\in\mathbb{R}\times\mathbb{R}^*$, in the open subset $W=\Omega\times\Omega$ of $SU(2)\times SU(2)$ with Euler angles coordinates $(\theta_1,\phi_1,\psi_1,\theta_2,\phi_2,\psi_2)$. The space $H_{\mathbb{C}}(W)$ of complex valued holomorphic functions on W consists of all complex smooth functions f on W which are annihilated by all

$$\widetilde{X}_j^{(k)\,-} = X_j^{(k)} + \mathrm{i}JX_j^{(k)},\quad 1\le j\le 3,\ 1\le k\le 2,$$

$\big(X_j^{(k)}\big)$ the left invariant vector fields associated to the basis $\big(J_1^{(1)}, J_2^{(1)}, J_3^{(1)}, J_1^{(2)}, J_2^{(2)}, J_3^{(2)}\big)$ of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. One has $\widetilde{X}_1^{(k)\,-} = X_1^{(k)} - \mathrm{i}X_2^{(k)}$, $\widetilde{X}_2^{(k)\,-} = = \mathrm{i}\,\widetilde{X}_1^{(k)\,-}$, $k=1,2$, $\widetilde{X}_3^{(2)\,-} = \mathrm{i}\eta X_3^{(1)} + (1-\mathrm{i}\xi)X_3^{(2)}$, $\widetilde{X}_3^{(1)\,-} - \mathrm{i}\frac{1+\mathrm{i}\xi}{\eta}\widetilde{X}_3^{(2)\,-}$. For $k=1,2$,

$$\begin{aligned}
X_1^{(k)} &= \cos\psi_k\frac{\partial}{\partial\theta_k} + \frac{\sin\psi_k}{\sin\theta_k}\frac{\partial}{\partial\varphi_k} - \cot\theta_k\sin\psi_k\frac{\partial}{\partial\psi_k},\\
X_2^{(k)} &= -\sin\psi_k\frac{\partial}{\partial\theta_k} + \frac{\cos\psi_k}{\sin\theta_k}\frac{\partial}{\partial\varphi_k} - \cot\theta_k\cos\psi_k\frac{\partial}{\partial\psi_k},\\
X_3^{(k)} &= \frac{\partial}{\partial\psi_k}\,.
\end{aligned}$$

Hence $f\in C^\infty(W)$ is in $H_{\mathbb{C}}(W)$ if and only if it satisfies the equations

$$\mathrm{i}\sin\theta_1\frac{\partial f}{\partial\theta_1} + \frac{\partial f}{\partial\varphi_1} - \cos\theta_1\frac{\partial f}{\partial\psi_1} = 0,$$

$$\mathrm{i}\sin\theta_2\,\frac{\partial f}{\partial\theta_2}+\frac{\partial f}{\partial\varphi_2}-\cos\theta_2\,\frac{\partial f}{\partial\psi_2}=0,\qquad \mathrm{i}\eta\,\frac{\partial f}{\partial\psi_1}+(1-\mathrm{i}\xi)\frac{\partial f}{\partial\psi_2}=0.$$

The functions

$$z^1=\mathrm{e}^{\mathrm{i}\varphi_1}\cot\frac{\theta_1}{2},\quad z^2=\mathrm{e}^{\mathrm{i}\varphi_2}\cot\frac{\theta_2}{2},\quad z^3=\mathrm{e}^{\mathrm{i}\frac{\psi_1}{2}}\,\mathrm{e}^{\frac{\eta(1+\mathrm{i}\xi)}{1+\xi^2}\,\frac{\psi_2}{2}}\sqrt{\sin\theta_1}\sqrt{\sin\theta_2}$$

are holomorphic on W. Let $Z : W \to \mathbb{C}^3$ defined by $Z=(z^1,z^2,z^3)$. Z is injective, with jacobian $-\frac{\eta}{4(1+\xi^2)}\,\mathrm{e}^{\frac{\eta}{1+\xi^2}\,\psi_2}(\cot\frac{\theta_1}{2})^2(\cot\frac{\theta_2}{2})^2\neq 0$, hence Z is a biholomorphic bijection of W onto an open subset of $\mathbb{C}^3$, i.e., (W,Z) is a local chart for $SU(2)\times SU(2)$ equipped with the complex structure $J(\xi,\eta)$.

REFERENCES

[1] T. Bröcker and T. Dieck, *Representations of Compact Lie Groups*. Graduate Texts in Math. **98**. Springer, New York, 1985.

[2] J.-Y. Charbonnel and H.O. Khalgui, *Classification des structures CR invariantes pour les groupes de Lie compacts*. J. Lie Theory **14** (2004), 165–198.

[3] `http://www.u-bourgogne.fr/monge/l.magnin/CSu2/CSu2index.html` or `http://math.u-bourgogne.fr/IMB/magnin/public_html/CSu2/CSu2index.html`

[4] N.A. Daurtseva, *Invariant complex structures on* $\mathbb{S}^3\times\mathbb{S}^3$. Electronic journal “Investigated in Russia”, 2004, 888–893.
English version `http://zhurnal.ape.relarn.ru/articles/2004/081e.pdf`
Russian version `http://zhurnal.ape.relarn.ru/articles/2004/081.pdf`

[5] S. Helgason, *Groups and Geometric Analysis* (*Integral Geometry, Invariant Differential Operators and Spherical Functions*). Academic Press, Orlando, 1984.

[6] L. Magnin, *Complex structures on indecomposable* 6*-dimensional nilpotent real Lie algebras*. Internat. J. Alg. Comput. **17** (2007), 77–113.

[7] L. Magnin, *Left invariant complex structures on real* 6*-dimensional simply connected indecomposable nilpotent Lie groups*. Internat. J. Alg. Comput. **17** (2007), 115–139.

[8] W. Miller Jr., *Lie Theory and Special Functions*. Academic Press, New York, 1968.

[9] W. Miller Jr., *Symmetry Groups and their Applications*. Academic Press, New York, 1972.

[10] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391–404.

[11] T. Sasaki, *Classification of left invariant complex structures on* $GL(2,\mathbb{R})$ *and* $U(2)$. Kumamoto J. Sci. (Math) **14** (1981), 115–123.

[12] N. Ja. Vilenkin, *Fonctions spéciales et théorie de la représentation des groupes*. Dunod, Paris, 1969.

Received 8 March 2010

University of Bourgogne
Mathematical Institute of Bourgogne
UMR CNRS 5584
BP 47870, 21078 Dijon Cedex, France
magnin@u-bourgogne.fr