G_{Δ_1,Δ_2} IN ACTION

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Set $\langle m\rangle = \{1, 2, \ldots, m\}$, $\forall m \geq 1$. We define, in the spirit of the general Δ-ergodic theory, the notions of a Δ- and a $[\Delta]$-stable matrix on Σ, where Δ and Σ are partitions of $\langle m\rangle$ and $\langle n\rangle$, respectively. Then these notions are generalized. We show that the notion of a $[\Delta]$-stable matrix on Σ has a basic role in the general Δ-ergodic theory (see [14–15] and [17] and the references therein for the general Δ-ergodic theory). Further, in Section 2, we define G_{Δ_1,Δ_2}, the set of $[\Delta_1]$-stable stochastic $m \times n$ matrices on Δ_2 (see also [13] or [16] for an equivalent definition), where Δ_1 and Δ_2 are partitions of $\langle m\rangle$ and $\langle n\rangle$, respectively. Then it is used to give some structure theorems for the finite products of stochastic matrices. An important special product is $P_1P_2\ldots P_n := \Pi$, where $P_1, P_2, \ldots, P_n, \Pi$ are stochastic $m \times m$ matrices and Π is a stable matrix (the given examples contain also ones from [4], [7, p. 94] (or [6]), [20] (see also [1] and [3, pp. 139–141]), and [21]). Also, we give a characterization of G_{Δ_1,Δ_2} by means of the ergodicity coefficients $\overline{\gamma}_{\Delta_1}$ and $\overline{\gamma}_{\Delta_2}$(see [11] for $\overline{\gamma}_{\Delta}$).

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1. Δ-STABLE MATRICES ON Σ

Set $\langle m\rangle = \{1, 2, \ldots, m\}$, $\forall m \geq 1$. In this section we define the notions of a Δ- and a $[\Delta]$-stable matrix on Σ, where Δ and Σ are partitions of $\langle m\rangle$ and $\langle n\rangle$, respectively, and, more generally, of a Δ- and a $[\Delta]$-stable matrix on $U \times V \times \Sigma$, where U and V are nonempty sets included in $\langle m\rangle$ and $\langle n\rangle$, respectively, and Σ is as above. Then some examples and results are given.

In this article, a vector x is a row vector and x' denotes its transpose. Set $e = e(n) = (1, 1, \ldots, 1) \in \mathbf{R}^n$, $\forall n \geq 1$.

Set

$$\mathrm{Par}(E) = \{\Delta \mid \Delta \text{ is a partition of } E\},$$

where E is a nonempty set. We shall agree that the partitions do not contain the empty set.

Definition 1.1. Let $\Delta_1, \Delta_2 \in \text{Par}(E)$. We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

Set

$$R_{m,n} = \{F \mid F \text{ is a real } m \times n \text{ matrix}\},$$
$$N_{m,n} = \{F \mid F \text{ is a nonnegative } m \times n \text{ matrix}\},$$
$$S_{m,n} = \{F \mid F \text{ is a stochastic } m \times n \text{ matrix}\},$$
$$R_n = R_{n,n}, \quad N_n = N_{n,n} \quad \text{and} \quad S_n = S_{n,n}.$$

Let $F = (F_{ij}) \in R_{m,n}$. (The entries of a matrix Z will be denoted Z_{ij}.) Let $\emptyset \neq U \subseteq \langle m \rangle$, $\emptyset \neq V \subseteq \langle n \rangle$, and $\Sigma = (K_1, K_2, \ldots, K_p) \in \text{Par}(\langle n \rangle)$. Suppose that Σ is an ordered set. Define

$$F_U = (F_{ij})_{i \in U,\, j \in \langle n \rangle}, \quad F^V = (F_{ij})_{i \in \langle m \rangle,\, j \in V}, \quad F_U^V = (F_{ij})_{i \in U,\, j \in V},$$

$$|||F|||_\infty = \max_{i \in \langle m \rangle} \sum_{j=1}^{n} |F_{ij}|$$

(the ∞-*norm of* F), and

$$F^+ = \left(F_{ij}^+\right), \quad F_{ij}^+ = \sum_{k \in K_j} F_{ik}, \quad \forall i \in \langle m \rangle, \ \forall j \in \langle p \rangle.$$

We call $F^+ = \left(F_{ij}^+\right)$ the *column-reduced matrix of* F (*on* Σ; $F^+ = F^+(\Sigma)$, i.e., it depends on Σ (if confusion can arise we write $F^{+\Sigma}$ instead of F^+)) (see [17] and, also, [15]). In this article, when we work with the operator $(\cdot)^+ = (\cdot)^+(\Sigma)$ we suppose that Σ is an ordered set, even if we omit to precise this.

Definition 1.2. Let $P \in N_{m,n}$. We say that P is a *generalized stochastic matrix* if $\exists a \geq 0$, $\exists Q \in S_{m,n}$ such that $P = aQ$.

The two definitions below are generalizations of Definition 1.3 in [12] and Definition 1.4 in [11], respectively. Note that they are given in the spirit of the general Δ-ergodic theory (see [14–15] and [17] and, also, the references therein).

Definition 1.3. Let $P \in N_{m,n}$. Let $\Delta \in \text{Par}(\langle m \rangle)$ and $\Sigma \in \text{Par}(\langle n \rangle)$. We say that P is a $[\Delta]$-*stable matrix on* Σ if P_K^L is a generalized stochastic matrix, $\forall K \in \Delta$, $\forall L \in \Sigma$. In particular, a $[\Delta]$-stable matrix on $(\{i\})_{i \in \langle n \rangle}$ is called $[\Delta]$-*stable* for short. $((\{i\})_{i \in \langle n \rangle} := (\{1\}, \{2\}, \ldots, \{n\}).)$

Definition 1.4. Let $P \in N_{m,n}$. Let $\Delta \in \text{Par}(\langle m \rangle)$ and $\Sigma \in \text{Par}(\langle n \rangle)$. We say that P is a Δ-*stable matrix on* Σ if Δ is the least fine partition for which P is a $[\Delta]$-stable matrix on Σ. In particular, a Δ-stable matrix on $(\{i\})_{i \in \langle n \rangle}$

is called Δ-*stable* while a $(\langle m\rangle)$-stable matrix on Σ is called *stable on* Σ for short. A stable matrix on $(\{i\})_{i\in\langle n\rangle}$ is called *stable* for short.

Note that the $[\Delta]$-stable matrices on Σ are encountered, but nowhere with this name so far, e.g., in the theory of grouped Markov chains (in the special case $\Delta = \Sigma$ (see, e.g., [2, p. 167, Proposition 5.9])) and in the general Δ-ergodic theory (see among other things the definition of G_{Δ_1,Δ_2} in [13], or [16], or, here, Section 2 (see also [12] for the definition of G_Δ)). Concerning the latter field we yet note. Let $(X_n)_{n\geq 0}$ be a finite Markov chain with state space $S = \langle r\rangle$, initial distribution p_0, and transition matrices $(P_n)_{n\geq 1}$. Let $\Sigma = (K_1, K_2, \ldots, K_p) \in \operatorname{Par}(S)$. Let $\emptyset \neq B \subseteq \mathbf{N}$. Set $P_{m,n} = P_{m+1}P_{m+2}\ldots P_n$, $\forall m \geq 0$, $\forall n > m$. Then the chain is weakly $[\Delta]$-ergodic on $\Sigma \times B$ (see [17] for this notion) if and only if $\forall m \in B$ there exist $[\Delta]$-stable $r \times p$ matrices $\Pi_{m,n}$, $m < n$, such that

$$\lim_{n\to\infty} \left[(P_{m,n})^+ - \Pi_{m,n}\right] = 0$$

(see [17, Theorem 1.16]; Σ is an ordered set). By this result, the chain is weakly $[\Delta]$-ergodic on $\Sigma\times B$ if and only if $\forall m \in B$ there exist $[\Delta]$-stable $r\times r$ matrices $\Pi'_{m,n}$, $m < n$, on Σ such that

$$\lim_{n\to\infty} \left(P_{m,n} - \Pi'_{m,n}\right)^+ = 0.$$

(For any $m \in B$ and $n > m$, we can take $\Pi'_{m,n}$ arbitrarily, but $[\Delta]$-stable on Σ and $\left(\Pi'_{m,n}\right)^+ = \Pi_{m,n}$ ($\Pi_{m,n}$ is given above).) The latter result says that $[\Delta]$-stable matrices on Σ have a basic role in the general Δ-ergodic theory.

Below we give some examples of $[\Delta]$-stable matrices on Σ.

Example 1.5. Let

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Obviously, P is a stable (stochastic) matrix.

Example 1.6 (see [5, p. 71]; the example here refer to the Gibbs sampler on discrete hypercube $\{0,1\}^m$ in the special case $m = 2$). Let

$$P_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{5}{8} \\ 0 & 0 & \frac{3}{8} & \frac{5}{8} \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{2}{7} & 0 & \frac{5}{7} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{2}{7} & 0 & \frac{5}{7} \end{pmatrix}.$$

Obviously, P_1 is a $(\{1,2\}, \{3,4\})$-stable matrix while P_2 is a $(\{1,3\}, \{2,4\})$-stable matrix.

Example 1.7. Let

$$P=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{5} & 0 & 0 & \frac{1}{5} & \frac{3}{5} \\ 0 & \frac{1}{7} & \frac{2}{7} & 0 & \frac{4}{7} \end{pmatrix}.$$

E.g., P is a $[\Delta_1]$-stable matrix on Δ_1, where $\Delta_1=(\{1\},\{2,3\},\{4\},\{5\})$, and a $[\Delta_2]$-stable matrix on Δ_2, where $\Delta_2=(\{1,2,3\},\{4\},\{5\})$. Note that P is a reducible stochastic matrix.

Example 1.8. Let

$$P=\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

E.g., P is a $[\Delta_1]$-stable matrix on Δ_1, where $\Delta_1=(\{1\},\{2,3\},\{4,5\})$, and a $[\Delta_2]$-stable matrix on Δ_2, where $\Delta_2=(\{1\},\{2\},\{3\},\{4,5\})$. Note that P is a cyclic stochastic matrix.

Remark 1.9. (a) A matrix $P\in N_{m,n}$ is $[\Delta]$-stable on Σ if and only if $P^{+\Sigma}$ is a $[\Delta]$-stable matrix.

(b) A matrix $P\in N_{m,n}$ is Δ-stable on Σ if and only if $P^{+\Sigma}$ is a Δ-stable matrix.

Remark 1.10. Let $\Delta\in\operatorname{Par}(\langle m\rangle)$ and $\Sigma=(K_1,K_2,\ldots,K_p)\in\operatorname{Par}(\langle n\rangle)$. Let $P\in N_{m,n}$ be a $[\Delta]$-stable matrix on Σ. Then P is a stable matrix on Σ if $\exists v\in\mathbf{R}^p$ such that $\forall K\in\Delta$, $\exists i\in K$ for which $(P^{+\Sigma})_{\{i\}}=v$. In particular, P is a stable matrix if $\exists v\in\mathbf{R}^n$ such that $\forall K\in\Delta$, $\exists i\in K$ for which $P_{\{i\}}=v$.

The two definitions below are generalizations of Definitions 1.3–4, respectively.

Definition 1.11. Let $P\in N_{m,n}$. Let $\emptyset\neq U\subseteq\langle m\rangle$ and $\emptyset\neq V\subseteq\langle n\rangle$. Let $\Delta\in\operatorname{Par}(U)$ and $\Sigma\in\operatorname{Par}(V)$. We say that P is a $[\Delta]$-*stable matrix on* $U\times V\times\Sigma$ if P_U^V is a $[\Delta]$-stable matrix on Σ. In particular, a $[\Delta]$-stable matrix on $U\times V\times(\{i\})_{i\in V}$ is called $[\Delta]$-*stable on* $U\times V$ for short.

Definition 1.12. Let $P\in N_{m,n}$. Let $\emptyset\neq U\subseteq\langle m\rangle$ and $\emptyset\neq V\subseteq\langle n\rangle$. Let $\Delta\in\operatorname{Par}(U)$ and $\Sigma\in\operatorname{Par}(V)$. We say that P is a Δ-*stable matrix on* $U\times V\times\Sigma$ if Δ is the least fine partition for which P_U^V is a $[\Delta]$-stable matrix on Σ. In particular, a Δ-stable matrix on $U\times V\times(\{i\})_{i\in V}$ is called Δ-*stable on* $U\times V$

while a (U)-stable matrix on $U\times V\times\Sigma$ is called *stable on* $U\times V\times\Sigma$ for short. A stable matrix on $U\times V\times(\{i\})_{i\in V}$ is called *stable on* $U\times V$ for short.

Remark 1.13. (a) $P\in N_{m,n}$ is a stable matrix on $U\times V$ if and only if P_U^V is a stable matrix.

(b) $P\in N_{m,n}$ is a stable matrix on $U\times V$ if and only if P is a $[(U,\{i\})_{i\in U^c}]$-stable matrix on $(V^c,\{j\})_{j\in V}$, where U^c is the complement of U, $(U,\{i\})_{i\in U^c}:=(U,\{i_1\},\{i_2\},\dots,\{i_l\})$ if $U^c=\{i_1,i_2,\dots,i_l\}$, a.s.o. (If $U^c=\emptyset$, then $(U,\{i\})_{i\in U^c}:=(U)$ while if $V^c=\emptyset$, then $(V^c,\{j\})_{j\in V}:=(\{j\})_{j\in V}$.)

Definition 1.14 ([7, p. 93] (see also [6])). Let $\emptyset\neq\mathcal{D}=\{P_1,P_2,\dots,P_t\}\subset S_m$ $(t\geq 1)$. We say that $\mathcal{D}$ is a *k-definite set* if

(i) $P_{i_1}P_{i_2}\dots P_{i_l}$ is a stable (stochastic) matrix, $\forall l\geq k$, $\forall i_1,i_2,\dots,i_l\in\langle t\rangle$;

(ii) k is the smallest number with the property (i).

The notion of a k-definite set is related to the theory of finite automata (see [7] and [18]). Concerning k-definite sets we give a special and simple result (this is in connection with deterministic finite automata because we use 0-1 stochastic matrices below (a matrix $P\in S_{m,n}$ is called 0-1 if $P_{ij}\in\{0,1\}$, $\forall i\in\langle m\rangle$, $\forall j\in\langle n\rangle$)).

THEOREM 1.15. *Let $\mathcal{D}\subset S_m$ be a k-definite set of* 0-1 *stochastic matrices. Suppose that $|\mathcal{D}|$, $m\geq 2$. Let $P\in\mathcal{D}$. Then $\exists s_1,s_2\in\langle m\rangle$, $s_1\neq s_2$, such that P is a stable matrix on $\{s_1,s_2\}\times\langle m\rangle$.*

Proof. Suppose that $\nexists s_1,s_2\in\langle m\rangle$, $s_1\neq s_2$, such that P is a stable matrix on $\{s_1,s_2\}\times\langle m\rangle$. Then P is a permutation matrix and we have reached a contradiction because a permutation matrix does not belong to $\mathcal{D}$ if $|\mathcal{D}|\geq 2$. □

Remark 1.16. The idea from the proof of Theorem 1.15, namely, a permutation matrix does not belong to $\mathcal{D}$ if $|\mathcal{D}|\geq 2$, can be used to prove that and other matrices belonging to S_m do not belong to $\mathcal{D}$, $\forall\mathcal{D}\subset S_m$. E.g., if

$$P=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix},$$

then $P\notin\mathcal{D}$, $\forall\mathcal{D}\subset S_4$, because $P_{\langle 3\rangle}^{\langle 3\rangle}=I_3$ (consequently, $(P^n)_{\langle 3\rangle}^{\langle 3\rangle}=I_3$, $\forall n\geq 1$).

2. G_{Δ_1,Δ_2} IN ACTION

In this section we define the set of stochastic matrices G_{Δ_1,Δ_2} (it was defined equivalently in [13] (see also [12] and [16])). Then it is used to give some structure theorems for the finite products of stochastic matrices (see among other things Remark 2.13(a)–(c); in particular, we obtain some structure theorems for k-definite sets (see among other things Remark 2.13(d)–(e))). Also, we give a characterization of G_{Δ_1,Δ_2} by means of the ergodicity coefficients $\overline{\gamma}_{\Delta_1}$ and $\overline{\gamma}_{\Delta_2}$.

Let $\Delta_1 \in \mathrm{Par}(\langle m\rangle)$ and $\Delta_2 \in \mathrm{Par}(\langle n\rangle)$. Define

$$G_{\Delta_1,\Delta_2} = \{P \mid P \in S_{m,n} \text{ and } P \text{ is a } [\Delta_1]\text{-stable matrix on } \Delta_2\}$$

(see [13] or [16] for an equivalent definition) and, if $m = n$,

$$G_{\Delta} = G_{\Delta,\Delta}$$

(see [12] for an equivalent definition).

Remark 2.1 (some basic results). (a)

$$S_{m,n} = G_{(\langle m\rangle),(\langle n\rangle)} = G_{\Delta,(\langle n\rangle)} = G_{(\{i\})_{i\in\langle m\rangle},\Delta'} = G_{(\{i\})_{i\in\langle m\rangle},(\{j\})_{j\in\langle n\rangle}} =$$
$$= \bigcup_{\substack{\Delta_1\in \mathrm{Par}(\langle m\rangle),\\ \Delta_2\in \mathrm{Par}(\langle n\rangle)}} G_{\Delta_1,\Delta_2}, \quad \forall \Delta \in \mathrm{Par}(\langle m\rangle),\ \forall \Delta' \in \mathrm{Par}(\langle n\rangle).$$

(b) If $\Delta_1 \preceq \Delta_2$, then $G_{\Delta_1,\Delta} \supseteq G_{\Delta_2,\Delta}$.

(c) If $\Delta_1 \preceq \Delta_2$, then $G_{\Delta,\Delta_1} \subseteq G_{\Delta,\Delta_2}$.

(d) (a generalization of (b) and (c)). If $\Delta_1 \succeq \Delta_3$ and $\Delta_2 \preceq \Delta_4$, then $G_{\Delta_1,\Delta_2} \subseteq G_{\Delta_3,\Delta_4}$.

(e) $G_{\Delta_1,\Delta_2} \cap G_{\Delta_3,\Delta_4} \neq \emptyset$, $\forall \Delta_1, \Delta_3 \in \mathrm{Par}(\langle m\rangle)$, $\forall \Delta_2, \Delta_4 \in \mathrm{Par}(\langle n\rangle)$. (See (f).)

(f) If $P \in S_{m,n}$ is a stable matrix, then $P \in G_{(\langle m\rangle),(\{j\})_{j\in\langle n\rangle}}$ and, more generally, $P \in G_{\Delta_1,\Delta_2}$, $\forall \Delta_1 \in \mathrm{Par}(\langle m\rangle)$, $\forall \Delta_2 \in \mathrm{Par}(\langle n\rangle)$ (obviously, $P \in G_{(\langle m\rangle),(\{j\})_{j\in\langle n\rangle}}$ implies $P \in G_{\Delta_1,\Delta_2}$, $\forall \Delta_1 \in \mathrm{Par}(\langle m\rangle)$, $\forall \Delta_2 \in \mathrm{Par}(\langle n\rangle)$).

(g) If $P \in S_{m,n}$ is a $[\Delta]$-stable matrix, then $P \in G_{\Delta,(\{j\})_{j\in\langle n\rangle}}$ and, more generally, $P \in G_{\Delta_1,\Delta_2}$, $\forall \Delta_1 \in \mathrm{Par}(\langle m\rangle)$ with $\Delta_1 \preceq \Delta$, $\forall \Delta_2 \in \mathrm{Par}(\langle n\rangle)$ (obviously, $P \in G_{\Delta,(\{j\})_{j\in\langle n\rangle}}$ implies $P \in G_{\Delta_1,\Delta_2}$, $\forall \Delta_1 \in \mathrm{Par}(\langle m\rangle)$ with $\Delta_1 \preceq \Delta$, $\forall \Delta_2 \in \mathrm{Par}(\langle n\rangle)$).

(h) (See (f) again.) $P \in S_{m,n}$ is a stable matrix if and only if $P \in G_{(\langle m\rangle),(\{j\})_{j\in\langle n\rangle}}$.

(i) (See (g) again.) $P \in S_{m,n}$ is a $[\Delta]$-stable matrix if and only if $P \in G_{\Delta,(\{j\})_{j\in\langle n\rangle}}$.

(j) (See Remark 1.13(b) again.) $P \in S_{m,n}$ is a stable matrix on $U \times V$ ($\emptyset \neq U \subseteq \langle m\rangle$ and $\emptyset \neq V \subseteq \langle n\rangle$) if and only if $P \in G_{\Delta_1,\Delta_2}$, where $\Delta_1 = (U, \{i\})_{i\in U^c}$ and $\Delta_2 = (V^c, \{j\})_{j\in V}$.

(k) $I_m \in G_\Delta$, $\forall \Delta \in \mathrm{Par}(\langle m\rangle)$. Therefore, $(G_\Delta, \cdot)$ is a monoid, $\forall \Delta \in \mathrm{Par}(\langle m\rangle)$. Moreover, this monoid is noncommutative if $m \geq 2$ because $PQ = Q$ and $QP = P$ if, e.g., $P, Q \in S_m$ with $P^{\{1\}} = Q^{\{m\}} = e'$ (in this case, obviously, $P, Q \in G_\Delta$, $\forall \Delta \in \mathrm{Par}(\langle m\rangle)$, see also (f)).

(l) G_{Δ_1,Δ_2} is a convex set, $\forall \Delta_1 \in \mathrm{Par}(\langle m \rangle)$, $\forall \Delta_2 \in \mathrm{Par}(\langle n \rangle)$.

Remark 2.2. Each of the matrix sets encountered in [8, Theorem 3.5], [9, p. 117], and [10, Section 4] is included in a certain G_Δ. (Note also that some results from [8–10] are generalized in [12].)

Let $P \in G_{\Delta_1,\Delta_2}$. Let $K \in \Delta_1$ and $L \in \Delta_2$. Then $\exists a_{K,L} \geq 0$, $\exists Q_{K,L} \in S_{|K|,|L|}$ such that $P_K^L = a_{K,L} Q_{K,L}$. Set

$$P^{-+} = (a_{K,L})_{K \in \Delta_1, L \in \Delta_2}.$$

If confusion can arise, we write $P^{-+(\Delta_1,\Delta_2)}$ instead of P^{-+}. We call P^{-+} the *row-and-column-reduced* (*reduced* for short) *matrix of* P *on* (Δ_1, Δ_2). In this article, when we work with the operator $(\cdot)^{-+} = (\cdot)^{-+}(\Delta_1, \Delta_2)$ we suppose that Δ_1 and Δ_2 are ordered sets, even if we omit to precise this.

The next result is the main one of this section; (i) is a generalization of Proposition 1.13 in [12].

THEOREM 2.3. *Let* $P \in G_{\Delta_1,\Delta_2} \subseteq S_{m,n}$ *and* $Q \in G_{\Delta_2,\Delta_3} \subseteq S_{n,p}$. *Then*
(i) $PQ \in G_{\Delta_1,\Delta_3} \subseteq S_{m,p}$;
(ii) $(PQ)^{-+} = P^{-+}Q^{-+}$.

Proof. (i) Let $P \in G_{\Delta_1,\Delta_2}$ and $Q \in G_{\Delta_2,\Delta_3}$. Then $\forall K \in \Delta_1$, $\forall U \in \Delta_2$, $\forall L \in \Delta_3$, $\exists a_{K,U} \geq 0$, $\exists A_{K,U} \in S_{|K|,|U|}$, $\exists b_{U,L} \geq 0$, $\exists B_{U,L} \in S_{|U|,|L|}$ such that $P_K^U = a_{K,U} A_{K,U}$ and $Q_U^L = b_{U,L} B_{U,L}$.

Let $K \in \Delta_1$ and $L \in \Delta_3$. Let $i \in K$. We have

$$\sum_{l \in L} (PQ)_{il} = \sum_{l \in L} \sum_{k \in \langle n \rangle} P_{ik} Q_{kl} = \sum_{k \in \langle n \rangle} P_{ik} \sum_{l \in L} Q_{kl} = \sum_{W \in \Delta_2} \sum_{k \in W} P_{ik} \sum_{l \in L} Q_{kl} =$$
$$= \sum_{W \in \Delta_2} \sum_{k \in W} P_{ik} b_{W,L} = \sum_{W \in \Delta_2} b_{W,L} \sum_{k \in W} P_{ik} = \sum_{W \in \Delta_2} a_{K,W} b_{W,L}.$$

It follows that $\sum_{l \in L} (PQ)_{il}$ only depends on constants $a_{K,W}$ and $b_{W,L}$, $W \in \Delta_2$, $\forall i \in K$. Therefore, $PQ \in G_{\Delta_1,\Delta_3}$.

(ii) See the proof of (i). $\square$

Remark 2.4. By Theorem 2.3(i) we have $G_{\Delta_1,\Delta_2} G_{\Delta_2,\Delta_3} \subseteq G_{\Delta_1,\Delta_3}$. Do we have $G_{\Delta_1,\Delta_2} G_{\Delta_2,\Delta_3} = G_{\Delta_1,\Delta_3}$? So far we know that the answer is in the affirmative in the special case $m = n = p := r$, $\Delta_1 = (\langle r \rangle)$, and $\Delta_3 = (\{i\})_{i \in \langle r \rangle}$ ($\Delta_2 \in \mathrm{Par}(\langle r \rangle)$ is arbitrary). Indeed, if $R \in G_{(\langle r \rangle),(\{i\})_{i \in \langle r \rangle}}$, then $R \in G_{\Delta,\Delta'}$, $\forall \Delta, \Delta' \in \mathrm{Par}(\langle r \rangle)$ (see Remark 2.1(f)). Now, taking $P = Q := R$, we have $P \in G_{(\langle r \rangle),\Delta_2}$ and $Q \in G_{\Delta_2,(\{i\})_{i \in \langle r \rangle}}$. Since $R = PQ$, we have $R \in G_{(\langle r \rangle),\Delta_2} G_{\Delta_2,(\{i\})_{i \in \langle r \rangle}}$.

Remark 2.5. Do we have $PQ \in G_{\Delta_1,\Delta_3} \subseteq S_{m,p}$ if and only if $\exists \Delta_2 \in \operatorname{Par}(\langle n\rangle)$ such that $P \in G_{\Delta_1,\Delta_2}$ ($G_{\Delta_1,\Delta_2} \subseteq S_{m,n}$) and $Q \in G_{\Delta_2,\Delta_3}$ ($G_{\Delta_2,\Delta_3} \subseteq S_{n,p}$)? The answer is in the negative. E.g., if

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

then

$$PQ = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

We have $PQ \in G_{(\langle 3\rangle),(\{1,2\},\{3\})}$. Suppose that $\exists \Delta \in \operatorname{Par}(\langle 3\rangle)$ such that $P \in G_{(\langle 3\rangle),\Delta}$ and $Q \in G_{\Delta,(\{1,2\},\{3\})}$. By $P \in G_{(\langle 3\rangle),\Delta}$, $\Delta = (\langle 3\rangle)$. It follows that $Q \in G_{(\langle 3\rangle),(\{1,2\},\{3\})}$, and we have reached a contradiction.

THEOREM 2.6. (i) (*A well-known result.*) *Let* $P \in S_{m,n}$ *and* $Q \in S_{n,p}$. *If* P *or* Q *is a stable matrix, then* PQ *is a stable matrix.*

(ii) *Let* $P \in S_{m,n}$ *and* $Q \in S_{n,p}$. *If* P *is a* $[\Delta]$*-stable matrix, then* PQ *is a* $[\Delta]$*-stable matrix.*

(iii) *Let* $P \in G_{\Delta_1,\Delta_2} \subseteq S_{m,n}$ *and* $Q \in S_{n,p}$. *If* Q *is a* $[\Delta_2]$*-stable matrix, then* PQ *is a* $[\Delta_1]$*-stable matrix. In particular, if* $P \in G_\Delta \subseteq S_m$ *and* $Q \in S_m$ *is a* $[\Delta]$*-stable matrix, then* PQ *is a* $[\Delta]$*-stable matrix.*

Proof. (i) (We give a proof by means of G_{Δ_1,Δ_2}.) *Case* 1. P is a stable matrix. Then $P \in G_{(\langle m\rangle),(\{i\})_{i\in\langle n\rangle}}$ (see Remark 2.1(h)). Obviously, $Q \in G_{(\{i\})_{i\in\langle n\rangle},(\{j\})_{j\in\langle p\rangle}}$. Now, by Theorem 2.3(i), $PQ \in G_{(\langle m\rangle),(\{j\})_{j\in\langle p\rangle}}$, i.e., is a stable matrix.

Case 2. Q is a stable matrix. Then $Q \in G_{(\langle n\rangle),(\{i\})_{i\in\langle p\rangle}}$. Obviously, $P \in G_{(\langle m\rangle),(\langle n\rangle)}$. Now, by Theorem 2.3(i), $PQ \in G_{(\langle m\rangle),(\{i\})_{i\in\langle p\rangle}}$, i.e., is a stable matrix.

(ii) Since P is a $[\Delta]$-stable matrix, we have $P \in G_{\Delta,(\{i\})_{i\in\langle n\rangle}}$ (see Remark 2.1(i)). Obviously, $Q \in G_{(\{i\})_{i\in\langle n\rangle},(\{j\})_{j\in\langle p\rangle}}$. Now, by Theorem 2.3(i), $PQ \in G_{\Delta,(\{j\})_{j\in\langle p\rangle}}$, i.e., is a $[\Delta]$-stable matrix.

(iii) Since Q is a $[\Delta_2]$-stable matrix, we have $Q \in G_{\Delta_2,(\{i\})_{i\in\langle p\rangle}}$. Now, since $P \in G_{\Delta_1,\Delta_2}$ and $Q \in G_{\Delta_2,(\{i\})_{i\in\langle p\rangle}}$, using Theorem 2.3(i), we have $PQ \in G_{\Delta_1,(\{i\})_{i\in\langle p\rangle}}$, i.e., is a $[\Delta_1]$-stable matrix. □

In connection with Theorem 2.6(iii) we have the next question.

Problem 2.7. Let $P \in G_\Delta \subseteq S_m$. Let $\Pi \in S_m$ be a $[\Delta]$-stable matrix. Is there a $[\Delta]$-stable matrix $Q \in S_m$ such that $PQ = \Pi$?

The answer to above question is in the affirmative if $\Delta = (\langle m \rangle)$. In this case, it is well-known that $Q = \Pi$. Instead, if $\Delta \neq (\langle m \rangle)$, then the answer to Problem 2.7 is in the negative. E.g., if

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{2}{4} & \frac{2}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} x & y & z \\ x & y & z \\ a & b & c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{pmatrix}$$

$(\Delta = (\{1,2\}, \{3\}))$, then from

$$\begin{cases} \frac{2}{4}x + \frac{2}{4}a = \frac{1}{2} \\ \frac{1}{4}x + \frac{3}{4}a = \frac{1}{4} \end{cases} \quad \text{and} \quad \begin{cases} \frac{2}{4}y + \frac{2}{4}b = \frac{1}{2} \\ \frac{1}{4}y + \frac{3}{4}b = \frac{1}{4}, \end{cases}$$

we have $x = y = 1$. We have reached a contradiction because $x + y + z = 1$. The answer to Problem 2.7 is in the negative even if Q is simply a matrix belonging to S_m. E.g., if

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{2}{4} & \frac{2}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} x & y & z \\ u & v & w \\ a & b & c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{pmatrix},$$

then from $\frac{1}{4}z + \frac{1}{4}w + \frac{2}{4}c = 0$, it follows that $z = w = c = 0$. We have reached a contradiction because $\frac{1}{4}z + \frac{3}{4}c = \frac{2}{4}$.

THEOREM 2.8. *Let* $P_1 \in G_{\Delta_1,\Delta_2} \subseteq S_{m_1,m_2}$, $P_2 \in G_{\Delta_2,\Delta_3} \subseteq S_{m_2,m_3}, \ldots,$ $P_n \in G_{\Delta_n,\Delta_{n+1}} \subseteq S_{m_n,m_{n+1}}$. *Then*

(i) $P_1P_2 \ldots P_n \in G_{\Delta_1,\Delta_{n+1}} \subseteq S_{m_1,m_{n+1}}$;

(ii) $(P_1P_2 \ldots P_n)^{-+} = P_1^{-+}P_2^{-+} \ldots P_n^{-+}$.

Proof. By induction (see Theorem 2.3). □

Definition 2.9. Under the assumptions of Theorem 2.8, we call

$$\left(G_{\Delta_1,\Delta_2}, G_{\Delta_2,\Delta_3}, \ldots, G_{\Delta_n,\Delta_{n+1}}\right)$$

a *linked structure of* (*the product*) $P_1P_2 \ldots P_n$.

An important special case of Theorem 2.8 is the next result.

THEOREM 2.10. *Let* $P_1 \in G_{(\langle m_1 \rangle),\Delta_2} \subseteq S_{m_1,m_2}$, $P_2 \in G_{\Delta_2,\Delta_3} \subseteq S_{m_2,m_3}$, $\ldots, P_{n-1} \in G_{\Delta_{n-1},\Delta_n} \subseteq S_{m_{n-1},m_n}$, $P_n \in G_{\Delta_n,(\{i\})_{i\in\langle m_{n+1}\rangle}} \subseteq S_{m_n,m_{n+1}}$. *Then*

(i) $P_1P_2 \ldots P_n$ *is a stable matrix*;

(ii) $\pi = P_1^{-+}P_2^{-+} \ldots P_n^{-+}$, *where* $e'\pi := P_1P_2 \ldots P_n$.

Proof. See Remark 2.1(h) and Theorem 2.8. □

Theorem 2.10 (i) can be used to see whether a finite set of matrices $\mathcal{D}$ is k-definite (see Section 1) or a Markov chain has a finite convergence time (compare the method here with those ones in [4] and [21] which are based on eigenvalues and eigenvectors). The problem of finite convergence time can be posed for each finite Markov chain, in particular, for the Markovian algorithms as, e.g., the simulated annealing (see, e.g., [19, p. 313]).

Example 2.11 (the uniform generation of random permutations of order n (see [3, pp. 139–141] for another solution; also see [1])). Let $\mathbb{S}_n$ be the set of permutations of order n. Define the matrices P_u, $u \in \langle n-1\rangle$, by

$$(P_u)_{\sigma\tau} = \begin{cases} \frac{1}{n-u+1} & \text{if } \tau = \sigma \circ (u,v) \text{ for some } v \in \{u, u+1, \ldots, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

$\forall u \in \langle n-1\rangle$ (see, e.g., [20]; (u,u) is the identity permutation and (u,v) is a transposition for $u \neq v$). Set $\Pi = P_1 P_2 \ldots P_{n-1}$. Then Π is a stable matrix and, moreover, $\Pi = \frac{1}{n!}e'e$. First, we show that Π is a stable matrix. Let $\mathbb{A}_n^u$ be the set of arrangements using u of n objects, $\forall u \in \langle n\rangle$. Set

$$K_{(i_1,i_2,\ldots,i_u)} = \{\sigma \mid \sigma \in \mathbb{S}_n \text{ and } \sigma(s) = i_s,\ \forall s \in \langle u\rangle\},\ \forall u \in \langle n-1\rangle,$$

$$\Delta_1 = (\mathbb{S}_n),$$

and

$$\Delta_{u+1} = \left(K_{(i_1,i_2,\ldots,i_u)}\right)_{(i_1,i_2,\ldots,i_u)\in\mathbb{A}_n^u}, \quad \forall u \in \langle n-1\rangle.$$

Obviously, we have $\Delta_n = (\{\sigma\})_{\sigma\in\mathbb{S}_n}$. Further, we show that $P_u \in G_{\Delta_u,\Delta_{u+1}}$, $\forall u \in \langle n-1\rangle$. Let $u \in \langle n-1\rangle$. Let $K \in \Delta_u$ and $L \in \Delta_{u+1}$. Let $\sigma \in K$. It follows that

$$\sum_{\tau\in L} (P_1)_{\sigma\tau} = \frac{1}{n}$$

and

$$\sum_{\tau\in L} (P_u)_{\sigma\tau} = \begin{cases} 0 & \text{if } K = K_{(i_1,i_2,\ldots,i_{u-1})},\ L = K_{(j_1,j_2,\ldots,j_u)} \\ & \text{and } \exists v \in \langle u-1\rangle \text{ such that } i_v \neq j_v, \\ \frac{1}{n-u+1} & \text{otherwise,} \end{cases}$$

if $u \geq 2$. Since $\sum\limits_{\tau\in L} (P_u)_{\sigma\tau}$ does not depend on σ, $P_u \in G_{\Delta_u,\Delta_{u+1}}$, $\forall u \in \langle n-1\rangle$. Now, by Theorem 2.10(i), Π is a stable matrix. Second, we show that $\Pi = \frac{1}{n!}e'e$. Since $(u,v)^{-1} = (u,v)$, P_u is a symmetric stochastic matrix, $\forall u \in \langle n-1\rangle$. Further, it follows that P_u is a bistochastic matrix, $\forall u \in \langle n-1\rangle$. But '$P_u$ is a bistochastic matrix, $\forall u \in \langle n-1\rangle$', implies '$\Pi$ is a bistochastic matrix'. Now, from Π is a stable and bistochastic matrix, we have $\Pi = \frac{1}{n!}e'e$.

Finally, if p_0 is a probability distribution on $\mathbb{S}_n$ and $p_{n-1} := p_0 P_1 P_2 \ldots P_{n-1}$, then

$$p_{n-1} = p_0 \Pi = p_0 \left(\frac{1}{n!} e' e \right) = \frac{1}{n!} \left(p_0 e' \right) e = \frac{1}{n!} e = \left(\frac{1}{n!}, \frac{1}{n!}, \ldots, \frac{1}{n!} \right),$$

i.e., p_{n-1} is the uniform probability distribution on $\mathbb{S}_n$.

Example 2.12. Let $\mathcal{D} = \{P_1, P_2\}$, where

$$P_1 = \begin{pmatrix} \frac{2}{8} & \frac{4}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{4}{8} & \frac{3}{8} \\ \frac{2}{8} & \frac{4}{8} & \frac{2}{8} \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{3}{6} & \frac{2}{6} \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \end{pmatrix}$$

(see [6] or [7, p. 94]). Since $P_1, P_2 \in G_{(\langle 3 \rangle),(\{1,3\},\{2\})} \cap G_{(\{1,3\},\{2\}),(\{i\})_{i\in\langle 3\rangle}}$, $\mathcal{D}$ is a 2-definite set.

Remark 2.13. (a) The linked structure

$$\left(G_{(\langle m_1 \rangle),\Delta_2}, G_{\Delta_2,\Delta_3}, \ldots, G_{\Delta_{n-1},\Delta_n}, G_{\Delta_n,(\{i\})_{i\in\langle m_{n+1}\rangle}} \right)$$

of $P_1 P_2 \ldots P_n$ from Theorem 2.10(i) is a sufficient condition for $P_1 P_2 \ldots P_n$ to be a stable matrix.

(b) The sufficient condition given in (a) for $P_1 P_2 \ldots P_n$ to be a stable matrix is not necessary. Indeed, let, e.g.,

$$P_1 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} \frac{1}{8} & \frac{4}{8} & \frac{3}{8} \\ \frac{2}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{7}{32} & \frac{13}{32} & \frac{12}{32} \end{pmatrix}.$$

Since

$$P_1 P_2 = \begin{pmatrix} \frac{7}{32} & \frac{13}{32} & \frac{12}{32} \\ \frac{7}{32} & \frac{13}{32} & \frac{12}{32} \\ \frac{7}{32} & \frac{13}{32} & \frac{12}{32} \end{pmatrix},$$

$P_1 P_2$ is a stable matrix. Suppose that $\exists \Delta \in \mathrm{Par}(\langle 3 \rangle)$ such that $P_1 \in G_{(\langle 3 \rangle),\Delta}$ and $P_2 \in G_{\Delta,(\{i\})_{i\in\langle 3\rangle}}$. By $P_1 \in G_{(\langle 3 \rangle),\Delta}$, $\Delta = (\langle 3 \rangle)$. It follows that $P_2 \in G_{(\langle 3 \rangle),(\{i\})_{i\in\langle 3\rangle}}$. We have reached a contradiction. Note that in the above example we can consider two parallel linked structures (P_1 is a reducible stochastic matrix). Indeed, if we write P_1 as

$$P_1 = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},$$

where

$$Q_1 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$

then $\left(G_{(\langle 2\rangle),(\{i\})_{i\in\langle 3\rangle}}, G_{(\{i\})_{i\in\langle 3\rangle},(\{i\})_{i\in\langle 3\rangle}}\right)$ is a linked structure of Q_1P_2 while $\left(G_{(\{3\}),(\{i\})_{i\in\langle 3\rangle}}, G_{(\{i\})_{i\in\langle 3\rangle},(\{i\})_{i\in\langle 3\rangle}}\right)$ is a linked structure of Q_2P_2. Obviously, we have

$$P_1P_2 = \begin{pmatrix} Q_1P_2 \\ Q_2P_2 \end{pmatrix}.$$

Now, see Theorem 2.6(ii) (for the products Q_1P_2 and Q_2P_2) and Remark 1.10 (for the matrix P_1P_2).

(c) There always exist at least m_1 parallel linked structures of $P_1P_2\ldots P_n$ if $P_1 \in S_{m_1,m_2}$, $P_2 \in S_{m_2,m_3}, \ldots,$ $P_n \in S_{m_n,m_{n+1}}$. Indeed, $\left(G_{(\{k\}),(\{i_2\})_{i_2\in\langle m_2\rangle}},\right.$ $\left.G_{(\{i_2\})_{i_2\in\langle m_2\rangle},(\{i_3\})_{i_3\in\langle m_3\rangle}}, \ldots, G_{(\{i_n\})_{i_n\in\langle m_n\rangle},(\{i_{n+1}\})_{i_{n+1}\in\langle m_{n+1}\rangle}}\right)$ is a linked structure of $(P_1)_{\{k\}} P_2 \ldots P_n$, $\forall k \in \langle m_1\rangle$ (these are trivial parallel linked structures).

(d) Let $\mathcal{D} = \{P_1, P_2, \ldots, P_t\} \subseteq S_m$ $(t \geq 1)$. By Theorem 2.10(i), if $\exists n \geq 2, \exists \Delta_1, \Delta_2, \ldots, \Delta_{n-1} \in \operatorname{Par}(\langle m\rangle)$ such that $P_i \in G_{(\langle m\rangle),\Delta_1} \cap G_{\Delta_1,\Delta_2} \cap \ldots \cap G_{\Delta_{n-2},\Delta_{n-1}} \cap G_{\Delta_{n-1},(\{i\})_{i\in\langle m\rangle}}$, $\forall i \in \langle t\rangle$, then $\exists k, 1 \leq k \leq n$, such that $\mathcal{D}$ is a k-definite set. (See, e.g., Example 2.12.)

(e) By (d) we have a sufficient condition for $\mathcal{D}$ to be a k-definite set. Is also this condition necessary? (see also Problem 2.14).

In the special case $P_1 = P_2 = \ldots = P_n := P$ it is possible as the linked structure of P^n to be as in the next question if P is a irreducible stochastic matrix (see Remark 2.13(b) again).

Problem 2.14. If $P \in S_m$ is a irreducible matrix and P^n is a stable matrix, then are there $\Delta_1, \Delta_2, \ldots, \Delta_{n-1} \in \operatorname{Par}(\langle m\rangle)$ such that $P \in G_{(\langle m\rangle),\Delta_1} \cap G_{\Delta_1,\Delta_2} \cap \ldots \cap G_{\Delta_{n-2},\Delta_{n-1}} \cap G_{\Delta_{n-1},(\{i\})_{i\in\langle m\rangle}}$? (A possible generalization of Problem 2.14 is: If $P_1, P_2, \ldots, P_n \in S_m$ are irreducible matrices and $P_1P_2\ldots P_n$ is a stable matrix, then are there $\Delta_1, \Delta_2, \ldots, \Delta_{n-1} \in \operatorname{Par}(\langle m\rangle)$ such that $P_1 \in G_{(\langle m\rangle),\Delta_1}$, $P_2 \in G_{\Delta_1,\Delta_2}, \ldots,$ $P_{n-1} \in G_{\Delta_{n-2},\Delta_{n-1}}$, $P_n \in G_{\Delta_{n-1},(\{i\})_{i\in\langle m\rangle}}$?)

Another way (it can also be viewed as one subway of that from Theorem 2.10(i)), to see if a product of stochastic matrices is a stable matrix, is given in the next result.

THEOREM 2.15. *Let $P_1, P_2, \ldots, P_n \in S_m$. Let $\Sigma = (K_1, K_2, \ldots, K_n) \in \operatorname{Par}(\langle m\rangle)$. If P_1 is a stable matrix on $\langle m\rangle \times K_n$, P_2 is a stable matrix on $(K_1 \cup K_2 \cup \ldots \cup K_{n-1}) \times (K_{n-1} \cup K_n)$, $\ldots$, P_{n-1} is a stable matrix on $(K_1 \cup K_2) \times (K_2 \cup K_3 \cup \ldots \cup K_n)$, and P_n is a stable matrix on $K_1 \times \langle m\rangle$, then $P_1P_2\ldots P_n$ is a stable matrix.*

Proof. Since P_1 is a stable matrix on $\langle m\rangle \times K_n$, P_2 is a stable matrix on $(K_1 \cup K_2 \cup \ldots \cup K_{n-1}) \times (K_{n-1} \cup K_n)$, $\ldots$, P_{n-1} is a stable matrix on $(K_1 \cup K_2) \times (K_2 \cup K_3 \cup \ldots \cup K_n)$, and P_n is a stable matrix on $K_1 \times \langle m\rangle$, we have

$$P_1 \in G_{(\langle m\rangle),(K_1\cup K_2\cup\ldots\cup K_{n-1},\{i\})_{i\in K_n}},$$
$$P_2 \in G_{(K_1\cup K_2\cup\ldots\cup K_{n-1},\{i\})_{i\in K_n},(K_1\cup K_2\cup\ldots\cup K_{n-2},\{i\})_{i\in K_{n-1}\cup K_n}},\ldots,$$
$$P_{n-1} \in G_{(K_1\cup K_2,\{i\})_{i\in K_3\cup\ldots\cup K_n},(K_1,\{i\})_{i\in K_2\cup K_3\cup\ldots\cup K_n}},$$

and

$$P_n \in G_{(K_1,\{i\})_{i\in K_2\cup K_3\cup\ldots\cup K_n},(\{i\})_{i\in\langle m\rangle}},$$

respectively. Now, by Theorem 2.10(i), $P_1P_2\ldots P_n$ is a stable matrix. □

Remark 2.16. (a) The condition 'P_1 is a stable matrix on $\langle m\rangle \times K_n$' from Theorem 2.15 implies that P_1 has at least a column with identical entries.

(b) Let $P \in S_m$. Suppose that $m \geq 2$. Let $\Sigma = (K_1, K_2, \ldots, K_n) \in \mathrm{Par}(\langle m\rangle)$. If $|K_1| = 1$ and P is a stable matrix on $(K_1 \cup K_2) \times (K_2 \cup K_3 \cup \ldots \cup K_n)$, then P is a stable matrix on $(K_1 \cup K_2) \times \langle m\rangle$.

(c) The number n from Theorem 2.15 can be taken at most $m - 1$ when $m \geq 2$. The proof is as follows. The worst case of Theorem 2.15 is $\Sigma = (\{i\})_{i\in\langle m\rangle}$. But, using (b), we can replace Σ by $\Sigma' = (\{1,2\}, \{i\})_{i\in\{3,4,\ldots,m\}}$.

(d) Theorem 2.15 is more restrictive than Theorem 2.10(i). This follows from the proof of Theorem 2.15 and, e.g., (a) and Example 2.11 (taking, e.g., $n = 3$).

Further, we give two example which use Theorem 2.15.

Example 2.17. Let

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{2}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

(see [4]). P is a stable matrix on $\langle 6\rangle \times \{6\}$, $\{1,2,3,4,5\} \times \{4,5,6\}$, $\{1,2,3\} \times \{3,4,5,6\}$, and $\{1,2\} \times \langle 6\rangle$. By Theorem 2.15 ($K_1 = \{1,2\}$, $K_2 = \{3\}$, $K_3 = \{4,5\}$, $K_4 = \{6\}$), we obtain that P^4 is a stable matrix. Also, by direct computation, we obtain that P^3 is a stable matrix (note that P is a reducible matrix). Indeed,

$$P^2 = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{7}{12} & \frac{1}{12} & \frac{4}{12} & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P^3 = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \\ \frac{2}{9} & \frac{4}{9} & \frac{3}{9} & 0 & 0 & 0 \end{pmatrix}.$$

From the above example, we draw the next conclusion.

Remark 2.18. If P is a reducible stochastic matrix we can have both linked structures and parallel linked structures of P^k for some $k \geq 1$. In the above example a linked structure is given by Theorem 2.15 while four parallel linked structures are obtained if we write P as

$$P = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix},$$

where

$$Q_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_4 = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{2}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

(see also Remark 1.13(b)–(c)).

Example 2.19. Let

$$P = \begin{pmatrix} \frac{\mu}{\mu+\lambda} & \frac{\lambda}{\mu+\lambda}q_1 & \frac{\lambda}{\mu+\lambda}q_2 & \cdots & \frac{\lambda}{\mu+\lambda}q_{a-2} & \frac{\lambda}{\mu+\lambda}q_{a-1} & \frac{\lambda}{\mu+\lambda}\sum\limits_{i=a}^{\infty} q_i \\ \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda}q_1 & \cdots & \frac{\lambda}{\mu+\lambda}q_{a-3} & \frac{\lambda}{\mu+\lambda}q_{a-2} & \frac{\lambda}{\mu+\lambda}\sum\limits_{i=a-1}^{\infty} q_i \\ \frac{\mu}{\mu+\lambda} & 0 & 0 & \cdots & \frac{\lambda}{\mu+\lambda}q_{a-4} & \frac{\lambda}{\mu+\lambda}q_{a-3} & \frac{\lambda}{\mu+\lambda}\sum\limits_{i=a-2}^{\infty} q_i \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \frac{\mu}{\mu+\lambda} & 0 & 0 & \cdots & 0 & \frac{\lambda}{\mu+\lambda}q_1 & \frac{\lambda}{\mu+\lambda}\sum\limits_{i=2}^{\infty} q_i \\ \frac{\mu}{\mu+\lambda} & 0 & 0 & \cdots & 0 & 0 & \frac{\lambda}{\mu+\lambda} \\ \frac{\mu}{\mu+\lambda} & 0 & 0 & \cdots & 0 & 0 & \frac{\lambda}{\mu+\lambda} \end{pmatrix}$$

(see [21]; this is the embedded Markov chain of a queueing model ($P \in S_{a+1}$, $\lambda, \mu > 0$, $\sum\limits_{i=1}^{\infty} q_i = 1$)). Set $K_1 = \{a, a+1\}$, $K_i = \{a-i+1\}$, $\forall i \in \{2, 3, \dots, a\}$. Then, by Theorem 2.15, P^a is a stable matrix. Thus, we obtained the same result as in [21, p. 259]: 'the embedded Markov chain attains its stationary distribution after at most a transitions'.

Further, we deal with a matter concerning the ergodicity coefficient $\overline{\gamma}_\Delta$.

Let $P \in R_{m,n}$. Let $\Delta \in \mathrm{Par}\left(\langle m\rangle\right)$. Define (see [11])

$$\overline{\gamma}_\Delta(P) = \frac{1}{2} \max_{\substack{K\in\Delta \\ i,j\in K}} \sum_{k=1}^{n} |P_{ik} - P_{jk}|$$

and

$$\overline{\alpha}(P) = \overline{\gamma}_{(\langle m\rangle)}(P).$$

The next result yields an important connection between G_{Δ_1,Δ_2} and the ergodicity coefficients $\overline{\gamma}_{\Delta_1}$ and $\overline{\gamma}_{\Delta_2}$.

THEOREM 2.20 ([16]; *see also* [13]). *Let* $P \in G_{\Delta_1,\Delta_2} \subseteq S_{m,n}$ *and* $Q \in S_{n,p}$. *Then*

$$\overline{\gamma}_{\Delta_1}(PQ) \le \overline{\gamma}_{\Delta_1}(P)\overline{\gamma}_{\Delta_2}(Q).$$

Proof. See [16, Theorem 1.18] (see also [13, Theorem 1.9]). □

Remark 2.21. (a) By Theorem 2.20,

$$\overline{\alpha}(PQ) \le \overline{\alpha}(P)\,\overline{\gamma}_{\Delta_2}(Q)$$

if $\Delta_1 = (\langle m\rangle)$. This is an inequality better than the well-known one, namely,

$$\overline{\alpha}(PQ) \le \overline{\alpha}(P)\,\overline{\alpha}(Q),$$

but more restrictive.

(b) If $P_1 \in G_{\Delta_1,\Delta_2} \subseteq S_{m_1,m_2}$, $P_2 \in G_{\Delta_2,\Delta_3} \subseteq S_{m_2,m_3}, \dots$, $P_{n-1} \in G_{\Delta_{n-1},\Delta_n} \subseteq S_{m_{n-1},m_n}$, $P_n \in S_{m_n,m_{n+1}}$, then, by Theorem 2.20, we have

$$\overline{\gamma}_{\Delta_1}(P_1P_2\dots P_n) \le \overline{\gamma}_{\Delta_1}(P_1)\,\overline{\gamma}_{\Delta_2}(P_2)\dots\overline{\gamma}_{\Delta_n}(P_n).$$

In particular,

$$\overline{\alpha}(P_1P_2\dots P_n) \le \overline{\alpha}(P_1)\,\overline{\gamma}_{\Delta_2}(P_2)\dots\overline{\gamma}_{\Delta_n}(P_n)$$

if $\Delta_1 = (\langle m_1\rangle)$. This is an inequality better than the well-known one, namely,

$$\overline{\alpha}(P_1P_2\dots P_n) \le \overline{\alpha}(P_1)\,\overline{\alpha}(P_2)\dots\overline{\alpha}(P_n),$$

but more restrictive.

(c) For our interest in the general Δ-ergodic theory, the inequality from Theorem 2.20 is too restrictive. Moreover, it cannot be generalized for any $P \in S_{m,n}$ (see [12, Remark 1.14]). Consequently, we need, if any, an ergodicity coefficient better than $\overline{\gamma}_\Delta$ which generalizes $\overline{\alpha}$.

The special inequality from Remark 2.21(b) can be applied, e.g., to some random walks on the symmetric group of permutations of order n. This idea is supported by Example 2.11 above and Example 2.22 below.

Example 2.22 (this refers to the top to random shuffle, namely, given a deck of n cards, remove the top card and put it back in the deck at random; here $n=3$). Let $\sigma_1=(1,2,3)$, $\sigma_2=(1,3,2)$, $\sigma_3=(2,1,3)$, $\sigma_4=(2,3,1)$, $\sigma_5=(3,1,2)$, and $\sigma_6=(3,2,1)$ be the permutations of order 3. Consider a Markov chain with state space $S=\mathbb{S}_3$, the set of permutations of order 3, and transition matrix

$$P=\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

We have $P\in G_{(\mathbb{S}_3),\Delta}$, where $\Delta=(\{\sigma_1,\sigma_6\},\{\sigma_2,\sigma_4\},\{\sigma_3,\sigma_5\})$. By Remark 2.1(b), $P\in G_\Delta$. By Remark 2.21(b),

$$\overline{\alpha}\left(P^n\right)\le\overline{\alpha}\left(P\right)\left(\overline{\gamma}_\Delta(P)\right)^{n-1}=\left(\frac{1}{3}\right)^{n-1}.$$

This inequality leads to others. Let p_n be the probability distribution of chain at time n, $\forall n\ge 0$. Let π be the limit probability distribution of chain. Since P is a bistochastic matrix, π is the uniform probability distribution, i.e., $\pi=\left(\frac{1}{6},\frac{1}{6},\dots,\frac{1}{6}\right)$. Using the equation $\pi P=\pi$ and the well-known inequality

$$\|\mu Q-\nu Q\|_1\le\|\mu-\nu\|_1\,\overline{\alpha}\left(Q\right),$$

where μ and ν are the probability distributions on $\langle m\rangle$, $Q\in S_{m,n}$, and $\|\cdot\|_1$ is the vector 1-norm (see, e.g., [13] or [16]), we have

$$\|p_n-\pi\|_1=\|p_0P^n-\pi P^n\|_1\le\|p_0-\pi\|_1\,\overline{\alpha}\left(P^n\right)\le 2\overline{\alpha}\left(P^n\right)\le 2\left(\frac{1}{3}\right)^{n-1}.$$

Set $\Pi=\lim_{n\to\infty}P^n$. Note that $\Pi=e'\pi$. Using the equation $\Pi P=\Pi$ and the well-known inequality

$$|||UQ-VQ|||_\infty\le|||U-V|||_\infty\,\overline{\alpha}\left(Q\right),$$

where $U,V\in S_{m,n}$ and $Q\in S_{n,p}$ (see, e.g., [13] or [16]), we have

$$|||P^n-\Pi|||_\infty=|||P^n-\Pi P^n|||_\infty\le|||I_6-\Pi|||_\infty\,\overline{\alpha}(P^n)\le 2\overline{\alpha}(P^n)\le 2\left(\frac{1}{3}\right)^{n-1}.$$

In connection with Remark 2.21(c) we shall give a result related to the limits of $\overline{\gamma}_{\Delta}$. We need the next result (which is closely related to Theorem 2.6(iii)).

THEOREM 2.23. *Let* $P \in S_{m,n}$ *and* $Q \in S_{n,p}$. *Let* $\Delta_1 \in \operatorname{Par}(\langle m \rangle)$. *Let* $\Delta_2 = (L_1, L_2, \ldots, L_t) \in \operatorname{Par}(\langle n \rangle)$ *with* $1 \le t \le p$. *If*

(i) $Q_{L_j}^{\{j\}} = e' = e'(|L_j|)$, $\forall j \in \langle t \rangle$ ((i) *implies that* Q *is a* Δ_2*-stable* 0-1 *matrix*);

(ii) PQ *is a* $[\Delta_1]$*-stable matrix,*

then $P \in G_{\Delta_1,\Delta_2}$.

Proof. Let $K \in \Delta_1$ and $w \in \langle t \rangle$. By (ii),

$$(PQ)_{uw} = (PQ)_{vw}, \quad \forall u, v \in K.$$

Further, by (i),

$$(PQ)_{uw} = \sum_{k \in \langle n \rangle} P_{uk} Q_{kw} = \sum_{k \in L_w} P_{uk} Q_{kw} = \sum_{k \in L_w} P_{uk}, \quad \forall u \in K,$$

so that

$$\sum_{k \in L_w} P_{uk} = \sum_{k \in L_w} P_{vk}, \quad \forall u, v \in K.$$

Consequently, $P \in G_{\Delta_1,\Delta_2}$. $\square$

Define

$$\Gamma_{\Delta_1,\Delta_2} = \Gamma_{\Delta_1,\Delta_2}(m,n,p) =$$
$$= \left\{ P \mid P \in S_{m,n} \text{ and } \overline{\gamma}_{\Delta_1}(PQ) \le \overline{\gamma}_{\Delta_1}(P)\overline{\gamma}_{\Delta_2}(Q),\ \forall Q \in S_{n,p} \right\},$$

where $\Delta_1 \in \operatorname{Par}(\langle m \rangle)$ and $\Delta_2 \in \operatorname{Par}(\langle n \rangle)$ with $1 \le |\Delta_2| \le p$, and, if $m = n$,

$$\Gamma_{\Delta} = \Gamma_{\Delta,\Delta}.$$

THEOREM 2.24. *We have*

$$\Gamma_{\Delta_1,\Delta_2} = G_{\Delta_1,\Delta_2}.$$

Proof. By Theorem 2.20,

$$G_{\Delta_1,\Delta_2} \subseteq \Gamma_{\Delta_1,\Delta_2}.$$

Let $Q \in S_{n,p}$. Suppose that Q is the same as in Theorem 2.23. Set

$$\Gamma_{\Delta_1,\Delta_2,Q} = \Gamma_{\Delta_1,\Delta_2,Q}(m,n,p) =$$
$$= \left\{ P \mid P \in S_{m,n} \text{ and } \overline{\gamma}_{\Delta_1}(PQ) \le \overline{\gamma}_{\Delta_1}(P)\overline{\gamma}_{\Delta_2}(Q) \right\}.$$

Obviously, $\Gamma_{\Delta_1,\Delta_2} \subseteq \Gamma_{\Delta_1,\Delta_2,Q}$. Further, we show that $\Gamma_{\Delta_1,\Delta_2,Q} \subseteq G_{\Delta_1,\Delta_2}$. Let $P \in \Gamma_{\Delta_1,\Delta_2,Q}$. Then

$$\overline{\gamma}_{\Delta_1}(PQ) \le \overline{\gamma}_{\Delta_1}(P)\overline{\gamma}_{\Delta_2}(Q) = 0$$

because Q is a Δ_2-stable matrix. Therefore, $\overline{\gamma}_{\Delta_1}(PQ) = 0$, i.e., PQ is a $[\Delta_1]$-stable matrix (see also Remark 2.26(a) below). Now, by Theorem 2.23, $P \in G_{\Delta_1,\Delta_2}$. Consequently, $\Gamma_{\Delta_1,\Delta_2,Q} \subseteq G_{\Delta_1,\Delta_2}$. Finally, from

$$G_{\Delta_1,\Delta_2} \subseteq \Gamma_{\Delta_1,\Delta_2} \subseteq \Gamma_{\Delta_1,\Delta_2,Q} \subseteq G_{\Delta_1,\Delta_2},$$

we have

$$\Gamma_{\Delta_1,\Delta_2} = G_{\Delta_1,\Delta_2} \quad (\Gamma_{\Delta_1,\Delta_2} = G_{\Delta_1,\Delta_2} = \Gamma_{\Delta_1,\Delta_2,Q}). \qquad \square$$

Remark 2.25. Theorem 2.24 says that $P \in G_{\Delta_1,\Delta_2}$ if and only if

$$\overline{\gamma}_{\Delta_1}(PQ) \leq \overline{\gamma}_{\Delta_1}(P)\overline{\gamma}_{\Delta_2}(Q), \ \forall Q \in S_{n,p} \text{ with } 1 \leq |\Delta_2| \leq p.$$

(In particular, we have $P \in G_{\Delta}$ if and only if

$$\overline{\gamma}_{\Delta}(PQ) \leq \overline{\gamma}_{\Delta}(P)\overline{\gamma}_{\Delta}(Q), \ \forall Q \in S_m \ (m = n = p).)$$

The implication "⇒" is closely related to Theorem 2.20. By implication "⇐", if we need the inequality $\overline{\gamma}_{\Delta_1}(PQ) \leq \overline{\gamma}_{\Delta_1}(P)\overline{\gamma}_{\Delta_2}(Q)$ and we only know that $Q \in S_{n,p}$ with $1 \leq |\Delta_2| \leq p$, we are limited to the case $P \in G_{\Delta_1,\Delta_2}$.

Remark 2.26. (a) $P \in G_{\Delta_1,\Delta_2}$ if and only if $\overline{\gamma}_{\Delta_1}\left(P^{+\Delta_2}\right) = 0$.

(b) If $P_1 \in G_{(\langle m_1 \rangle),\Delta_2} \subseteq S_{m_1,m_2}$, $P_2 \in G_{\Delta_2,\Delta_3} \subseteq S_{m_2,m_3}, \ldots, P_{n-1} \in G_{\Delta_{n-1},\Delta n} \subseteq S_{m_{n-1},m_n}$, $P_n \in G_{\Delta n,(\{i\})_{i\in\langle m_{n+1}\rangle}} \subseteq S_{m_n,m_{n+1}}$, and $\Pi = P_1P_2\ldots P_n$ (the above conditions implies that Π is a stable matrix), then, by Theorem 2.20, we have ($\overline{\gamma}_{(\langle m_1 \rangle)} = \overline{\alpha}$)

$$\overline{\alpha}\left(P_1P_2\right) \leq \overline{\alpha}\left(P_1\right)\overline{\gamma}_{\Delta_2}\left(P_2\right),$$
$$\overline{\alpha}\left(P_1P_2P_3\right) \leq \overline{\alpha}\left(P_1\right)\overline{\gamma}_{\Delta_2}\left(P_2\right)\overline{\gamma}_{\Delta_3}\left(P_3\right), \ldots,$$
$$\overline{\alpha}\left(P_1P_2\ldots P_{n-1}\right) \leq \overline{\alpha}\left(P_1\right)\overline{\gamma}_{\Delta_2}\left(P_2\right)\ldots\overline{\gamma}_{\Delta_{n-1}}\left(P_{n-1}\right),$$
$$0 = \overline{\alpha}\left(\Pi\right) = \overline{\alpha}\left(P_1P_2\ldots P_n\right) \leq \overline{\alpha}\left(P_1\right)\overline{\gamma}_{\Delta_2}\left(P_2\right)\ldots\overline{\gamma}_{\Delta_n}\left(P_n\right) = 0$$

(by Remark 2.26(a), $\overline{\gamma}_{\Delta_n}\left(P_n\right) = 0$). The above inequalities in turn measure the closeness of rows of matrices $P_1P_2, P_1P_2P_3, \ldots, P_1P_2\ldots P_{n-1}, P_1P_2\ldots P_n$ ($\overline{\alpha}(P)$ is a measure of the closeness of rows of matrix P, where $P \in S_{m,n}$). Note also that

$$0 = \overline{\alpha}\left(P_1P_2\ldots P_n\right) \leq \overline{\alpha}\left(P_1P_2\ldots P_{n-1}\right) \leq \ldots \leq \overline{\alpha}\left(P_1P_2\right) \leq \overline{\alpha}\left(P_1\right).$$

Obviously, the above things can be generalized for $[\Delta]$-stable products of stochastic matrices.

We conclude this article with a challenging remark.

Remark 2.27. An important and hard problem in the theory of finite Markov chains refers to their speed of convergence, especially when the state space is very large. Therefore, it is an important and interesting task to find more general sets than G_{Δ_1,Δ_2} to give necessary and/or sufficient structure theorems in the next cases ($P_1, P_2, \ldots, P_n \in S_m$ and $0 \leq \varepsilon \leq 1$):

1. $|||P_1P_2 \dots P_n - \Pi|||_\infty \leq \varepsilon$, where $\Pi \in S_m$ is a stable matrix or, more generally, a $[\Delta]$-stable matrix;

2. $\overline{\gamma}_\Delta (P_1P_2 \dots P_n) \leq \varepsilon$.

Both cases were considered in this article when $\varepsilon = 0$. Concerning $\varepsilon > 0$ see also Theorem 2.20, Remark 2.21, Example 2.22, Theorem 2.24, and Remarks 2.25–26.

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