We consider a quasistatic frictionless contact problem for viscoelastic bodies with long memory. The contact is modelled with normal compliance in such a way that the penetration is limited and restricted to unilateral constraints. The adhesion between contact surfaces is taken into account and the evolution of the bonding field is described by a first order differential equation. We derive a variational formulation of the mechanical problem and we establish an existence and uniqueness result by using arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem. Moreover, using compactness properties we study a regularized problem which has a unique solution and we obtain the solution of the original model by passing to the limit as the regularization parameter converges to zero.

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Key words: viscoelastic, normal compliance, adhesion, frictionless, contact, weak solution, fixed point.

1. INTRODUCTION

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws lead to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study contact problems within the framework of variational inequalities was made in [6]. The mathematical, mechanical and numerical state of the art can be found in [19]. In this reference we find results on the mathematical analysis and numerical studies of various adhesive contact problems. Recently a new book [21] introduces the reader into the theory of variational inequalities with emphasis on the study of contact mechanics and more specifically, on antiplane frictional contact problems. Also, recently existence results for the continuous case were established in [1, 5, 7, 16] in the study of unilateral
and frictional contact problems for linear elastic materials. In this paper as in [13] we study a mathematical model which describes a frictionless and adhesive contact problem between a viscoelastic body with long memory and a foundation. The elasticity operator is assumed to vanish for zero strain, to be Lipschitz continuous and strictly monotone. As in [12] the contact is modeled with normal compliance in such a way that the penetration is limited and restricted to unilateral constraints. We recall that models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and a foundation have been studied in [3, 4, 8, 18, 19, 20, 22]. Following [9, 10] we use the bonding field as an additional state variable $\beta$, defined on the contact surface of the boundary. The variable satisfies the restrictions $0 \leq \beta \leq 1$. At a point on the boundary contact surface, when $\beta = 1$ the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction $\beta$ of the bonds is active. We refer the reader to the extensive bibliography on the subject in [2, 9, 10, 11, 15, 17, 18, 19]. According to [12], the method presented here considers a compliance model in which the compliance term doesn’t represent necessarily a compact perturbation of the original problem without contact. This leads to study such models, where a strictly limited penetration is permitted to perform the limit procedure to the Signorini contact problem. In this work as in [23] we extend the result established in [22] to the unilateral contact problem with a modified normal compliance when the penetration is finite and the adhesion between contact surfaces is taken into account. We derive a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution, and obtain a partial regularity result for the solution. Moreover, we study a regularized problem which we consider as a frictionless contact problem and adhesion with unlimited penetration. We prove its unique weak solvability and show that the solution of the original model is obtained by passing to the limit as the regularization parameter converges to zero.

The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 3.1. Finally, in Section 4, we prove a convergence result of a regularized problem, Theorem 4.2.

2. PROBLEM STATEMENT
AND VARIATIONAL FORMULATION

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), be a domain initially occupied by a viscoelastic body with long memory. $\Omega$ is supposed to be open, bounded, with a sufficiently regular boundary $\Gamma$. We assume that $\Gamma$ is composed of three sets $\Gamma_1, \Gamma_2,$ and
\[ \Gamma_3, \text{ with the mutually disjoint relatively open sets } \Gamma_1, \Gamma_2 \text{ and } \Gamma_3, \text{ such that } \text{meas}(\Gamma_1) > 0. \text{ The body is acted upon by a volume force of density } \varphi_1 \text{ on } \Omega \text{ and a surface traction of density } \varphi_2 \text{ on } \Gamma_2. \text{ On } \Gamma_3 \text{ the body is in adhesive frictionless contact with a foundation.} \]

Thus, the classical formulation of the mechanical problem is written as follows.

**Problem** \( P_1. \) Find a displacement \( u : \Omega \times [0,T] \to \mathbb{R}^d \) and a bonding field \( \beta : \Gamma_3 \times [0,T] \to [0,1] \) such that

\[
\begin{align*}
(2.1) & \quad \sigma(t) = A \varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s) \varepsilon(u(s)) \, ds \text{ in } \Omega, \\
(2.2) & \quad \text{div } \sigma(t) + \varphi_1(t) = 0 \text{ in } \Omega, \\
(2.3) & \quad u(t) = 0 \text{ on } \Gamma_1, \\
(2.4) & \quad \sigma(t)\nu = \varphi_2(t) \text{ on } \Gamma_2, \\
(2.5) & \quad \begin{cases} 
    u_\nu(t) \leq g, & \sigma_\nu(t) + p(u_\nu(t)) - c_\nu \beta^2(t) R_\nu(u_\nu(t)) \leq 0 \\
    \sigma_\nu(t) + p(u_\nu(t)) - c_\nu \beta^2(t) R_\nu(u_\nu(t)) (u_\nu(t) - g) = 0 
\end{cases} \text{ on } \Gamma_3, \\
(2.6) & \quad \sigma_\tau(t) = 0 \text{ on } \Gamma_3, \\
(2.7) & \quad \dot{\beta}(t) = -(c_\nu \beta(t) (R_\nu(u_\nu(t)))^2 - \varepsilon_a) + \text{ on } \Gamma_3, \\
(2.8) & \quad \beta(0) = \beta_0 \text{ on } \Gamma_3.
\end{align*}
\]

Equation (2.1) represents the viscoelastic constitutive law of the material in which \( A \) and \( \mathcal{F} \) denote the elasticity operator and the relaxation fourth-order respectively. Equation (2.2) represents the equilibrium equation while (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which \( \nu \) denotes the unit outward normal vector on \( \Gamma \) and \( \sigma \nu \) represents the Cauchy stress vector. Condition (2.5) represents the unilateral contact with adhesion in which \( c_\nu \) is a given adhesion coefficient which may dependent on \( x \in \Gamma_3 \) and \( R_\nu \) is the truncation operator defined by

\[
R_\nu(s) = \begin{cases} 
    L & \text{if } s < -L, \\
    -s & \text{if } -L \leq s \leq 0, \\
    0 & \text{if } s > 0.
\end{cases}
\]

Here \( L > 0 \) is the characteristic length of the bond, beyond which it does not offer any additional traction (see [20]) and \( p \) is a normal compliance function which satisfies the assumption (2.17); \( g \) denotes the maximum value of the penetration which satisfies \( g \geq 0. \) When \( u_\nu < 0, \) i.e., when there is separation
between the body and the foundation, condition (2.5) combined with assumption (2.17) and definition of $R_{\nu}$ implies that $\sigma_{\nu} = c_{\nu}\beta^2 R_{\nu}(u_{\nu})$ and does not exceed the value $L \|c_{\nu}\|_{L^\infty(\Gamma_3)}$. When $g > 0$, the body may interpenetrate into the foundation, but the penetration is limited that is $u_{\nu} \leq g$. In this case of penetration (i.e., $u_{\nu} \geq 0$), when $0 \leq u_{\nu} < g$ then $-\sigma_{\nu} = p(u_{\nu})$ which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_{\nu} \leq 0$. Since $p$ is an increasing function then the reaction is increasing with the penetration. When $u_{\nu} = g$ then $-\sigma_{\nu} \geq p(g)$ and $\sigma_{\nu}$ is not uniquely determined. When $g > 0$ and $p = 0$, from (2.5) we recover the contact conditions

$$u_{\nu} \leq g, \quad \sigma_{\nu} - c_{\nu}\beta^2 R_{\nu}(u_{\nu}) \leq 0, \quad (\sigma_{\nu} - c_{\nu}\beta^2 R_{\nu}(u_{\nu}))(u_{\nu} - g) = 0.$$ 

When $g = 0$, conditions (2.5) combined with hypothesis (2.16) lead to the contact conditions

$$u_{\nu} \leq 0, \quad \sigma_{\nu} - c_{\nu}\beta^2 R_{\nu}(u_{\nu}) \leq 0, \quad (\sigma_{\nu} - c_{\nu}\beta^2 R_{\nu}(u_{\nu}))u_{\nu} = 0.$$ 

These contact conditions were used in [20, 22] to model the unilateral contact with adhesion. It follows from (2.5) that there is no penetration between the body and the foundation, since $u_{\nu} \leq 0$ during the process. Also, note that when the bonding field vanishes, then the contact conditions (2.5) become the classical Signorini contact conditions with zero gap function, that is,

$$u_{\nu} \leq 0, \quad \sigma_{\nu} \leq 0, \quad \sigma_{\nu}u_{\nu} = 0.$$ 

Equation (2.6) represents a frictionless contact condition and shows that the tangential stress vanishes on the contact surface during the process. Also it means that the glue does not provide any resistance to the tangential motion of the body on the foundation. Equation (2.7) represents the ordinary differential equation which describes the evolution of the bonding field, in which $r_+ = \max\{r, 0\}$, and it was already used in [3]. Since $\beta \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [14] it must be pointed out clearly that condition (2.7) does not allow for complete debonding in finite time. Finally, (2.8) is the initial condition, in which $\beta_0$ denotes the initial bonding field. In (2.7) a dot above a variable represents its derivative with respect to time. We denote by $S_d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$) while $\| \cdot \|$ represents the Euclidean norm on $\mathbb{R}^d$ and $S_d$. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $\|v\| = (v.v)^{\frac{1}{2}}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $\|\tau\| = (\tau.\tau)^{\frac{1}{2}}$. Here and below, the indices $i$ and $j$ run between 1 and $d$ and the summation convention over repeated indices is adopted. Now, to proceed
with the variational formulation, we need the function spaces

\[ H = (L^2(\Omega))^d, \quad H_1 = (H^1(\Omega))^d, \]

\[ Q = \{ \tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega) \}, \quad Q_1 = \{ \sigma \in Q; \text{div} \sigma \in H \}. \]

Note that \( H \) and \( Q \) are real Hilbert spaces endowed with the respective canonical inner products

\[ (u,v)_H = \int_{\Omega} u_i v_i dx, \quad \langle \sigma, \tau \rangle_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} dx. \]

The strain tensor is

\[ \varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i}); \]

\( \text{div} \sigma = (\sigma_{ij,j}) \) is the divergence of \( \sigma \). For every \( v \in H_1 \) we denote by \( v_\nu \) and \( v_\tau \) the normal and tangential components of \( v \) on the boundary \( \Gamma \) given by

\[ v_\nu = v.\nu, \quad v_\tau = v - v_\nu. \]

Also, we denote by \( \sigma_\nu \) and \( \sigma_\tau \) the normal and the tangential traces of a function \( \sigma \in Q_1 \), and when \( \sigma \) is a regular function then

\[ \sigma_\nu = (\sigma \nu) \nu, \quad \sigma_\tau = \sigma - \sigma_\nu, \]

and the following Green's formula holds

\[ \langle \sigma, \varepsilon(u) \rangle_Q + (\text{div} \sigma, v)_H = \int_{\Gamma} \sigma_\nu v_\nu da \quad \forall v \in H_1, \]

where \( da \) is the surface measure element. Now, let \( V \) be the closed subspace of \( H_1 \) defined by

\[ V = \{ v \in H_1; v = 0 \text{ on } \Gamma_1 \}, \]

and let the convex subset of admissible displacements given by

\[ K = \{ v \in V; v_\nu \leq g \text{ a.e. on } \Gamma_3 \}. \]

Since \( \text{meas}(\Gamma_1) > 0 \), Korn’s inequality [6], holds

\[ \| \varepsilon(v) \|_Q \geq c_\Omega \| v \|_{H_1} \quad \forall v \in V, \]

where \( c_\Omega > 0 \) is a constant which depends only on \( \Omega \) and \( \Gamma_1 \). We equip \( V \) with the inner product

\[ (u,v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Q \]

and \( \| \cdot \|_V \) is the associated norm. It follows from Korn’s inequality (2.7) that the norms \( \| \cdot \|_{H_1} \) and \( \| \cdot \|_V \) are equivalent on \( V \). Then \((V, \| \cdot \|_V)\) is a real Hilbert space. Moreover by Sobolev’s trace theorem, there exists \( d_\Omega > 0 \) which only depends on the domain \( \Omega \), \( \Gamma_1 \) and \( \Gamma_3 \) such that

\[ \| v \|_{(L^2(\Gamma_3))^d} \leq d_\Omega \| v \|_V \quad \forall v \in V. \]
For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1, \infty}(0, T; V)$ equipped with the norm

$$
\|v\|_{W^{1, \infty}(0, T; V)} = \|v\|_{L^\infty(0, T; V)} + \|\dot{v}\|_{L^\infty(0, T; V)}.
$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to $X$; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$
\|x\|_{C([0, T]; X)} = \max_{t \in [0, T]} \|x(t)\|_X.
$$

We suppose that the body forces and surface tractions have the regularity

$$
\varphi_1 \in C([0, T]; H), \quad \varphi_2 \in C([0, T]; (L^2(\Gamma_2))^d).
$$

Next, we denote by $f : [0, T] \rightarrow V$ the function defined by

$$
(f(t), v)_V = \int_{\Omega} \varphi_1(t).v dx + \int_{\Gamma_2} \varphi_2(t).v da \quad \forall v \in V, \ t \in [0, T],
$$

and we note that (2.11) and (2.12) imply

$$
f \in C([0, T]; V).
$$

In the study of the mechanical problem $P_1$ we assume that the elasticity operator $A : \Omega \times S_d \rightarrow S_d$, satisfies

$$
(2.13) \begin{cases}
(a) \text{ there exists } M > 0 \text{ such that } \|A(x, \varepsilon_1) - A(x, \varepsilon_2)\| \leq M \|\varepsilon_1 - \varepsilon_2\| \\
\text{ for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d, \ a.e. \ x \text{ in } \Omega; \\
(b) \text{ there exists } m > 0 \text{ such that } (A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m \|\varepsilon_1 - \varepsilon_2\|^2, \\
\text{ for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d, \ a.e. \ x \text{ in } \Omega; \\
(c) \text{ the mapping } x \rightarrow A(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for any } \varepsilon \text{ in } S_d; \\
(d) A(x, 0) = 0 \text{ for a.e. } x \text{ in } \Omega.
\end{cases}
$$

Also, we need to introduce the space of the tensors of fourth order defined by

$$
Q_\infty = \{ \mathcal{E} = (\mathcal{E}_{ijkl}); \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega) \},
$$

which is the real Banach space with the norm

$$
\|\mathcal{E}\|_{Q_\infty} = \max_{0 \leq i, j, k, l \leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.
$$

We assume that the tensor of relaxation $\mathcal{F}$ satisfies

$$
(2.14) \quad \mathcal{F} \in C([0, T]; Q_\infty).
$$

The adhesion coefficients satisfy

$$
(2.15) \quad c_{\nu}, \varepsilon_a \in L^\infty(\Gamma_3) \text{ and } c_{\nu}, \varepsilon_a \geq 0 \text{ a.e. on } \Gamma_3,
$$
and we assume that the initial bonding field satisfies
\begin{equation}
(2.16) \quad \beta_0 \in L^2(\Gamma_3); \quad 0 \leq \beta_0 \leq 1 \text{ a.e. on } \Gamma_3.
\end{equation}

Next, we define the functional \( j : L^2(\Gamma_3) \times V \times V \rightarrow \mathbb{R} \) by
\begin{equation}
(2.17) \quad j(\beta, u, v) = \int_{\Gamma_3} \left( p(u_v) - c_p \beta^2 R_v(u_v) \right) v \, da \quad \forall \beta \in L^2(\Gamma_3), \forall u, v \in V.
\end{equation}

As in [12] we assume that the normal compliance function \( p \) satisfies
\begin{equation}
(2.18) \quad j(\beta_1, u_1, u_2 - u_1) + j(\beta_2, u_2, u_1 - u_2) \leq c \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} \|u_1 - u_2\|_V,
\end{equation}
\begin{equation}
(2.19) \quad j(\beta, u_1, u_2 - u_1) + j(\beta, u_2, u_1 - u_2) \leq 0,
\end{equation}
\begin{equation}
(2.20) \quad j(\beta, u_1, v) - j(\beta, u_2, v) \leq c \|u_1 - u_2\|_V \|v\|_V,
\end{equation}
where \( c \) is a positive constant.

Finally, we need to introduce the set
\begin{equation}
(2.21) \quad j(\beta, v, v) \geq 0.
\end{equation}

Following [19], the properties satisfied by the functional \( j \) are
\begin{equation}
(2.22) \quad \beta(t) = - \left( c_p \beta(t)(R_v(u_v(t)))^2 - \varepsilon_a \right)_+ \text{ a.e. } t \in (0, T),
\end{equation}
\begin{equation}
(2.23) \quad \beta(0) = \beta_0.
\end{equation}

We now use Green’s formula and techniques similar to those presented in [20] to obtain the following variational formulation of the problem \( P_1 \).

**Problem \( P_2 \).** Find a displacement field \( u \in C([0, T]; V) \) and a bonding field \( \beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B \) such that
\begin{equation}
(2.22) \quad u(t) \in K, \quad (A\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)))_Q +
\end{equation}
\begin{equation}
(2.23) \quad + \left( \int_0^t \mathcal{F}(t-s) \varepsilon(u(s)) \, ds, \varepsilon(v) - \varepsilon(u(t)) \right)_Q +
\end{equation}
\begin{equation}
(2.24) \quad + j(\beta(t), u(t), v - u(t)) \geq (f(t), v - u(t))_V \quad \forall v \in K, \quad t \in [0, T],
\end{equation}
\begin{equation}
(2.25) \quad \dot{\beta}(t) = - \left( c_p \beta(t)(R_v(u_v(t)))^2 - \varepsilon_a \right)_+ \text{ a.e. } t \in (0, T),
\end{equation}
\begin{equation}
(2.26) \quad \beta(0) = \beta_0.
\end{equation}
3. EXISTENCE AND UNIQUENESS RESULT

The main result in this section is the following existence and uniqueness theorem.

**Theorem 3.1.** Let (2.11), (2.13), (2.14), (2.15), (2.16), and (2.17) hold. Then Problem $P_2$ has a unique solution.

The proof of Theorem 2.1 is carried out in several steps. In the first step, let $k > 0$ and consider the space $X$ defined as

$$X = \left\{ \beta \in C([0, T]; L^2(\Gamma_3)) : \sup_{t \in [0, T]} \left[ \exp(-kt) \| \beta(t) \|_{L^2(\Gamma_3)} \right] < +\infty \right\}.$$ 

It is well known that $X$ is a Banach space with the norm

$$\| \beta \|_X = \sup_{t \in [0, T]} \left[ \exp(-kt) \| \beta(t) \|_{L^2(\Gamma_3)} \right].$$

Next for a given $\beta \in X$, we consider the following variational problem.

**Problem** $P_{1\beta}$. Find $u_{\beta} \in C([0, T]; V)$ such that

$$u_{\beta}(t) \in K, \langle A \varepsilon(u_{\beta}(t)), \varepsilon(v) - \varepsilon(u_{\beta}(t)) \rangle_Q +$$

$$+ \left( \int_0^t \mathcal{F}(t-s) \varepsilon(u_{\beta}(s)) \, ds, \varepsilon(v) - \varepsilon(u_{\beta}(t)) \right)_Q + j(u_{\beta}(t), v) -$$

$$- j(u_{\beta}(t), u_{\beta}(t)) \geq (f(t), v - u_{\beta}(t))_V \quad \forall v \in K, \ t \in [0, T].$$

We have the following result.

**Proposition 3.2.** Problem $P_{1\beta}$ has a unique solution.

For the proof of this proposition we consider the following intermediate problem.

**Problem** $P_{1\beta\eta}$. For $\eta \in C([0, T]; Q)$, find $u_{\beta\eta} \in C([0, T]; V)$ such that

$$u_{\beta\eta}(t) \in K, \langle A \varepsilon(u_{\beta\eta}(t)), \varepsilon(v) - \varepsilon(u_{\beta\eta}(t)) \rangle_Q +$$

$$+ \left( \int_0^t \mathcal{F}(t-s) \varepsilon(u_{\beta\eta}(s)) \, ds, \varepsilon(v) - \varepsilon(u_{\beta\eta}(t)) \right)_Q + j(u_{\beta\eta}(t), v) -$$

$$- j(u_{\beta\eta}(t), u_{\beta\eta}(t)) \geq (f(t), v - u_{\beta\eta}(t))_V \quad \forall v \in K, \ t \in [0, T].$$

We have the following result.

**Lemma 3.3.** Problem $P_{1\beta\eta}$ has a unique solution.

**Proof.** Riesz’s representation theorem leads to the existence of an element $f_{\eta} \in C([0, T]; V)$ such that

$$(f_{\eta}(t), v)_V = (f(t), v)_V - \langle \eta, \varepsilon(v) \rangle_Q.$$
Let $t \in [0, T]$ and let $A_t : V \to V$ be the operator defined by

$$(A_t u, v)_V = (A\varepsilon (u), \varepsilon (v))_Q + j (\beta (t), u, v) \quad \forall u, v \in V.$$  

Using the hypotheses on $A$ and (2.18)–(2.21) we see that $A_t$ is strongly monotone and Lipschitz continuous. Then from [21], since $K$ is a nonempty closed convex subset of $V$, using the standard results for elliptic variational inequalities, we deduce that there exists a unique element $u_{\beta \eta}(t) \in K$ which satisfies the inequality

$$(3.3) \quad (A\varepsilon (u_{\beta \eta}(t)), \varepsilon (v - u_{\beta \eta}(t)))_Q + j (\beta (t), u_{\beta \eta}(t), v - u_{\beta \eta}(t)) \geq (f_\eta(t), v - u_{\beta \eta}(t))_V \quad \forall v \in K, \ t \in [0, T],$$

and then it satisfies (3.2). To show that $u_{\beta \eta} \in C ([0, T]; V)$, it suffices to see [19]. □

Now, to end the proof we need to introduce the operator

$$\Lambda_\beta : C ([0, T]; Q) \to C ([0, T]; Q)$$

defined by

$$(3.4) \quad \Lambda_\beta \eta(t) = \int_0^t F (t - s) \varepsilon (u_{\beta \eta}(s)) \, ds \quad \forall \eta \in C ([0, T]; Q), \ t \in [0, T].$$

**Lemma 3.4.** The operator $\Lambda_\beta$ has a unique fixed point $\eta_\beta$.

**Proof.** Let $\eta_1, \eta_2 \in C ([0, T]; Q)$. Relations (3.3), (3.4) and assumption (2.14) on $F$ imply that there exists a constant $c > 0$ such that

$$\|\Lambda_\beta \eta_1(t) - \Lambda_\beta \eta_2(t)\|_Q \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_Q \, ds \quad \forall t \in [0, T].$$

We deduce that for a positive integer $m$, $\Lambda_\beta^m$ is a contraction; then it admits a unique fixed point $\eta_\beta$ which is a unique fixed point of $\Lambda_\beta$, i.e.,

$$(3.5) \quad \Lambda_\beta \eta_\beta(t) = \eta_\beta (t) \quad \forall t \in [0, T].$$

Then by (3.3) and (3.5) we conclude that $u_{\beta \eta_\beta}$ is the unique solution of (3.1) and Proposition 3.2 is proved. □

Denote $u_\beta = u_{\beta \eta_\beta}$. In the next step we consider the following problem.

**Problem $P_{2\beta}$.** Find $\beta^* : [0, T] \to L^\infty (\Gamma_3)$ such that

$$(3.6) \quad \dot{\beta}^*(t) = - \left( c_\nu \beta^* (t) (R_\nu (u_{\beta \nu}(t)))^2 - \varepsilon_a \right)_+ \quad a.e. \ t \in (0, T),$$

$$(3.7) \quad \beta^* (0) = \beta_0.$$  

We obtain the following result.


**Lemma 3.5.** Problem $P_{2\beta}$ has a unique solution $\beta^*$ which satisfies

$$
\beta^* \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B.
$$

**Proof.** Consider the mapping $T:X \to X$ given by

$$
T\beta(t) = \beta_0 - \int_0^t (c_\nu \beta(s) (R_\nu (u_{\beta\nu}(s)))^2 - \varepsilon \alpha)_+ \, ds,
$$

where $u_\beta$ is the solution of Problem $P_{1\beta}$. Then for $\beta_1, \beta_2 \in X$, we have

$$
\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \leq 
$$

$$
\int_0^t \left\| (c_\nu \beta_1(s)(R_\nu(u_{\beta\nu}(s)))^2 - \varepsilon \alpha)_+ - (c_\nu \beta_2(s)(R_\nu(u_{\beta\nu}(s)))^2 - \varepsilon \alpha)_+ \right\|_{L^2(\Gamma_3)} \, ds.
$$

Using the definition of $R_\nu$ and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we deduce as in [19] that there exists a constant $c > 0$ such that

$$
\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} \, ds.
$$

On the other hand, we have

$$
\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} \, ds \leq \|\beta_1 - \beta_2\|_X \frac{\exp(kt)}{k}.
$$

Therefore,

$$
\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \leq c \|\beta_1 - \beta_2\|_X \frac{\exp(kt)}{k} \quad \forall t \in [0, T],
$$

which yields

$$
\exp(-kt) \|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \leq \frac{c}{k} \|\beta_1 - \beta_2\|_X \quad \forall t \in [0, T].
$$

Hence

$$
(3.8) \quad \|T\beta_1 - T\beta_2\|_X \leq \frac{c}{k} \|\beta_1 - \beta_2\|_X.
$$

Inequality (3.8) shows that for $k > c$, $T$ is a contraction. Then it has a unique fixed point $\beta^*$ which satisfies (3.6) and (3.7). To prove that $\beta^* \in B$, it suffices to see [20, Remark 3.1]. Finally, as in [20, 22], we conclude that

$$
(u_{\beta^*}, \beta^*) \in C([0, T]; K) \times (W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B)
$$

is the unique solution to Problem $P_2$ and Theorem 3.1 is proved. \[\square]\
4. THE REGULARIZED PROBLEM

In this section we study a frictionless contact problem with normal compliance and adhesion with unlimited penetration. The contact condition (2.5) is replaced by the contact condition
\[-\sigma_\delta \nu = p_\delta (u_\delta \nu) - c_\nu \beta^2 R_\nu (u_\delta \nu) \quad \text{on} \quad \Gamma_3 \times (0, T),\]
where as in [12] the regularized functional $p_\delta : \mathbb{R} \to \mathbb{R}$ is defined by
\[p_\delta (r) = \begin{cases} p(r) & \text{if } r \leq g, \\ r - g + \frac{p(g)}{\delta} & \text{if } r > g, \end{cases}\]
and $\delta > 0$ denotes the regularization parameter. We study the behavior of the solution as $\delta \to 0$ and prove that in the limit we obtain the solution of adhesive frictionless contact problem with normal compliance and finite penetration. Next, we define the functional $j_\delta : L^2 (\Gamma_3) \times V \times V \to \mathbb{R}$ by
\[j_\delta (\beta, u, v) = \int_{\Gamma_3} (p_\delta (u_\nu \nu) - c_\nu \beta^2 R_\nu (u_\nu \nu)) v_\nu \, da \quad \forall \beta \in L^2 (\Gamma_3), \quad \forall u, v \in V.\]
With this notation, the formulation of the regularized problem with frictionless contact and adhesion is the following.

Problem $P_{1,\delta}$. Find a displacement field $u_\delta : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta_\delta : \Gamma_3 \times [0, T] \to [0, 1]$ such that
\[
\begin{align*}
(4.1) \quad & \sigma_\delta (t) = A \varepsilon (u_\delta (t)) + \int_0^t F(t - s) \varepsilon (u_\delta (s)) \, ds \quad \text{in} \quad \Omega, \\
(4.2) \quad & \text{div} \sigma_\delta (t) + \varphi_1 (t) = 0 \quad \text{in} \quad \Omega, \\
(4.3) \quad & u_\delta (t) = 0 \quad \text{on} \quad \Gamma_1, \\
(4.5) \quad & \sigma_\delta (t) \nu = \varphi_2 (t) \quad \text{on} \quad \Gamma_2, \\
(4.6) \quad & -\sigma_\delta \nu = p_\delta (u_\delta \nu (t)) - c_\nu \beta^2 (t) R_\nu (u_\delta \nu (t)) \quad \text{on} \quad \Gamma_3, \\
(4.7) \quad & \dot{\beta}_\delta (t) = - (c_\nu \beta_\delta (t) (R_\nu (u_\delta \nu (t)))^2 - \varepsilon_\alpha)_+ \quad \text{on} \quad \Gamma_3, \\
(4.8) \quad & \beta_\delta (0) = \beta_0 \quad \text{on} \quad \Gamma_3.
\end{align*}
\]
Problem $P_{1,\delta}$ has the following variational formulation.
Problem $P_{2\delta}$. Find $(u_\delta, \beta_\delta) \in C([0, T]; V) \times (W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B)$ such that

\begin{equation}
\langle A\varepsilon (u_\delta(t)), \varepsilon(v) \rangle_Q + \left\langle \int_0^t F(t-s) \varepsilon (u_\delta(s)) \, ds, \varepsilon(v) \right\rangle_Q + 
+j_\delta (\beta_\delta(t), u_\delta(t), v) = (f(t), v)_V \quad \forall v \in V, \, t \in [0, T],
\end{equation}

\begin{equation}
\dot{\beta_\delta}(t) = - \left( c_v \beta_\delta(t) (R_v (u_{\delta\nu}(t)))^2 - \varepsilon_a \right)_+ \quad \text{a.e. } t \in (0, T),
\end{equation}

\begin{equation}
\beta_\delta(0) = \beta_0.
\end{equation}

We have the following result.

**Theorem 4.1.** Problem $P_{2\delta}$ has a unique solution.

**Proof.** The proof of Theorem 4.1 is similar to the proof of Theorem 3.1 and it is carried out in several steps. So, we omit the details of the proof. We recall that the steps are as follows.

(i) For any $\beta \in X$, we prove that there exists a unique $u_\delta \in C([0, T]; V)$ such that

\begin{equation}
\langle A\varepsilon (u_\delta(t)), \varepsilon(v) \rangle_Q + \left\langle \int_0^t F(t-s) \varepsilon (u_\delta(s)) \, ds, \varepsilon(v) \right\rangle_Q + 
+j_\delta (\beta_\delta(t), u_\delta(t), v) = (f(t), v)_V \quad \forall v \in V, \, t \in [0, T].
\end{equation}

To provide this step for all $t \in [0, T]$ we consider the operator $T_t : V \to V$ defined by

\[ (T_t u, v)_V = \langle A\varepsilon (u), \varepsilon(v) \rangle_Q + j_\delta (\beta(t), u, v) \quad \forall u, v \in V. \]

We use the hypotheses on $A$, the properties (2.18)–(2.21) satisfied by the functional $j$ and the definition of the function $p_\delta$ to see that the operator $T_t$ is strongly monotone and Lipschitz continuous. Then we conclude by using arguments similar to those used in the proof of Lemma 3.3.

(ii) There exists a unique $\beta_\delta$ such that

\begin{equation}
\beta_\delta \in W^{1,\infty}(0, T; L^2(\Gamma_3)),
\end{equation}

\begin{equation}
\dot{\beta_\delta}(t) = - \left( c_v \beta_\delta(t) (R_v (u_{\delta\nu}(t)))^2 - \varepsilon_a \right)_+ \quad \text{a.e. } t \in (0, T),
\end{equation}

\begin{equation}
\beta_\delta(0) = \beta_0.
\end{equation}

(iii) Let $\beta_\delta$ defined in (ii) and denote again by $u_\delta$ the function obtained in step (i) for $\beta = \beta_\delta$. Then, by using (4.13)–(4.15) it is easy to see that $(u_\delta, \beta_\delta)$ is the unique solution to Problem $P_{2\delta}$ and it satisfies

\[ (u_\delta, \beta_\delta) \in C([0, T]; V) \times (W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B). \]
The behavior of the solution \((u_\delta, \beta_\delta)\) as \(\delta \to 0\) is given by the following theorem.

**Theorem 4.2.** Assume that (2.11), (2.13), (2.14), (2.15), (2.16), and (2.17) hold. Then the solution \((u_\delta, \beta_\delta)\) of Problem \(P_{2\delta}\) converges to the solution \((u, \beta)\) of Problem \(P_2\), that is,

\[
\begin{align*}
\lim_{\delta \to 0} \|u_\delta(t) - u(t)\|_V &= 0 \quad \text{for all } t \in [0, T], \\
\lim_{\delta \to 0} \|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} &= 0 \quad \text{for all } t \in [0, T].
\end{align*}
\]

The proof is carried out in several steps. In the first step, we show the following lemma.

**Lemma 4.3.** For each \(t \in [0, T]\), there exists \(\bar{u}(t) \in K\) such that after passing to a subsequence still denoted \((u_\delta(t))\) we have

\[
\lim_{\delta \to 0} u_\delta(t) \to \bar{u}(t) \quad \text{weakly in } V.
\]

**Proof.** Let \(t \in [0, T]\). Take \(v = u_\delta(t)\) in (4.9) then

\[
\begin{align*}
\langle A\varepsilon (u_\delta(t)) \varepsilon (u_\delta(t)) \rangle_Q + j (\beta_\delta(t), u_\delta(t)) + \\
+ \left\langle \int_0^t \mathcal{F}(t - s) \varepsilon (u_\delta(s)) \, ds, \varepsilon (u_\delta(t)) \right\rangle_Q = (f(t), u_\delta(t))_V.
\end{align*}
\]

Since

\[
j_\delta (\beta_\delta(t), u_\delta(t), u_\delta(t)) \geq 0,
\]

from (4.19) we deduce that

\[
\begin{align*}
\langle A\varepsilon (u_\delta(t)) \varepsilon (u_\delta(t)) \rangle_Q + \left\langle \int_0^t \mathcal{F}(t - s) \varepsilon (u_\delta(s)) \, ds, \varepsilon (u_\delta(t)) \right\rangle_Q &\leq (f(t), u_\delta(t))_V.
\end{align*}
\]

Now, we use Riesz’z representation theorem to define the operator \(D : [0, T] \to L(V)\) by

\[
(D(t)u, v)_V = \left\langle \int_0^t \mathcal{F}(t - s) \varepsilon (u(s)) \, ds, \varepsilon (v) \right\rangle_Q \quad \forall u, v \in V.
\]

As in [21] it is clear that

\[
D \in C ([0, T]; L(V)).
\]

Moreover from (4.20) it follows that

\[
m \|u_\delta(t)\|^2_V - \|D\|_{C([0, T]; L(V))} \|u_\delta(t)\|^2_V \leq \|f(t)\|_V \|u_\delta(t)\|_V.
\]
This inequality implies that if
\[
\|D\|_{C([0,T];\mathcal{L}(V))} < m,
\]
there exists a constant \( c > 0 \) such that
\[
\|u_\delta(t)\|_V \leq c \|f(t)\|_V.
\]
The sequence \((u_\delta(t))\) is bounded in \( V \). Then there exists \( \bar{u}(t) \in V \) and a subsequence again denoted \((u_\delta(t))\) such that (4.18) holds. Also from (4.19) we have
\[
j_\delta (\beta_\delta(t), u_\delta(t), u_\delta(t)) \leq (f(t), u_\delta(t))_V - \left\langle \int_0^t \mathcal{F}(t-s) \varepsilon(u_\delta(s)) \, ds, \varepsilon(u_\delta(t)) \right\rangle_Q.
\]
On the other hand,
\[
j_\delta (\beta_\delta(t), u_\delta (t), u_\delta(t)) = \int_{\Gamma_3} \left( p_\delta (u_\delta(t)) - c_\nu \beta_\delta^2(t) R_\nu (u_\delta(t)) \right) u_\delta(t) \, da.
\]
Since
\[
\int_{\Gamma_3} c_\nu \beta_\delta^2(t) R_\nu (u_\delta(t)) u_\delta(t) \, da \leq 0,
\]
we have
\[
\int_{\Gamma_3} p_\delta (u_\delta(t)) u_\delta(t) \, da \leq (f(t), u_\delta(t))_V - \left\langle \int_0^t \mathcal{F}(t-s) \varepsilon(u_\delta(s)) \, ds, \varepsilon(u_\delta(t)) \right\rangle_Q.
\]
Now, according to [12],
\[
\int_{\Gamma_3} p_\delta (u_\delta(t)) u_\delta(t) \, da = \int_{\Gamma_3 \cap \{u_\delta(t) \leq g\}} p_\delta (u_\delta(t)) u_\delta(t) \, da + \int_{\Gamma_3 \cap \{u_\delta(t) > g\}} p_\delta (u_\delta(t)) u_\delta(t) \, da.
\]
As
\[
\int_{\Gamma_3 \cap \{u_\delta(t) \leq g\}} p_\delta (u_\delta(t)) u_\delta(t) \, da = \int_{\Gamma_3 \cap \{u_\delta(t) \leq g\}} p (u_\delta(t)) u_\delta(t) \, da \geq 0,
\]
we find that
\[
\int_{\Gamma_3 \cap \{u_\delta(t) > g\}} p_\delta (u_\delta(t)) u_\delta(t) \, da \leq (f(t), u_\delta(t))_V - \left\langle \int_0^t \mathcal{F}(t-s) \varepsilon(u_\delta(s)) \, ds, \varepsilon(u_\delta(t)) \right\rangle_Q.
\]
The left hand side of the previous inequality can be written as
\[
\int_{\Gamma_3 \cap \{u_{\delta \nu}(t) > g\}} p_\delta(u_{\delta \nu}(t)) u_{\delta \nu}(t) \, da + \int_{\Gamma_3 \cap \{u_{\delta \nu}(t) > g\}} \frac{g}{\delta} (u_{\delta \nu}(t) - g) \, da + \int_{\Gamma_3 \cap \{u_{\delta \nu}(t) > g\}} p(g) u_{\delta \nu}(t) \, da,
\]
from which we deduce
\[
\int_{\Gamma_3 \cap \{u_{\delta \nu}(t) > g\}} (u_{\delta \nu}(t) - g) \left( u_{\delta \nu}(t) - g \right) \, da \leq \left( f(t), u_{\delta}(t) \right)_V - \left( \int_0^t F(t - s) \varepsilon(u_{\delta}(s)) \, ds, \varepsilon(u_{\delta}(t)) \right)_Q.
\]
This inequality implies that
\[
\|(u_{\delta \nu}(t) - g)_+\|_{L^2(\Gamma_3)}^2 \leq \delta c,
\]
where \(c > 0\). Hence, using (4.18), we deduce that
\[
(4.21) \quad \|(\bar{u}_\nu(t) - g)_+\|_{L^2(\Gamma_3)} \leq \liminf_{\delta \to 0} \|(u_{\delta \nu}(t) - g)_+\|_{L^2(\Gamma_3)} = 0.
\]
Therefore, it follows from (4.21) that \((\bar{u}_\nu(t) - g)_+ = 0\) a.e. on \(\Gamma_3\), i.e., \(\bar{u}_\nu(t) \leq g\) a.e. on \(\Gamma_3\) and then \(\bar{u}(t) \in K\).

Next, we prove the following auxiliary problem.

**Problem** \(P_3\). Find \(\beta_a : [0, T] \to L^\infty(\Gamma_3)\) such that
\[
\dot{\beta}_a(t) = - \left( c_\nu \beta_a(t) (R_\nu(\bar{u}_\nu(t)))^2 - \varepsilon_a \right)_+ \quad \text{a.e. } t \in (0, T), \quad \beta_a(0) = \beta_0.
\]

As in [22, Lemma 3.2] we have the following result.

**Lemma 4.4.** Problem \(P_3\) has a unique solution
\[
\beta_a \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B.
\]

Also as in [22, Lemma 4.5] we have the following convergence result.

**Lemma 4.5.** We have
\[
(4.22) \lim_{\delta \to 0} \|\beta_\delta(t) - \beta_a(t)\|_{L^2(\Gamma_3)} = 0 \text{ for all } t \in [0, T].
\]

Next, we prove the following lemma.

**Lemma 4.6.** We have \(\bar{u}(t) = u(t)\) for all \(t \in [0, T]\).
Proof. Let \( v \in K \) and take \( v - u_\delta(t) \) in (4.9), we have
\[
\langle A\varepsilon(u_\delta(t)), \varepsilon(v - u_\delta(t)) \rangle_Q + j_\delta(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) + \int_0^t \mathcal{F}(t - s) \varepsilon(u_\delta(s)) ds, \varepsilon(v - u_\delta(t)) \rangle_Q = (f(t), v - u_\delta(t))_V.
\]
Since
\[
j_\delta(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) =
\int_{\Gamma^3 \cap \{u_\delta(t) \leq g\}} \left( p(u_\delta(t)) - c_\nu \beta_\delta^2(t) R_\nu(u_\delta(t)) \right) (v_\nu - u_\delta(t)) \, da
\]
\[
+ \int_{\Gamma^3 \cap \{u_\delta(t) > g\}} \left( p_\delta(u_\delta(t)) - c_\nu \beta_\delta^2(t) R_\nu(u_\delta(t)) \right) (v_\nu - u_\delta(t)) \, da,
\]
we use the definition of the operator \( R_\nu \) to see that
\[
\int_{\Gamma^3 \cap \{u_\delta(t) > g\}} \left( p_\delta(u_\delta(t)) - c_\nu \beta_\delta^2(t) R_\nu(u_\delta(t)) \right) (v_\nu - u_\delta(t)) \, da =
\int_{\Gamma^3 \cap \{u_\delta(t) > g\}} p_\delta(u_\delta(t)) (v_\nu - u_\delta(t)) \, da.
\]
Therefore, using the definition of the function \( p_\delta \), the right hand side of the previous equality is written as
\[
\int_{\Gamma^3 \cap \{u_\delta(t) > g\}} p_\delta(u_\delta(t)) (v_\nu - u_\delta(t)) \, da =
\int_{\Gamma^3 \cap \{u_\delta(t) > g\}} \left( \frac{u_\delta(t) - g}{\delta} \right) ((v_\nu - g) - (u_\delta(t) - g)) \, da
\]
\[
+ \int_{\Gamma^3 \cap \{u_\delta(t) > g\}} (p(g)) ((v_\nu - g) - (u_\delta(t) - g)) \, da,
\]
which implies that
\[
\int_{\Gamma^3 \cap \{u_\delta(t) > g\}} p_\delta(u_\delta(t)) (v_\nu - u_\delta(t)) \, da \leq 0.
\]
Then we deduce that
\[
(4.24) \quad j_\delta(\beta_\delta(t), u_\delta(t), v - u_\delta(t)) \leq
\int_{\Gamma^3 \cap \{u_\delta(t) \leq g\}} \left( p(u_\delta(t)) - c_\nu \beta_\delta^2(t) R_\nu(u_\delta(t)) \right) (v_\nu - u_\delta(t)) \, da. \quad \square
\]
Now, take $v = \bar{u}(t)$ in (4.23) to obtain by using the assumption (2.13) (b) on $A$ that

\[(4.25)\quad m \| u_\delta(t) - \bar{u}(t) \|^2_V \leq \]

\[\leq \left( \int_0^t \mathcal{F}(t-s) \left( \varepsilon(u_\delta(s)) - \varepsilon(\bar{u}(s)) \right) ds, \varepsilon(\bar{u}(t)) - \varepsilon(u_\delta(t)) \right)_Q + \]

\[+ \int_{\Gamma_\delta \cap \{u_\delta(t) \leq \bar{u} \}} \left( p(u_\delta(t)) - c_\nu \beta_\delta^2(t) R_\nu(u_\delta(t)) \right) (\bar{u}(t) - u_\delta(t)) \, da + \]

\[+ \left( \int_0^t \mathcal{F}(t-s) \varepsilon(\bar{u}(s)) ds, \varepsilon(u_\delta(t)) - \varepsilon(\bar{u}(t)) \right)_Q + (f(t), u_\delta(t) - \bar{u}(t))_V . \]

Next, we denote

\[c_\delta(t) = \int_{\Gamma_\delta \cap \{u_\delta(t) \leq \bar{u} \}} \left( p(u_\delta(t)) - c_\nu \beta_\delta^2(t) R_\nu(u_\delta(t)) \right) (\bar{u}(t) - u_\delta(t)) \, da + \]

\[+ (A \varepsilon(\bar{u}(t)) , \varepsilon(\bar{u}(t)) - \varepsilon(u_\delta(t)))_Q + \]

\[+ \left( \int_0^t \mathcal{F}(t-s) \varepsilon(\bar{u}(s)) ds, \varepsilon(u_\delta(t)) - \varepsilon(\bar{u}(t)) \right)_Q + (f(t), u_\delta(t) - \bar{u}(t))_V . \]

As in [21], we have

\[\left( \int_0^t \mathcal{F}(t-s) \left( \varepsilon(\bar{u}(s)) - \varepsilon(u_\delta(s)) \right) ds, \varepsilon(u_\delta(t)) - \varepsilon(\bar{u}(t)) \right)_Q \leq \]

\[\leq m \left( \int_0^t \| u_\delta(s) - \bar{u}(s) \|_V \, ds \right) \| u_\delta(t) - \bar{u}(t) \|_V . \]

Using the elementary inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we find that

\[(4.26)\quad \left( \int_0^t \mathcal{F}(t-s) \left( \varepsilon(u(s)) - \varepsilon(u_\delta(s)) \right) ds, \varepsilon(u_\delta(t)) - u(t) \right)_Q \leq \]

\[\leq \frac{m}{2} \left( \int_0^t \| u_\delta(s) - u(s) \|_V \, ds \right)^2 + \frac{m}{2} \| u_\delta(t) - u(t) \|^2_V . \]

Now, we combine inequalities (4.25) and (4.26) to obtain

\[\frac{m}{2} \| u_\delta(t) - \bar{u}(t) \|^2_V \leq \frac{m}{2} \left( \int_0^t \| u_\delta(s) - \bar{u}(s) \|_V \, ds \right)^2 + | c_\delta(t) | , \]

which yields

\[\| u_\delta(t) - \bar{u}(t) \|^2_V \leq c \left( \int_0^t \| u_\delta(s) - \bar{u}(s) \|^2_V \, ds + | c_\delta(t) | \right) . \]
where $c > 0$. The Gronwall argument implies that there exists a constant $c_1 > 0$ such that
\begin{equation}
\|u_\delta(t) - \bar{u}(t)\|_V \leq c_1 \sqrt{|c_\delta(t)|}.
\end{equation}
Now, using Lemma 4.3, we get
\begin{align*}
\int_{\Gamma_3 \cap \{u_\delta, (t) \leq g\}} (p(u_\delta, (t)) - c_\nu \beta_\delta^2(t) R_\nu(u_\delta, (t)) \left(\bar{u}_\nu(t) - u_\delta, (t)\right)) \text{da} &\to 0, \\
\langle A\varepsilon(\bar{u},(t)), \varepsilon(\bar{u}(t)) - \varepsilon(\bar{u}(t)) \rangle_Q + \\
\left\langle \int_0^t \mathcal{F}(t - s) \varepsilon(u_\delta(s)) \text{ds}, \varepsilon(u_\delta(t)) - \varepsilon(\bar{u}(t)) \right\rangle_Q &\to 0,
\end{align*}
which imply
\[c_\delta(t) \to 0 \text{ as } \delta \to 0.\]
Hence
\begin{equation}
\|u_\delta(t) - \bar{u}(t)\|_V \to 0 \text{ for all } t \in [0, T].
\end{equation}
Now, from (4.23) and (4.24) we deduce the inequality
\begin{align*}
\langle A\varepsilon(u_\delta(t)), \varepsilon(v - u_\delta(t)) \rangle_Q + & \\
+ \int_{\Gamma_3 \cap \{u_\delta, (t) \leq g\}} (p(u_\delta, (t)) - c_\nu \beta_\delta^2(t) R_\nu(u_\delta, (t)) \left(\bar{u}_\nu(t) - u_\delta, (t)\right)) \text{da} + \\
+ \left\langle \int_0^t \mathcal{F}(t - s) \varepsilon(u_\delta(s)) \text{ds}, \varepsilon(v - u_\delta(t)) \right\rangle_Q &\geq (f(t), v - u_\delta(t))_V \forall v \in K.
\end{align*}
Therefore, using (4.22), (4.28), (2.17) (b) and the properties of $R_\nu$, we obtain by passing to the limit as $\delta \to 0$ in inequality (4.29) that
\begin{equation}
\langle A\varepsilon(\bar{u}(t)), \varepsilon(v) - \varepsilon(\bar{u}(t)) \rangle_Q + \\
+ \int_0^t \mathcal{F}(t - s) \varepsilon(u_\delta(s)) \text{ds}, \varepsilon(v) - \varepsilon(\bar{u}(t)) \rangle_Q + \\
+ j(\beta_\alpha(t), \bar{u}(t), v - \bar{u}(t)) \geq (f(t), v - \bar{u}(t))_V \forall v \in K, t \in [0, T].
\end{equation}
Hence we combine Lemma 4.3, (4.30) and Problem $P_3$ to deduce that $(\bar{u}, \beta_\alpha)$ satisfies (2.22)–(2.24) and so it is a solution of Problem $P_2$. As this problem has a unique solution, Lemma 4.6 is proved.

Moreover from (4.28), Lemmas 4.5 and 4.6, we deduce (4.16) and (4.17) and then Theorem 4.2 is proved.

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