A QUASISTATIC UNILATERAL CONTACT PROBLEM WITH NORMAL COMPLIANCE AND NONLOCAL FRiction

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We consider a mathematical model which describes the quasistatic unilateral and frictional contact of a linear elastic body with a foundation. The contact is modelled with a normal compliance condition of such a type that the penetration is restricted with unilateral constraint and associated to the nonlocal friction law. We present the classical formulation of the problem, the variational formulation and establish the existence of a weak solution, when the coefficient of friction is sufficiently small. The proofs are based on arguments of time-discretization, compactness and lower semicontinuity.

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1. INTRODUCTION

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Recall that unilateral contact problems including Signorini’s condition were studied by many authors, we refer for instance the reader to [2, 4, 5, 6, 7, 9, 10, 12, 13, 15, 18, 19]. A first attempt to study frictional contact problems within the framework of variational inequalities was made in [8]. The mathematical, mechanical and numerical state of the art can be found in [17]. In [13] we find a detailed analysis and numerical approximations of contact problems concerning elastic materials with or without friction. In this paper we consider a frictional unilateral contact between a linear elastic body and a foundation. We assume that the contact is modelled with a normal compliance condition in such a way that the penetration is allowed, restricted with unilateral constraint and associated to the nonlocal Coulomb law. This normal compliance condition is similar to the one in [12, 18] where was respectively studied a dynamic frictionless contact problem for elastic-visco-plastic materials and a quasistatic frictional contact problem for
nonlinear elastic materials. We assume that the forces applied to the body vary slowly in time so that the acceleration in the system is negligible. In this case we can study a quasistatic approach of the process. We recall that for linear elastic materials the quasistatic contact problem using a normal compliance law has been studied in [1] by considering incremental problems and in [14] by another method using a time-regularization. The quasistatic contact problem with local or nonlocal friction has been solved respectively in [15] and in [4] by using a time-discretization. In [2] the quasistatic contact problem with Coulomb friction was solved by an established shifting technique used to obtain increased regularity at the contact surface and by the aid of auxiliary problems involving regularized friction terms and a so-called normal compliance penalization technique. In the static case Signorini’s problem with friction for nonlinear elastic materials has been solved in [6] by using a fixed point method. In the quasistatic case it was solved in [19] by using a time-discretization method. Also, in viscoelasticity, the quasistatic contact problem with normal compliance and friction has been solved in [16] for nonlinear viscoelastic materials by the same fixed point argument. In [11] the authors resolve the quasistatic contact problems in viscoelasticity and viscoplasticity. Carrying out the variational analysis, the authors systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators.

In this work we propose a variational formulation written in the form of two variational inequalities. By means of Euler’s implicit scheme as in [18, 19], the quasistatic contact problem leads us to solve a well-posed variational inequality at each time step. Finally, if the coefficient of friction is small enough we prove by using lower semicontinuity and compactness arguments that the limit of the discrete solution is a solution to the continuous problem.

2. PROBLEM STATEMENT AND VARIATIONAL FORMULATION

We consider a linear elastic body which occupies a regular domain $\Omega$ of $\mathbb{R}^d$ ($d = 2, 3$) with boundary $\Gamma$ that is partitioned into three parts such that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ are disjoint open sets and $\text{meas}(\Gamma_1) > 0$. The body is subjected to volume forces of density $\varphi_1$, prescribed zero displacements and tractions $\varphi_2$ on the part $\Gamma_1$ and $\Gamma_2$, respectively. On $\Gamma_3$ the body is in unilateral and frictional contact with a foundation. We denote by $u = (u_i)$, the displacement field,

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$
the strain tensor, $\sigma = \mathcal{E} \varepsilon(u)$, the stress tensor where $\mathcal{E}$ denotes the fourth order tensor of elasticity coefficients. Let $\nu$ be the unit outward normal vector on $\Gamma$. We denote the normal and tangential components of the displacement vector and stress vector by

$$u_\nu = u \cdot \nu, \quad u_\tau = u - u_\nu \nu, \quad \sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma_\tau = \sigma \nu - \sigma_\nu \nu.$$  

With these preliminaries, the classical formulation of the problem of frictional contact we consider is the following.

**Problem** $P_1$. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$ such that

1. $(2.1)$ $\sigma = \mathcal{E} \varepsilon(u)$ in $\Omega \times (0, T)$,
2. $(2.2)$ $\text{div} \sigma + \varphi_1 = 0$ in $\Omega \times (0, T)$,
3. $(2.3)$ $u = 0$ on $\Gamma_1 \times (0, T)$,
4. $(2.4)$ $\sigma \nu = \varphi_2$ on $\Gamma_2 \times (0, T)$,
5. $(2.5)$ $u_\nu \leq g$, $\sigma_\nu + p(u_\nu) \leq 0$, $(\sigma_\nu + p(u_\nu))(u_\nu - g) = 0$ on $\Gamma_3 \times (0, T)$,
6. $(2.6)$ $\begin{cases} |\sigma_\tau| \leq \mu |R\sigma_\nu| \\
|\sigma_\tau| < \mu |R\sigma_\nu| \Rightarrow \dot{u}_\tau = 0 \quad \text{on } \Gamma_3 \times (0, T), \\
|\sigma_\tau| = \mu |R\sigma_\nu| \Rightarrow \exists \lambda \geq 0 \text{ s.t. } \sigma_\tau = -\lambda \dot{u}_\tau \end{cases}$
7. $(2.7)$ $u(0) = u_0$ in $\Omega$.

Equation (2.1) represents the linear elastic constitutive law. Equation (2.2) represents the equilibrium equation and (2.3), (2.4) are the displacement-traction boundary conditions, respectively. Condition (2.5) represents the contact condition with finite penetration, introduced in [12], in which $g \geq 0$ represents the maximum value of the penetration and $p$ is a prescribed function. Condition (2.6) represents the nonlocal friction law in which $\mu$ denotes the coefficient of friction, $R$ is a regularization operator (see [7]) and $u_\tau$ is the tangential part of velocity field. The tangential shear cannot exceed the maximal frictional resistance $\mu |R\sigma_\nu|$. Then, if the strict inequality is satisfied, the surface adheres to the foundation and is in the so-called stick state, and when equality is satisfied there is relative sliding, the so-called slip state.

In the study of the mechanical problem $P_1$ we adopt the following notations and hypotheses:

We denote by $S^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $S^d$ are given by

$$u \cdot v = u_i v_i, \quad |v| = (v \cdot v)^{1/2} \quad \forall u = (u_i), \ v = (v_i) \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij}, \quad |\tau| = (\tau \cdot \tau)^{1/2} \quad \forall \sigma = (\sigma_{ij}), \ \tau = (\tau_{ij}) \in S^d.$$
respectively. In (2.6) and below, a dot above a variable represents its derivative with respect to time. To proceed with the variational formulation, we need some function spaces

\[ H = (L^2(\Omega))^d, \quad Q = \{ \tau = (\tau_{ij}) \in (L^2(\Omega))^{d \times d} : \tau_{ij} = \tau_{ji}, 1 \leq i, j \leq d \}, \]

\[ H_1 = (H^1(\Omega))^d. \]

\( H, Q \) are real Hilbert spaces equipped with the respective inner products

\[ \langle u, v \rangle_H = \int_{\Omega} u_i v_i dx, \quad \langle \sigma, \tau \rangle_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} dx. \]

Let \( V \) be the closed subspace of \( H_1 \) defined by

\[ V = \{ v \in H_1 : v = 0 \text{ on } \Gamma_1 \}. \]

and the set of admissible displacements fields given by

\[ K = \{ v \in V : v_{\nu} \leq g \text{ on } \Gamma_3 \}. \]

Since \( \text{meas}(\Gamma_1) > 0 \), the following Korn’s inequality holds [8],

\[ \| \varepsilon(v) \|_Q \geq c_\Omega \| v \|_{H_1} \quad \forall v \in V, \]

where \( c_\Omega > 0 \) is a constant which depends only on \( \Omega \) and \( \Gamma_1 \). We equip \( V \) with the inner product given by

\[ (u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Q \]

and let \( \| \cdot \|_V \) be the associated norm. It follows from (2.8) that the norms \( \| \cdot \|_{H_1} \) and \( \| \cdot \|_V \) are equivalent and \( (V, \| \cdot \|_V) \) is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant \( d_\Omega > 0 \) depending only on the domain \( \Omega, \Gamma_1 \) and \( \Gamma_3 \) such that

\[ \| v \|_{(L^2(\Gamma_3))^d} \leq d_\Omega \| v \|_V \quad \forall v \in V. \]

Recall that it is well known that, if \( \sigma \) is a regular function, then the following Green’s formula holds

\[ \langle \sigma, \varepsilon(v) \rangle_Q + \langle \text{div} \sigma, v \rangle_H = \int_{\Gamma} \sigma v \cdot v da \quad \forall v \in H_1. \]

Let now

\[ H^{\frac{1}{2}}(\Gamma_3) = \{ w_{\Gamma_3} : w \in H^{\frac{1}{2}}(\Gamma), w = 0 \text{ on } \Gamma_1 \} \]

and let the brackets \( \langle \cdot, \cdot \rangle_{\Gamma_3} \) stand for the duality between the dual spaces \( H^{-\frac{1}{2}}(\Gamma_3) \) and \( H^{\frac{1}{2}}(\Gamma_3) \). Before we start with the variational formulation of Problem \( P_1 \) let us state in which sense the duality pairing \( \langle \cdot, \cdot \rangle_{\Gamma_3} \) is taken.
Indeed, let $\sigma \in Q_1 = \{ \tau \in Q : \text{div} \tau \in H \}$ and $h \in (L^2(\Gamma_2))^d$ such that $\sigma \nu = h$ on $\Gamma^2$. Then as in [19] the normal stress $\sigma_{\nu}$ is defined by

$$
(2.10) \quad \forall w \in H^\frac{1}{2}(\Gamma_3) : \langle \sigma_{\nu}, w \rangle_{\Gamma_3} = \langle \sigma, \varepsilon(v) \rangle_Q + \langle \text{div} \sigma, v \rangle_H - \int_{\Gamma_2} h \cdot v \, d\alpha
$$

$\forall v \in V; \ v_{\nu} = w$ and $v_\tau = 0$ on $\Gamma_3$.

We suppose that $R : H^{-\frac{1}{2}}(\Gamma_3) \to L^2(\Gamma_3)$ is a compact linear operator (see [7]). The linearity of $R$ implies that there exists a constant $C_R > 0$ such that

$$
(2.11) \quad \| R \tau \|_{L^2(\Gamma_3)} \leq C_R \| \tau \|_{H^{-\frac{1}{2}}(\Gamma_3)} \quad \forall \tau \in H^{-\frac{1}{2}}(\Gamma_3).
$$

Next, in the study of the mechanical problem $P_\xi$, we assume that the linear elasticity operator $E = (E_{ijkl}) : \Omega \times S^d \to S^d$ satisfies the following conditions:

\begin{itemize}
  \item $E_{ijkl} \in L^{\infty}(\Gamma_3)$,
  \item $E_{ijkl} = E_{jikl} = E_{iljk}, \ 1 \leq i, j, k, l \leq d$,
  \item $\exists \alpha > 0$ such that $\forall \xi = (\xi_{ij}) \in \mathbb{R}^{d \times d}$ with $\xi_{ij} = \xi_{ji}, \ 1 \leq i, j, k, l \leq d$,
    \begin{equation}
    E_{ijkl}\xi_{ij}\xi_{kl} \geq \alpha |\xi|^2.
    \end{equation}
\end{itemize}

We define the bilinear form $a(\cdot, \cdot)$ on $V \times V$ by

$$
(2.12) \quad a(u, v) = \int_\Omega E_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v) \, dx.
$$

It follows from (2.12) and (2.8) that $a$ is continuous and coercive, that is,

$$
(2.13) \quad \begin{cases}
  \text{there exists } M > 0 \text{ such that } |a(u, v)| \leq M\|u\|_V\|v\|_V \forall u, v \in V; \\
  \text{there exists } m > 0 \text{ such that } a(u, v) \geq m\|v\|_V^2 \forall v \in V.
\end{cases}
$$

For every real Banach space $(X, \| \cdot \|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to $X$; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$
\|x\|_{C([0, T]; X)} = \max_{t \in [0, T]} \|x(t)\|_X.
$$

For $p \in [1, \infty]$ we use the standard notation of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$
\|v\|_{W^{1,\infty}(0, T; V)} = \|v\|_{L^{\infty}(0, T; V)} + \|\dot{v}\|_{L^{\infty}(0, T; V)},
$$

where a dot now represents the weak derivative with respect to the time variable.

The forces are assumed to satisfy

$$
(2.14) \quad \varphi_1 \in W^{1,\infty}(0, T; H), \quad \varphi_2 \in W^{1,\infty}(0, T; (L^2(\Gamma_2))^d).
$$
Let $f : [0, T] \to V$ given by
\[
(f(t), v)_V = \int_{\Omega} \varphi_1 \cdot vdx + \int_{\Gamma_2} \varphi_2 \cdot vda \quad \forall v \in V, \ t \in [0, T].
\]
The assumption (2.14) implies that $f \in W^{1, \infty}(0, T; V)$.

As in [12] we assume that the contact function $p$ satisfies
\[
\begin{cases}
(a) p : ] - \infty, g [ \to \mathbb{R}; \\
(b) \text{there exists } L_p > 0 \text{ such that } |p(r_1) - p(r_2)| \leq L_p|r_1 - r_2| \text{ for all } r_1, r_2 \leq g; \\
(c) (p(r_1) - p(r_2))(r_1 - r_2) \geq 0 \text{ for all } r_1, r_2 \leq g; \\
(d) p(r) = 0 \text{ for all } r < 0.
\end{cases}
\]

We note that when $u_\nu < 0$, i.e., when there is separation between the body and the obstacle then the condition (2.5) combined with hypotheses (2.15) shows that the reaction of the foundation vanishes ($\sigma_\nu = 0$). When $0 \leq u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement. When $u_\nu = g$ then $-\sigma_\nu \geq p(g)$ and $\sigma_\nu$ is not uniquely determined. We note that when $g = 0$ then the condition (2.5) becomes the classical Signorini contact condition without a gap
\[
u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu u_\nu = 0,
\]
and when $g > 0$ and $p = 0$, condition (2.5) becomes the classical Signorini contact condition with a gap.
\[
u_\nu \leq g, \quad \sigma_\nu \leq 0, \quad \sigma_\nu(u_\nu - g) = 0.
\]

The last two conditions are used to model the unilateral conditions with a rigid foundation. We recall that examples of normal compliance functions can be found in [1, 11, 12, 16, 17]. Next, we define the set
\[
V_0 = \{ v \in H_1 : \text{div}\sigma(v) \in H \}
\]
and the functionals
\[
j_c : V \times V \to \mathbb{R}, \quad j_f : V_0 \times V \to \mathbb{R}
\]
by
\[
j_c(v, w) = \int_{\Gamma_3} p(v_\nu)w_\nu da \quad \forall v, w \in V,
\]
\[
j_f(v, w) = \int_{\Gamma_3} \mu|\text{R}\sigma_\nu(v)||w_\tau| da \quad \forall (v, w) \in V_0 \times V,
\]
where the coefficient of friction $\mu$ is assumed to satisfy
\begin{equation}
\mu \in L^\infty(\Gamma_3) \text{ and } \mu \geq 0 \text{ a.e. on } \Gamma_3.
\end{equation}
We also assume that the initial data $u_0$ satisfies
\begin{equation}
(2.17) \quad u_0 \in K \cap V_0
\end{equation}
and the compatibility condition
\begin{equation}
(2.18) \quad a(u_0, v - u_0) + j(u_0, v - u_0) \geq (f(0), v - u_0) \quad \forall v \in K,
\end{equation}
where
\[ j = j_c + j_f. \]
Now, in order to establish the weak formulation of Problem $P_1$, we assume that $u$ is a smooth function satisfying (2.1)–(2.7). Indeed, let $v \in V$ and multiply the equilibrium of forces (2.2) by $v - \dot{u}(t)$, integrate the result over $\Omega$ and use Green’s formula to obtain

\[ \int_\Omega \sigma(t) \cdot (\varepsilon(v) - \varepsilon(\dot{u}(t))) \, dx = \int_\Omega \varphi_1(t) \cdot (v - \dot{u}(t)) \, dx + \int_{\Gamma_3} \sigma(t) \nu \cdot (v - \dot{u}(t)) \, da. \]

Taking into account the boundary conditions (2.3) and $v = 0$ on $\Gamma_1$, we see that

\[ \int_{\Gamma_3} \sigma(t) \nu \cdot (v - \dot{u}(t)) \, da = \int_{\Gamma_2} \varphi_2(t) \cdot (v - \dot{u}(t)) \, da + \int_{\Gamma_3} \sigma(t) \nu \cdot (v - \dot{u}(t)) \, da. \]

Moreover, we have
\[ \int_{\Gamma_3} \sigma(t) \nu \cdot (v - \dot{u}(t)) \, da = \]
\[ = \int_{\Gamma_3} \sigma_\nu(u(t)) (v_\nu - \dot{u}_\nu(t)) \, da + \int_{\Gamma_3} \sigma_\tau(t) \cdot (v_\tau - \dot{u}_\tau(t)) \, da, \]
and
\[ \int_{\Gamma_3} \sigma_\nu(u(t)) (v_\nu - \dot{u}_\nu(t)) \, da = \]
\[ = \int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t))) (v_\nu - \dot{u}_\nu(t)) \, da - \int_{\Gamma_3} p(u_\nu(t)) (v_\nu - \dot{u}_\nu(t)) \, da. \]

The law of friction (2.6) leads to the following relation
\[ \sigma_\nu \cdot (v_\nu - \dot{u}_\nu) + \mu|R\sigma_\nu(u)|(|v_\nu| - |\dot{u}_\nu|) \geq 0 \quad \forall v_\nu, \]
from which we deduce that the function $u$ satisfies the inequality
\begin{equation}
(2.19) \quad a(u(t), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \geq \]
\[ (f(t), v - \dot{u}(t)) + \int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t))) (v_\nu - \dot{u}_\nu(t)) \, da \quad \forall v \in V. \]
On the other hand, we have
\[
\int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t))(z_\nu - u_\nu(t)))da =
\]
\[
= \int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t))((z_\nu - g) - (u_\nu(t) - g)))da =
\]
\[
= \int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t)))(z_\nu - g)da - \int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t))(u_\nu(t) - g))da.
\]
Using the conditions (2.5) it follows that
\[
\int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t)))(z_\nu - g)da \geq 0 \quad \forall z \in K,
\]
and
\[
\int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t))(u_\nu(t) - g))da = 0.
\]
Hence we deduce that
\[
(2.20) \quad \int_{\Gamma_3} (\sigma_\nu(u(t)) + p(u_\nu(t)))(z_\nu - u_\nu(t)) \geq 0 \quad \forall z \in K.
\]
Finally we combine (2.7), (2.19) and (2.20) to derive the variational formulation of Problem $P_1$.

**Problem $P_2$.** Find a displacement field $u \in W^{1,\infty}(0,T;V)$ such that
\[
u(t) = u_0, \quad u(t) \in K \cap V_0, \quad \text{for all } t \in [0,T], \quad \text{and for almost all } t \in [0,T],
\]
(2.21) \quad \begin{align*}
&\quad + (\sigma_\nu(u(t)), v_\nu - \dot{u}_\nu(t))_{\Gamma_3} + (p(u_\nu(t)), v_\nu - \dot{u}_\nu(t))_{L^2(\Gamma_3)} \quad \forall v \in V,
&\quad (2.22) \quad (\sigma_\nu(u(t)), z_\nu - u_\nu(t))_{\Gamma_3} + (p(u_\nu(t)), z_\nu - u_\nu(t))_{L^2(\Gamma_3)} \geq 0 \quad \forall z \in K.
\end{align*}
Our main result is contained in the following theorem.

**Theorem 2.1.** Let (2.12), (2.14), (2.15), (2.16), (2.17) and (2.18) hold. Then, there exists at least one solution of Problem $P_2$ if
\[
\|\mu\|_{L^\infty(\Gamma_3)} < m/C_RMd\Omega.
\]

**Remark 2.2.** It is interesting to note that as in [4] any element $u$ such that $u(t) \in K$ for all $t \in [0,T]$ and satisfies the inequality (2.22) verifies
\[
(\sigma_\nu(u(t)), \frac{u_\nu(t + \Delta t) - u_\nu(t)}{\Delta t})_{\Gamma_3} + (p(u_\nu(t)), \frac{u_\nu(t + \Delta t) - u_\nu(t)}{\Delta t})_{L^2(\Gamma_3)} \geq 0
\]
and
\[
(\sigma_\nu(u(t)), \frac{u_\nu(t - \Delta t) - u_\nu(t)}{-\Delta t})_{\Gamma_3} + (p(u_\nu(t)), \frac{u_\nu(t - \Delta t) - u_\nu(t)}{-\Delta t})_{L^2(\Gamma_3)} \leq 0
\]
for all $\Delta t > 0$. Moreover, using the assumption (2.15) (b) on $p$, one obtains that when $\Delta t \rightarrow 0$,

$$\langle \sigma_\nu(u(t)), \dot{u}_\nu(t) \rangle_{\Gamma_3} + (p(u_\nu(t)), \dot{u}_\nu(t))_{L^2(\Gamma_3)} = 0 \quad a.e. \ t \in (0, T).$$

### 3. INCREMENTAL FORMULATION

In order to solve Problem $P_2$, we consider an incremental formulation obtained by using an implicit time discretization scheme for (2.21) and (2.22). For $n \in \mathbb{N}^*$, we need a partition of the time interval $[0, T]$, with $0 = t_0 < t_1 < \cdots < t_n = T$, where $t_i = i\Delta t$, $i = 0, 1, \ldots, n$, with step size $\Delta t = T/n$. We denote by $u^i$ the approximation of $u$ at time $t_i$ and we set $\Delta u^i = u^{i+1} - u^i$, $i = 0, \ldots, n - 1$. For a continuous function $w \in C([0, T]; X)$ where $X$ is a Banach space we use the notation $w^i = w(t_i)$. Then we obtain the following sequence of incremental problems $P^i_n$ defined for $u^0 = u_0$ by

**Problem $P^i_n$**. Find $u^{i+1} \in K \cap V_0$ such that

$$\begin{aligned}
& a(u^{i+1}, w - u^{i+1}) + j(u^{i+1}, w - u^i) - j(u^{i+1}, \Delta u^i) \geq (f^{i+1}, w - u^{i+1})_V \\
& \quad \langle \sigma_\nu(u^{i+1}), w_\nu - u_\nu^{i+1} \rangle_{\Gamma_3} + (p(u_\nu^{i+1}), w_\nu - u_\nu^{i+1})_{L^2(\Gamma_3)} \quad \forall w \in V, \\
& \quad \langle \sigma_\nu(u^{i+1}), w_\nu - u_\nu^{i+1} \rangle_{\Gamma_3} + (p(u_\nu^{i+1}), w_\nu - u_\nu^{i+1})_{L^2(\Gamma_3)} \geq 0 \quad \forall w \in K.
\end{aligned}$$

(3.1)

As in [4] Problem $P^i_n$ is equivalent to Problem $Q^i_n$ defined as follows.

**Problem $Q^i_n$**. Find $u^{i+1} \in K \cap V_0$ such that

$$\begin{aligned}
& a(u^{i+1}, w - u^{i+1}) + j(u^{i+1}, w - u^i) - j(u^{i+1}, \Delta u^i) \\
& \geq (f^{i+1}, w - u^{i+1})_V \quad \forall w \in K.
\end{aligned}$$

(3.2)

We have the following result.

**Proposition 3.1.** There exists a unique solution to Problem $Q^i_n$ if

$$\|\mu\|_{L^\infty(\Gamma_3)} < m/C_R M d_Q.$$

We shall prove this proposition in several steps. In the first step, we introduce the following intermediate problem. Indeed we define the following non-empty closed subset

$$C_+ = \{\eta \in L^2(\Gamma_3); \ \eta \geq 0 \ a.e. \text{ on } \Gamma_3\}$$

and for a fixed $\eta \in C_+$ let us consider the following contact problem with given friction.

**Problem $Q^{i\eta}_n$**. Find $u^{i+1}_\eta \in K$ such that

$$\begin{aligned}
& (A u^{i+1}_\eta, w - u^{i+1}_\eta)_V + j_\eta(w - u^i) - j_\eta(u^{i+1}_\eta - u^i) \\
& \geq (f^{i+1}, w - u^{i+1}_\eta)_V \quad \forall w \in K,
\end{aligned}$$

(3.3)
where the operator $A : V \to V$ and the functional $j_\eta : V \to \mathbb{R}$ are respectively defined by

$$(Au, v)_V = a(u, v) + j_c(u, v), \quad j_\eta(v) = \int_{\Gamma_3} \mu|v_r|da.$$ 

We now prove the following lemma.

**Lemma 3.2.** Problem $Q_{i\eta}$ has a unique solution.

**Proof.** By the assumptions (2.13), (2.15)(b) and (2.15)(c), the operator $A$ is strongly monotone and Lipschitz continuous; on the other hand the functional $j_\eta$ is a semi-norm continuous. Since $K$ is a non-empty closed convex subset of $V$, it follows from the theory of elliptic variational inequalities [3] that the inequality (3.3) has a unique solution.

In the second step we define the following mapping

$$T : C_+ \to C_+ \quad \text{as} \quad \eta \to T(\eta) = u_\eta.$$ 

Then the following lemma holds.

**Lemma 3.3.** $T$ has a unique fixed point $\eta^*$ and $u_{\eta^*}$ is a unique solution of Problem $Q^i_\eta$.

**Proof.** We set $v = u_{\eta_1}$ in inequality of Problem $Q^i_{\eta_2}$ and $v = u_{\eta_2}$ in inequality of Problem $Q^i_{\eta_1}$. Using (2.15)(c), we find after adding the resulting inequalities that

$$a(u_{\eta_1} - u_{\eta_2}, u_{\eta_1} - u_{\eta_2}) \leq j_\eta(u_{\eta_2} - u^i) - j_\eta(u_{\eta_1} - u^i) + j_{\eta_2}(u_{\eta_1} - u^i) - j_{\eta_2}(u_{\eta_2} - u^i).$$

Hence using (2.15)(c), (2.9), (2.10) and (2.13)(b), we get

$$\|T(\eta_1) - T(\eta_2)\|_{L^2(\Gamma_3)} \leq \frac{Md_\Omega C_R}{m} \|\mu\|_{L^\infty(\Gamma_3)} \|\eta_1 - \eta_2\|_{L^2(\Gamma_3)}.$$ 

Then it follows that for $\|\mu\|_{L^\infty(\Gamma_3)} < m/Md_\Omega C_R$, $T$ is contractive; thus it admits a unique fixed point $\eta^*$ and $u_{\eta^*}$ is a unique solution of Problem $Q^i_\eta$.

**Remark 3.4.** As $u_{\eta^*} \in V_0$ then $u^{i+1} \in V_0$.

4. **EXISTENCE RESULT**

The main result of this section is to show the existence of a solution obtained as a limit of the interpolate function of the discrete solution. For thus it is necessary at first to establish the following lemma.
Lemma 4.1. There exists a constant $c > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < m/M\Omega C_R$, we have
\begin{equation}
\|u^{i+1}\|_V \leq c\|f^{i+1}\|_V, \quad \|\Delta u^i\|_V \leq c\|\Delta f^i\|_V.
\end{equation}

Proof. Take $w = 0$ in inequality (3.2); then, using (2.12)(b) and (2.9), one obtains that there exists a constant $c > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < m/M\Omega C_R$, the first inequality (4.1) holds.

To prove the second inequality, set $v = u^i$ in inequality (3.2) and then $v = u^{i+1}$ in the translated inequality satisfied by $u^i$. We find after adding the resulting inequalities that
\[a(\Delta u^i, \Delta u^i) - j(u^i, u^{i+1} - u^{i-1}) + j(u^i, u^{i-1} - u^i) + j(u^{i+1}, \Delta u^i) \leq (\Delta f^i, \Delta u^i)_V.
\]

On the other hand, we have
\[j(u^i, u^{i+1} - u^{i-1}) - j(u^i, u^{i-1} - u^i) = \int_{\Gamma_3} (p(u^i_v) - p(u^{i+1}_v))\Delta u^i_v da + \int_{\Gamma_3} \mu|\mathbf{R}\sigma_v(u^i)|(|u^{i+1}_\tau - u^{i-1}_\tau| - |u^{i-1}_\tau - u^i_\tau|)da - \int_{\Gamma_3} \mu|\mathbf{R}\sigma_v(u^{i+1})|\Delta u^i_\tau da.
\]
Using (2.15)(c), we have
\[\int_{\Gamma_3} (p(u^i_v) - p(u^{i+1}_v))\Delta u^i_v da \leq 0,
\]
and moreover as
\[\|u^{i+1}_\tau - u^{i-1}_\tau| - |u^{i-1}_\tau - u^i_\tau\| \leq |\Delta u^i_\tau|,
\]
we obtain
\[a(\Delta u^i, \Delta u^i) \leq \int_{\Gamma_3} \mu|\mathbf{R}\sigma_v(\Delta u^i)|\Delta u^i_\tau da + (\Delta f^i, \Delta u^i)_V.
\]

Then using (2.11), (2.9) and (2.13)(b), this last inequality implies
\[m\|\Delta u^i\|^2_V \leq a(\Delta u^i, \Delta u^i) \leq \Omega MC_R\|\mu\|_{L^\infty(\Gamma_3)}\|\Delta u^i\|^2_V + \|\Delta f^i\|_V\|\Delta u^i\|_V.
\]

Therefore, it follows that there exists a constant $c > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < m/M\Omega C_R$,\n\[\|\Delta u^i\|_V \leq c\|\Delta f^i\|_V.
\]

Next, we define the function $u^n(t)$, continuous in $[0, T] \rightarrow V$ by
\[u^n(t) = u^i + \frac{t-t_i}{\Delta t} \Delta u^i \text{ on } [t_i, t_{i+1}], \quad i = 0, \ldots , n - 1.
\]

Then as in [19] we have the following lemmas.
Lemma 4.2. There exists a function $u$, such that, by selecting, a subsequence still denoted $(u^n)$, we have

$$u^n \to u \text{ weak * in } W^{1,\infty}(0, T; V).$$

On the other hand, we introduce the following piecewise constant functions

$$\tilde{u}^n : [0, T] \to V, \quad \tilde{f}^n : [0, T] \to V,$$

defined by

$$\tilde{u}^n(t) = u^{i+1}, \quad \tilde{f}^n(t) = f(t_{i+1}) \quad \forall t \in (t_i, t_{i+1}], \ i = 0, \ldots, n - 1.$$

Lemma 4.3. There exists a subsequence of $(\tilde{u}^n)$ still denoted $(\tilde{u}^n)$ such that the following results hold

\begin{align*}
(\text{i}) \quad & \tilde{u}^n \to u \text{ weak * in } L^\infty(0, T; V), \\
(\text{ii}) \quad & \tilde{u}^n(t) \to u(t) \text{ weakly in } V \ a.e. \ t \in [0, T], \\
(\text{iii}) \quad & u(t) \in K \cap \overline{V}_0, \text{ for all } t \in [0, T].
\end{align*}

We now have the following proposition.

Proposition 4.4. The weak limit $u$ is a solution to Problem $P_2$.

Proof. In the first step we show the inequality (2.22). Indeed, from inequality (3.2) we deduce the inequality

$$a(u^{i+1}, w - u^{i+1}) + j(u^{i+1}, w - u^{i+1}) \geq (f^{i+1}, w - u^{i+1})_V \quad \forall w \in K,$$

from which we deduce that for almost all $t \in (0, T)$,

$$a(\tilde{u}^n(t), w - \tilde{u}^n(t)) + j(\tilde{u}^n(t), w - \tilde{u}^n(t)) \geq (\tilde{f}^n(t), w - \tilde{u}^n(t))_V \quad \forall w \in K.$$

By standard arguments passing to the limit as $n \to +\infty$, we have

$$a(u(t), w - u(t)) + j(u(t), w - u(t)) \geq (f(t), w - u(t))_V \quad \forall w \in K$$

which implies by Green’s formula that for all $t \in [0, T]$,

$$(\sigma_\nu(u(t)), w_\nu - u_\nu(t))_{\Gamma_3} + (p(u_\nu(t)), w_\nu - u_\nu(t))_{L^2(\Gamma_3)} \geq 0 \quad \forall w \in K$$

and then (2.22) follows.

Now, in order to prove the inequality (2.21), in the first inequality (3.1) take, for $v \in V$, $w = u^i + v\Delta t$ and divide by $\Delta t$, one obtains

$$a\left(u^{i+1}, v - \frac{\Delta u^i}{\Delta t}\right) + j(u^{i+1}, v) - j\left(u^{i+1}, \frac{\Delta u^i}{\Delta t}\right) \geq \left(f^{i+1}, v - \frac{\Delta u^i}{\Delta t}\right)_V +$$

$$+ \left(\sigma_\nu(u^{i+1}), v_\nu - \frac{\Delta u_\nu^i}{\Delta t}\right)_{\Gamma_3} + \left(p(u_\nu^{i+1}), v_\nu - \frac{\Delta u_\nu^i}{\Delta t}\right)_{L^2(\Gamma_3)} \forall v \in V.$$
prove the following lemmas.

Now, in order to pass to the limit in the previous inequality, we must at first

\[ (4.4) \]

\[ (4.5) \]

Since from the second inequality (3.1) we have

\[
\left< \sigma \nu(u^{i+1}), v \nu - \frac{\Delta u^i}{\Delta t} \right>_{\Gamma_3} + (p(u^{i+1}), v \nu - u^{i+1})_{L^2(\Gamma_3)} \geq
\]

\[
\geq \langle \sigma \nu(u^{i+1}), v \nu \rangle_{\Gamma_3} + (p(u^{i+1}), v \nu)_{L^2(\Gamma_3)},
\]

then it follows that

\[
a \left( u^{i+1}, v - \frac{\Delta u^i}{\Delta t} \right) + j(u^{i+1}, v) - j \left( u^{i+1}, \frac{\Delta u^i}{\Delta t} \right) \geq
\]

\[
\geq (f^{i+1}, v - \frac{\Delta u^i}{\Delta t})_V + \langle \sigma \nu(u^{i+1}), v \nu \rangle_{\Gamma_3} + (p(u^{i+1}), v \nu)_{L^2(\Gamma_3)} \quad \forall v \in V.
\]

This inequality implies that for any \( v \in L^2(0, T; V) \),

\[
\left\{ \begin{array}{l}
a(\tilde{u}^n(t), v(t) - \dot{u}^n(t)) + j(\tilde{u}^n(t), v(t)) - j(\tilde{u}^n(t), \dot{u}^n(t)) \\
\geq (\tilde{f}^n(t), v(t) - \dot{u}^n(t))_V + \langle \sigma \nu(\tilde{u}^n(t)), v \nu(t) \rangle_{\Gamma_3} \\
+ (p(\tilde{u}^n(t)), v \nu(t))_{L^2(\Gamma_3)} \text{ for a.a. } t \in (0, T).
\end{array} \right.
\]

Integrating both sides of the previous inequality on \((0, T)\) we obtain the inequality

\[
(4.3) \quad \int_0^T a(\tilde{u}^n(t), v(t) - \dot{u}^n(t))dt + \int_0^T j(\tilde{u}^n(t), v(t))dt -
\]

\[
- \int_0^T j(\tilde{u}^n(t), \dot{u}^n(t))dt \geq \int_0^T (\tilde{f}^n(t), v(t) - \dot{u}^n(t))_V dt +
\]

\[
+ \int_0^T \langle \sigma \nu(\tilde{u}^n(t)), v \nu(t) \rangle_{\Gamma_3} + (p(\tilde{u}^n(t)), v \nu(t))_{L^2(\Gamma_3)}dt.
\]

Now, in order to pass to the limit in the previous inequality, we must at first prove the following lemmas.

**Lemma 4.5.** We have

\[
(4.4) \quad \liminf_{n \to \infty} \int_0^T a(\tilde{u}^n(t), u^n(t))dt \geq \int_0^T a(u(t), \dot{u}(t))dt,
\]

\[
(4.5) \quad \liminf_{n \to \infty} \int_0^T j(\tilde{u}^n(t), \dot{u}^n(t))dt \geq \int_0^T j(u(t), \dot{u}(t))dt.
\]

**Proof.** For the proof of (4.4) it suffices to see [4]. To prove (4.5) we have

\[
\int_0^T j(\tilde{u}^n(t), \dot{u}^n(t))dt = \int_0^T j_0(\tilde{u}^n(t), \dot{u}^n(t))dt + \int_0^T j_1(\tilde{u}^n(t), \dot{u}^n(t))dt.
\]
We now see that
\[ \int_0^T j_c(\tilde{u}^n(t), \tilde{u}^n(t))dt = \]
\[ = \int_0^T (j_c(\tilde{u}^n(t), \tilde{u}^n(t)) - j_c(u(t), \tilde{u}^n(t)))dt + \int_0^T j_c(u(t), \tilde{u}^n(t))dt. \]
Next we have
\[ \liminf_{n \to +\infty} \int_0^T j_c(\tilde{u}^n(t), \tilde{u}^n(t))dt \geq \]
\[ \geq \liminf_{n \to +\infty} \int_0^T (j_c(\tilde{u}^n(t), \tilde{u}^n(t)) - j_c(u(t), \tilde{u}^n(t)))dt + \liminf_{n \to +\infty} \int_0^T j_c(u(t), \tilde{u}^n(t))dt. \]
Using standard arguments, the first term of the right hand side of the previous inequality is estimated as
\[ \left| \int_0^T (j_c(\tilde{u}^n(t), \tilde{u}^n(t)) - j_c(u(t), \tilde{u}^n(t)))dt \right| \leq c\|\tilde{u}_\nu^n - u_\nu\|_{L^2(0,T;L^2(\Gamma_3))} \]
which implies as \( \tilde{u}_\nu^n \to u_\nu \) strongly in \( L^2(0,T;L^2(\Gamma_3)) \),
\[ \lim_{n \to +\infty} \int_0^T (j_c(\tilde{u}^n(t), \tilde{u}^n(t)) - j_c(u(t), \tilde{u}^n(t)))dt = 0. \]
The functional \( v \to \int_0^T j_c(u(t), v(t))dt \) is convex and continuous on \( L^2(0,T;V) \) so it is sequentially weakly lower semicontinuous, which implies that
\[ \liminf_{n \to +\infty} \int_0^T j_c(u(t), \tilde{u}^n(t))dt \geq \int_0^T j_c(u(t), \tilde{u}^n(t))dt. \]
Using now (4.7), (4.8) and keeping in mind (see [4]) that
\[ \liminf_{n \to +\infty} \int_0^T j_f(\tilde{u}^n(t), \tilde{u}^n(t))dt \geq \int_0^T j_f(u(t), \tilde{u}^n(t))dt, \]
we deduce from (4.6) that
\[ \liminf_{n \to +\infty} \int_0^T j(\tilde{u}^n(t), \tilde{u}^n(t))dt \geq \int_0^T j_c(u(t), \tilde{u}^n(t))dt + \int_0^T j_f(u(t), \tilde{u}^n(t))dt = \]
\[ = \int_0^T j(u(t), \tilde{u}(t))dt. \]

**Lemma 4.6.** For all \( v \in L^2(0,T;V) \) the following properties hold:
\[ \lim_{n \to +\infty} \int_0^T a(\tilde{u}^n(t), v(t))dt = \int_0^T a(u(t), v(t))dt, \]
\[ \lim_{n \to +\infty} \int_0^T b(\tilde{u}^n(t), v(t))dt = \int_0^T b(u(t), v(t))dt, \]
\[ \lim_{n \to +\infty} \int_0^T c(\tilde{u}^n(t), \tilde{u}^n(t))dt = \int_0^T c(u(t), u(t))dt. \]
Similarly, we have
\[
\lim_{n \to \infty} \int_0^T (\tilde{F}^n(t), v(t)) \, dt = \int_0^T (f(t), v(t) - \dot{u}(t)) \, dt,
\]
\[
\lim_{n \to \infty} \int_0^T \left( \vec{F}^n(t), v(t) - \ddot{u}^n(t) \right) \, dt = \int_0^T (f(t), v(t) - \dot{u}(t)) \, dt,
\]
\[
\lim_{n \to \infty} \int_0^T \left( (\sigma_\nu(\tilde{u}^n(t)), v_\nu(t))_{\Gamma_3} + (p(\tilde{u}^n(t)), v_\nu(t))_{L^2(\Gamma_3)} \right) \, dt = \\
\int_0^T (\sigma_\nu(u(t)), v_\nu(t))_{\Gamma_3} + (p(u_\nu(t)), v_\nu(t))_{L^2(\Gamma_3)} \, dt.
\]

\textbf{Proof.} To prove (4.9) it suffices to see [4]. To show (4.10) we write for \( t \in (0, T) \),
\[
j(\tilde{u}^n(t), v(t)) = (j(\tilde{u}^n(t), v(t)) - j(u(t), v(t))) + j(u(t), v(t)).
\]
Since
\[
j(\tilde{u}^n(t), v(t)) - j(u(t), v(t)) = j_c(\tilde{u}^n(t), v(t)) - j_c(u(t), v(t)) + \\
j_f(\tilde{u}^n(t), v(t)) - j_f(u(t), v(t)),
\]
using (2.15)(b) and (2.9), we find
\[
\left| \int_0^T (j_c(\tilde{u}^n(t), v(t)) - j_c(u(t), v(t))) \, dt \right| \leq c \| \tilde{u}^n_\nu - u_\nu \|_{L^2(0, T; L^2(\Gamma_3))} \| v \|_{L^2(0, T; V)},
\]
which implies
\[
\lim_{n \to \infty} \int_0^T (j_c(\tilde{u}^n(t), v(t)) - j_c(u(t), v(t))) \, dt = 0.
\]
Similarly, we have
\[
\lim_{n \to \infty} \int_0^T (j_f(\tilde{u}^n(t), v(t)) - j_f(u(t), v(t))) \, dt = 0,
\]
as
\[
\left| \int_0^T (j_f(\tilde{u}^n(t), v(t)) - j_f(u(t), v(t))) \, dt \right| \leq c \| R\sigma_\nu(\tilde{u}^n_\nu - u_\nu) \|_{L^2(0, T; L^2(\Gamma_3))} \| v \|_{L^2(0, T; V)},
\]
where (see [4])
\[
\lim_{n \to +\infty} \| R\sigma_\nu(\tilde{u}^n_\nu - u_\nu) \|_{L^2(0, T; L^2(\Gamma_3))} = 0.
\]
Then (4.10) follows. To show (4.11), we use (4.2)(i) and that $\tilde{f}^n \to f$ strongly in $L^2(0,T; V)$. To prove (4.12), we use (2.10), (4.2)(ii) and that $\tilde{u}^n \to u_\nu$ strongly in $L^2(0,T; L^2(\Gamma_3))$.

Next, by going to the limit in inequality (4.5) using Lemmas 4.5 and 4.6, we obtain the following inequality

$$
\int_0^T (a(u(t), v(t) - \dot{u}(t)) + j(u(t), v(t)) - j(u(t), \dot{u}(t))) dt \geq \int_0^T (f(t), v(t) - \dot{u}(t))v dt + \int_0^T (\sigma_\nu(u(t)), v_\nu(t))_{\Gamma_3} dt + \\
+ \int_0^T (p(u_\nu(t)), v_\nu(t))_{L^2(\Gamma_3)} dt \quad \forall v \in L^2(0,T; V).
$$

Finally, we plug (2.23) in (4.13), and by a standard argument from the inequality obtained one obtains (2.21).

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