Dedicated to Professor Radu Miron
on the occasion of his 85^{th} birthday

A NONLINEAR CONNECTION FOR HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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A geometric setting for studying higher order ordinary differential equations (HODE) is obtained by choosing a nonlinear connection associated to the HODE. One such nonlinear connection was introduced in local coordinates by Miron and Atanasiu [23]. In this note we summarize results from [8], and show that this nonlinear connection has many of the useful properties of the canonical nonlinear connection associated to a system of second order differential equations. For example, (1) the nonlinear connection has simple coordinate free characterizations, (2) using the connection, key objects like the Jacobi endomorphism, dynamical covariant derivatives and variation equations can be written using simple equations and (3) the connection can be used to geometrically express some of the invariants that appear in problems like equivalence problems.

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1. INTRODUCTION

To a system of second order ordinary differential equations (SODE) one can associate a *nonlinear connection* in a canonical way. The advantage of this nonlinear connection is that it provides a geometric setting for the study of the SODE. From the nonlinear connection one can derive geometric structures such as the dynamical covariant derivative, torsion and curvature tensors, and the Jacobi endomorphism. Using these objects one can further write down simple geometric equations for the behavior of parallel transport, symmetries and first order geodesic variations. This geometric setting is well developed and has been used by numerous authors for the study of solutions to SODE, [2, 12, 25, 28, 29].

Due to these applications, it would be attractive to have a similar geometric framework also for systems of higher order ordinary differential equations

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(HODE). However, associating such a geometric structure, or nonlinear connection, to a HODE turns out to be more involved that in the second order case. It is now known that to a HODE one can associate a number of different nonlinear connections, and each of these offer different information about the HODE [1, 3, 5, 6, 10, 11, 19, 20, 23, 21]. The purpose of this paper is to summarize our results in [8] on this topic. In [8] we consider the nonlinear connection introduced in local coordinates by Miron and Atanasiu in [23], and proved that this connection has many of the key properties associated with the canonical nonlinear connection of an SODE. Such properties include:

1. simple coordinate invariant characterizations;
2. simple expressions for key objects like the Jacobi endomorphism and the Jacobi equation;
3. simple expressions for some of the invariants that appear, for example, in equivalence problems;
4. in the second order case, the connection reduces to the usual nonlinear connection associated to a SODE.

Below these properties for the nonlinear connection in [23] are summarized in more detail. We will not give any proofs, but refer the interested reader to [8] for a more extensive discussion.

The approach in [8] is inspired by ideas of Kosambi [17], who states that a "system of differential equations can be dealt with geometrically by means of the tensorial operator of differentiation, called the bi-derivative". Therefore, we propose a global expression, which is called the dynamical covariant derivative, for this bi-derivative operator introduced by Kosambi. Below this dynamical covariant derivative ∇ in defined by equation (2.9), and the novelty of this definition is that it depends on both on a HODE and a nonlinear connection and these need not be related. In Theorem 3.1 we will see that a unique nonlinear connection is determined by a suitable compatibility condition on higher order iterates of ∇. Moreover, this unique nonlinear connection is the nonlinear connection introduced by Miron and Atanasiu in [23].

2. PRELIMINARIES

In this section we describe a natural setting for studying systems of $k+1$ order ordinary differential equations on a manifold M. We will see that such a system can naturally be identified with a suitable vector field called a semispray on the kth order tangent bundle of M, that is, on a suitable jet bundle on M. We also define the notion of a nonlinear connection and the Jacobi endomorphism associated with a kth order semispray and a nonlinear connection.

Semisprays

Throughout this work we assume that M is a real, C^∞-smooth, and n-dimensional manifold. We also assume that all objects are smooth where defined. Vector fields on a manifold M are denoted by $\mathfrak{X}(M)$, smooth functions by $C^\infty(M)$, and for vector fields X, Y the Lie bracket is denoted by $\mathcal{L}_X Y$.

The natural setting for studying systems of $k+1$ order ordinary differential equations on M is the tangent bundle $T^kM = J^k_0 M$ of order $k \geq 1$ [10, 21]. This is the kth order jet bundle of curves c from a neighborhood of $0 \in \mathbb{R}$ to M. If $c : I \to M$, $c(t) = (x^i(t))$ is a smooth curve on M, let $j^k c : I \to T^k M$ be its jet lift of order k, that is,

$$j^k c(t) = \left(x^i(t), \frac{1}{1!}\frac{\mathrm{d}x^i}{\mathrm{d}t}(t), \dots, \frac{1}{k!}\frac{\mathrm{d}^k x^i}{\mathrm{d}t^k}(t) \right).$$

If (x^i) are local coordinates on M then induced coordinates $(x^i, y^{(1)i}, \dots, y^{(k)i})$ for T^kM are given by

$$y^{(\alpha)i}(j^k_0 c) = \frac{1}{\alpha!}\frac{\mathrm{d}^\alpha (x^i(c(t)))}{\mathrm{d}t^\alpha}\bigg|_{t=0}, \quad \alpha \in \{1, \dots, k\}.$$

Let us also denote $y^{(0)i} = x^i$ and $T^0 M = M$.

The *tangent structure* (or vertical endomorphism) of order k is the $(1,1)$-type tensor field on T^kM defined as

$$J = \frac{\partial}{\partial y^{(1)i}} \otimes \mathrm{d}x^i + \frac{\partial}{\partial y^{(2)i}} \otimes \mathrm{d}y^{(1)i} + \dots + \frac{\partial}{\partial y^{(k)i}} \otimes \mathrm{d}y^{(k-1)i}, \tag{2.1}$$

whence $J^{k+1} = 0$. For each $\alpha \in \{0, 1, \dots, k-1\}$, the canonical submersion $\pi^k_\alpha : T^kM \to T^\alpha M$ induces a natural foliation of T^kM. This foliated structure of T^kM determines k regular vertical distributions on T^kM that are defined by the tangent structure J as

$$V_\alpha(u) = \operatorname{Im} J^\alpha_u \quad \text{for } u \in T^kM \text{ and } \alpha \in \{1, \dots, k\},$$

whence $V_k(u) \subset V_{k-1}(u) \subset \dots \subset V_1(u)$ for all $u \in T^kM$.

A system of $(k+1)$ order ordinary differential equations on M can be represented using a *semispray of order* k on M. It is a vector field $S \in \mathfrak{X}(T^kM)$ such that any integral curve $\gamma : I \to T^kM$ of S is of the form $\gamma = j^k(\pi^k_0 \circ \gamma)$. In induced coordinates for T^kM, a semispray of order k is locally given by

$$S = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i\frac{\partial}{\partial y^{(k)i}}, \tag{2.2}$$

for some functions G^i defined on the domain of the induced local chart.

For an integral curve $\gamma : I \to T^kM$ of S, the curve $c = \pi^k_0 \circ \gamma$ is a *geodesic* of S. Therefore, a curve $c : I \to M$ is a geodesic of S if and only if $S \circ j^k c = (j^k c)'$. Locally, a curve $c : I \to M$, $c(t) = (x^i(t))$, is a geodesic of S if and only if it locally satisfies the system of $(k+1)$ order ordinary differential equations

$$\frac{1}{(k+1)!}\frac{\mathrm{d}^{k+1}x^i}{\mathrm{d}t^{k+1}} + G^i\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \dots, \frac{1}{k!}\frac{\mathrm{d}^k x}{\mathrm{d}t^k}\right) = 0. \tag{2.3}$$

Hence a semispray of order k describes systems of HODE with curves on M as solutions. Conversely, local systems of HODE as in equation (2.3) determine a semispray of order k provided that functions G^i satisfy suitable transformation rules on overlapping charts.

Nonlinear connection

A *nonlinear connection* (or horizontal distribution) is an n-dimensional distribution $H_0 : u \in T^kM \mapsto H_0(u) \subset T_uT^kM$ that is supplementary to the vertical distribution $V_1(u)$, so that $T_uT^kM = H_0(u) \oplus V_1(u)$ for all $u \in T^kM$. For a nonlinear connection H_0, consider also the n-dimensional distributions $H_\alpha = J^\alpha(H_0)$ for $\alpha \in \{1, \dots, k\}$, where $H_k = V_k$. It follows that

$$T_uT^kM = \oplus_{\alpha=0}^{k-1} H_\alpha(u) \oplus V_k(u) = \oplus_{\alpha=0}^{k} H_\alpha(u), \quad \forall u \in T^kM. \tag{2.4}$$

We will denote by $h_0, h_1, \dots, h_{k-1}, v_k$ the projectors that correspond to decomposition (2.4), and these projectors characterize the nonlinear connection. Sometimes, we will also denote $h_k = v_k$. Decomposition (2.4) induces also a decomposition of the vertical subspaces

$$V_\alpha(u) = \oplus_{\beta=\alpha}^{k-1} H_\beta(u) \oplus V_k(u), \quad \forall u \in T^kM,\ \forall \alpha \in \{1, \dots, k-1\}. \tag{2.5}$$

In this work we will use bases for $\mathfrak{X}(T^kM)$, which are adapted to the decomposition (2.4). Such a basis is given by

$$\frac{\delta}{\delta x^i} = h_0\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \sum_{\beta=1}^{k} N^j_{(\beta)i}\frac{\partial}{\partial y^{(\beta)j}}, \quad \frac{\delta}{\delta y^{(\alpha)i}} = J^\alpha\left(\frac{\delta}{\delta x^i}\right), \tag{2.6}$$

for $\alpha \in \{1, \dots, k\}$. Functions $N^i_{(\alpha)j}$ are local functions on T^kM and called the *coefficients of the nonlinear connection.*

The dual basis to the basis (2.6) is given by the following locally defined 1-forms on T^kM

$$\mathrm{d}x^i, \delta y^{(\alpha)i} = \mathrm{d}y^{(\alpha)i} + \sum_{\beta=1}^{\alpha} M^i_{(\beta)j}\mathrm{d}y^{(\alpha-\beta)j}, \quad \alpha \in \{1, \dots, k\}, \tag{2.7}$$

where *the dual coefficients of the nonlinear connection* $M^i_{(\beta)j}$ are given by [21]

$$M^i_{(1)j} = N^i_{(1)j}, \quad M^i_{(\alpha)j} = N^i_{(\alpha)j} + \sum_{\beta=1}^{\alpha-1} N^i_{(\alpha-\beta)s} M^s_{(\beta)j}, \quad \alpha \in \{2, \dots, k\}.$$

Let us emphasize that bases (2.6) and (2.7) are determined by the nonlinear connection alone. With respect to these adapted bases, the tangent structure J in equation (2.1) can be rewritten as

$$J = \frac{\delta}{\delta y^{(1)i}} \otimes \mathrm{d}x^i + \frac{\delta}{\delta y^{(2)i}} \otimes \delta y^{(1)i} + \cdots + \frac{\partial}{\partial y^{(k)i}} \otimes \delta y^{(k-1)i}.$$

The Jacobi endomorphism

Next we define the *Jacobi endomorphism*, which is a curvature type tensor that will play a key role in the next section. If (h_α, v_k) is a nonlinear connection and S is a semispray of order k, then the *Jacobi endomorphism* is the vertically valued $(1,1)$-type tensor field on $T^k M$ defined as

$$\Phi = \sum_{\alpha=0}^{k-1} v_k \circ \mathcal{L}_S \circ h_\alpha.$$

The novelty of this definition (introduced in [8]) is that it considers (h_α, v_k) and S as independent objects. In Theorem 3.1 we will see that a unique nonlinear connection can be determined by assuming that (h_α, v_k), S and Φ are suitably compatible.

With respect to the adapted bases (2.6) and (2.7) the Jacobi endomorphism can locally be written as

$$\Phi = \sum_{\alpha=0}^{k-1} R^i_{(\alpha)j} \frac{\partial}{\partial y^{(k)i}} \otimes \delta y^{(\alpha)j}. \tag{2.8}$$

Under a change of induced local coordinates on $T^k M$, components $R^i_{(\alpha)j}$ in the Jacobi endomorphism transform as the components of a $(1,1)$-type tensor field on the base manifold M.

Formulae for components $R^i_{(\alpha)j}$ are known explicitly and for an arbitrary nonlinear connection and semispray, these were first derived in [8]. We here omit the explicit expressions as they are rather involved. In [11], the authors explained how these components can be computed for a particular nonlinear connection, but without explicit formulae. In Theorem 3.1, we will see that when the non-linear connection is the canonical nonlinear connection introduced by Miron and Atanasiu in [23], these expressions can be written using simple formulae. See Equation (3.1).

Two dynamical covariant derivatives

For use in the next section, we define two differential operators on tensor fields on T^kM: the dynamical covariant derivative and the dynamical covariant derivative of order α where $\alpha \in \{1, \dots, k\}$.

As in the definition of Φ, let S be a semispray of order k and let (h_α, v_k) be a nonlinear connection. Then the *dynamical covariant derivative* for the pair $(S, (h_\alpha, v_k))$ is the differential operator $\nabla : \mathfrak{X}(T^kM) \to \mathfrak{X}(T^kM)$ defined as

$$\nabla = \sum_{\alpha=0}^{k-1} h_\alpha \circ \mathcal{L}_S \circ h_\alpha + v_k \circ \mathcal{L}_S \circ v_k. \tag{2.9}$$

For $f \in C^\infty(T^kM)$, let also $\nabla f = S(f)$. By insisting that ∇ satisfies the Leibniz rule and that ∇ commutes with tensor contractions, the action of ∇ extends to arbitrary tensor fields on T^kM.

Formula (2.9) implies that $\nabla h_\alpha = 0$ for every $\alpha \in \{0, \dots, k\}$. Hence ∇ preserves all distributions H_α. From formula (2.5) it also follows that ∇ preserves all vertical distributions V_α. Dynamical covariant derivative ∇ acts in a similar way on distributions H_α, but has a different action on the last vertical distribution V_k. Indeed,

$$\nabla \frac{\delta}{\delta y^{(\alpha)i}} = N^j_{(1)i} \frac{\delta}{\delta y^{(\alpha)j}}, \quad \alpha \in \{0, \dots, k-1\},$$
$$\nabla \frac{\partial}{\partial y^{(k)i}} = \left((k+1) \frac{\partial G^j}{\partial y^{(k)i}} - k N^j_{(1)i} \right) \frac{\partial}{\partial y^{(k)j}}.$$

For a vector field $X \in \mathfrak{X}(T^kM)$, $X = \sum_{\beta=0}^{k} X^{(\beta)i} \delta / \delta y^{(\beta)i}$ and $\alpha \in \{1, \dots, k\}$ the *dynamical covariant derivative of order* α is defined as the differential operator

$$\nabla^{(\alpha)} X = \sum_{\beta=0}^{k} \nabla^{(\alpha)} X^{(\beta)i} \frac{\delta}{\delta y^{(\beta)i}}, \quad \alpha \in \{1, \dots, k\},$$

where

$$\frac{1}{\alpha!} \nabla^{(\alpha)} X^i = \frac{1}{\alpha!} S^\alpha(X^i) + \sum_{\beta=1}^{\alpha} \frac{1}{(\alpha-\beta)!} M^i_{(\beta)j} S^{\alpha-\beta}(X^j).$$

Let also $\nabla^{(0)} = \mathrm{Id}$ and for $f \in C^\infty(T^kM)$ and $\alpha \in \{1, \dots, k\}$, let $\nabla^{(\alpha)} f = S^\alpha(f)$. Then $\nabla^{(\alpha)}$ satisfies the Leibniz-type rule of order α, that is,

$$\nabla^{(\alpha)}(fX) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \nabla^{(\beta)} f \cdot \nabla^{(\alpha-\beta)} X.$$

By requiring that $\nabla^{(\alpha)}$ satisfies the Leibniz-type rule of order α for arbitrary tensors, and that $\nabla^{(\alpha)}$ commutes with tensor contractions, the action of $\nabla^{(\alpha)}$ extends to arbitrary tensor fields on T^kM. Let us emphasise that differential operator $\nabla^{(\alpha)}$ is not *a priori* related to the αth iteration ∇^α of the dynamical covariant derivative ∇.

3. THE CANONICAL NONLINEAR CONNECTION

The next theorem (proven in [8, Theorem 3.10]) gives conditions that assign a unique nonlinear connection to any semispray S of order k. We will say that this is the *canonical nonlinear connection* associated to S.

THEOREM 3.1. *Let S be a semispray of order $k \geq 1$ and let (h_α, v_k) be a nonlinear connection. Moreover, let ∇ and $\nabla^{(\alpha)}$ be the dynamical covariant derivative and the dynamical covariant derivative of order α associated to the pair $(S, (h_\alpha, v_k))$. Then the following conditions are equivalent:*

i) $\mathcal{L}_S J + \mathrm{Id} - (k+1)v_k = i_J\Phi$;

ii) $\nabla^{(\alpha)} = \nabla^\alpha$ *for all* $\alpha \in \{1, 2, \ldots, k\}$;

iii) $M^j_{(1)i} = \partial G^j/\partial y^{(k)i}$ *and* $\alpha M^i_{(\alpha)j} = S\big(M^i_{(\alpha-1)j}\big) + M^p_{(\alpha-1)j} M^i_{(1)p}$ *for all* $\alpha \in \{2, \ldots, k\}$;

iv) $N^j_{(1)i} = \partial G^j/\partial y^{(k)i}$ *and* $\alpha N^i_{(\alpha)j} = S\big(N^i_{(\alpha-1)j}\big) - N^i_{(\alpha-1)p} N^p_{(1)j}$ *for all* $\alpha \in \{2, \ldots, k\}$.

Let us first note that dual coefficients $M^i_{(\alpha)j}$ given by item iii) in Theorem 3.1 are the same coefficients considered first by Miron and Atanasiu in [23]. For this canonical non-linear connection, the components of the Jacobi endomorphism simplify considerably and take the simple form

$$\begin{aligned} R^i_{(\alpha)j} &= (k+1)\left(\frac{\delta G^i}{\delta y^{(\alpha)j}} - N^i_{(k+1-\alpha)j}\right), \quad \alpha \in \{1, \ldots, k-1\}, \\ R^i_{(0)j} &= (k+1)\frac{\delta G^i}{\delta x^j} - S\big(N^i_{(k)j}\big) + N^i_{(k)l} N^l_{(1)j}. \end{aligned} \tag{3.1}$$

Let us also note that the $(1,1)$-type tensor $\mathcal{L}_S J - \mathrm{Id} + (k+1)v_k$ in item i) has been used to characterize other nonlinear connections associated to a semispray of order k. For example if we require it to vanish, we obtain the nonlinear connection studied in [1, 3, 10, 11]. When $k = 1$, we have $i_J\Phi = 0$ and both of these nonlinear connections coincide with the usual nonlinear connection $v = \frac{1}{2}(\mathrm{Id} + \mathcal{L}_S J)$ associated to a semispray. However, for $k > 1$ the different nonlinear connections need not coincide.

4. APPLICATIONS OF THE CANONICAL NONLINEAR CONNECTION

In this section we describe some applications of the canonical nonlinear connection in Theorem 3.1. For a further applications and discussions, see [8].

Symmetries for geodesics

Let us consider symmetries of geodesics for a semispray S of order k. A vector field $X \in \mathfrak{X}(T^kM)$ is a *dynamical symmetry* if $\mathcal{L}_S X = 0$, that is, if the geodesic flow of S commutes with the flow of vector field X. If locally

$$X = X^i\big(x, y^{(\beta)}\big)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{k} Y^{(\alpha)i}\big(x, y^{(\beta)}\big)\frac{\partial}{\partial y^{(\alpha)i}}, \tag{4.1}$$

then writing out $\mathcal{L}_S X = 0$ shows that X is a dynamical symmetry if and only if the components satisfy

$$(k+1)!X(G^i) + S^{k+1}(X^i) = 0, \tag{4.2}$$

$$\alpha! Y^{(\alpha)i} = S^\alpha(X^i) \quad \text{for } \alpha \in \{1, \dots, k\}. \tag{4.3}$$

The next theorem (see [8, Theorem 3.11]) geometrically characterizes dynamical symmetries using the canonical nonlinear connection in Theorem 3.1.

THEOREM 4.1. *Let (h_α, v_k) be the canonical nonlinear connection induced by a semispray S, and let ∇ be the corresponding dynamical covariant derivative. Then a vector field $X \in \mathfrak{X}(T^kM)$ is a dynamical symmetry if and only if*

i) $X = \pi_S^k(X)$ *where* $\pi_S^k : \mathfrak{X}(T^kM) \to \mathfrak{X}(T^kM)$ *is the differential operator*

$$\pi_S^k = \mathrm{Id} + \frac{1}{1!}J \circ \mathcal{L}_S + \frac{1}{2!}J^2 \circ \mathcal{L}_S^2 + \cdots + \frac{1}{k!}J^k \circ \mathcal{L}_S^k;$$

ii) *vector field X satisfies the Jacobi-type equation*

$$\frac{1}{k!}\nabla^{k+1}J^kX + \Phi(X) = 0.$$

A vector fields $X \in \mathfrak{X}(T^kM)$ that satisfies $X = \pi_S^k(X)$ as in condition i) is known as a *newtonoid* [8]. The expression for π_S^k is motivated by the generalized Cartan operator acting on 1-forms on T^kM [10, 26]. Locally, X is a newtonoid if X satisfies condition (4.3). With this terminology, Theorem 4.1 states that a vector field X is a dynamical symmetry if and only if X is a newtonoid and X satisfies the Jacobi-type equation in condition ii). Locally, condition ii) can also be rewritten as

$$\frac{1}{k!}\nabla^{k+1}X^i + \sum_{\alpha=0}^{k-1}\frac{1}{\alpha!}R^i_{(\alpha)j}\nabla^\alpha X^j = 0,$$

where $X^i = X^i(x, y^{(\beta)})$ are components for the horizontal components of X and $\nabla X^i = S(X^i) + M^i_{(1)j}X^j$.

Theorem 4.1 illustrates how dynamical symmetries can be described using the canonical nonlinear connection. In [8] we show that the equations for Lie symmetries and variation fields for geodesic variations can also be written using the canonical nonlinear connection. In both cases we obtain similar Jacobi-type equations as in Theorem 4.1. (A Lie symmetry is a vector field on M that lifts to a dynamical symmetry.)

Invariants

On an interval $I \subset \mathbb{R}$, let us consider the third-order differential equation for a scalar function $x\colon I \to \mathbb{R}$,

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + 3!G\left(x, \frac{1}{1!}\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{1}{2!}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right) = 0, \tag{4.4}$$

and let S be the associated second order semispray $S \in \mathfrak{X}(T^2 I)$. To this equation the *Wuenschmann invariant* is given by [24, Equation (4)]

$$W_3 = -\frac{1}{2}S^2\left(\frac{\partial G}{\partial y^{(2)}}\right) - 3\frac{\partial G}{\partial y^{(2)}}S\left(\frac{\partial G}{\partial y^{(2)}}\right) + 3S\left(\frac{\partial G}{\partial y^{(1)}}\right) - \\ -2\left(\frac{\partial G}{\partial y^{(2)}}\right)^3 + 6\frac{\partial G}{\partial y^{(1)}}\frac{\partial G}{\partial y^{(2)}} - 6\frac{\partial G}{\partial x}.$$

This invariant was introduced by Wuenschmann in his Ph.D. Thesis of 1905. One property of the Wuenschmann invariant W_3 is that there exists an associated conformal Lorentzian structure on the 3-dimensional solution space of Equation (4.4) if and only if W_3 vanishes [11, 15]. For a geometric derivation of the Wuenschmann invariant, see [11], and for the role of the Wuenschmann invariant under contact transformations, see [24].

Suppose $R_{(0)}$ and $R_{(1)}$ are the components that appear in the local formula (2.8) for the Jacobi endomorphism for S. Then the Wuenschmann invariant W_3 can be rewritten as [8]

$$W_3 = \nabla R_{(1)} - 2R_{(0)}. \tag{4.5}$$

Let us emphasize that formula (4.5) was obtained in [11] for two other different nonlinear connections.

For a fourth order scalar ordinary differential equation, there is also an analogue of the Wuenschmann-type invariant. This invariant appears, for example, in the equivalence problem for scalar fourth order differential equations under fiber preserving diffeomorphisms studied by Dridi and Neut in [13]. Up to scaling the same invariant also appears in [14], where Fels showed that a scalar fourth-order ordinary differential equation admits a variational multiplier if and only if two invariants vanish. In [8] it is shown that using the canonical nonlinear connection, this invariant can be written using an analogue of equation (4.5), that is, using components for the Jacobi endomorphism and the dynamical covariant derivative.

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