Dedicated to Professor Radu Miron
on the occasion of his 85^{th} birthday

FINSLER MANIFOLDS WITH REVERSIBLE GEODESICS

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We study geometrical properties of Finsler spaces with reversible geodesics focusing especially on (α, β)-metrics.

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1. **INTRODUCTION**

If (M, a) is a Riemannian manifold, then it is easy to see that its geodesics are **reversible**, i.e., if $\gamma : [0, 1] \to M$ is a geodesic of a, then $\overline{\gamma} : [0, 1] \to M$, $\overline{\gamma}(t) := \gamma(1 - t)$ is also a geodesic of a.

In the more general case of an arbitrary Finsler manifold (M, F), this property is not always true. However, there are special Finsler structures on smooth manifolds whose geodesics are reversible. Among these, except the Riemannian ones, we mention: the absolute homogeneous Finsler structures, Randers metrics obtained from a Riemannian structure and a closed 1-form β, etc. (see [5], [7], [11]).

Our main interest is to characterize Finsler structures which have reversible geodesics and emphasize the difference with the absolute homogeneous and Riemannian cases, which we regard hereafter as trivial Finsler structures. We have studied in the past this problem in the Finsler manifolds (M, F) with (α, β)-metrics, obtaining necessary and sufficient conditions for F to be with reversible geodesics. Namely, in any dimension, F is with reversible geodesics if and only if F is a Randers change of an absolute homogeneous Finsler structure on M, by a closed 1-form, or a Minkowsky structure obtained from a flat Riemannian metric and a 1-form β with constant coefficients ([11], [12]).

In the following, we will consider the most general family of Finsler structures with reversible geodesics known, namely the Randers change $F = F_0 + \beta$, where F_0 is an absolute homogeneous Finsler structure on M, and $\beta = b_i(x)y^i$ is a linear form on TM, whose associated differentiable 1-form $\hat{\beta} := b_i(x)dx^i$

is closed. In the present paper, we consider $\hat{\beta}$ to be a 1-form on M and regard $\beta : TM \to \mathbb{R}$ as a map, throughly.

A ubiquitous example is the polynomial (α, β)-metric $\phi(s) = \sum_{k=0}^{p} a_{2k} \cdot s^{2k} + \varepsilon \cdot s$, $\forall p \in \mathbb{N}^*$, where a_{2k} and $\varepsilon \neq 0$ are constants (see Section 2 for details on the ϕ-notation).

Special cases of this polynomial (α, β)-metrics family are the **Randers** metric $\phi(s) = 1 + \varepsilon \cdot s$, and the **quadric** metric $\phi(s) = 1 + \varepsilon \cdot s + s^2$ ([1], [6] and maybe others).

In the present paper we will study some geometrical properties of (α, β)-metrics with reversible geodesics, as well as some global constructions on warped products and symplectic manifolds of these structures. From our present study, one can see that there is plenty of naturally induced nontrivial Finsler structures with reversible geodesics and that their geometry worth more detailed investigation.

2. DEFINITIONS AND BASIC FACTS ON FINSLER MANIFOLDS

Let (M, F) be an n-dimensional connected Finsler manifold (see [2] for definition). Hereafter $TM = \underset{x \in M}{\cup} T_x M$ denotes the tangent bundle of M with local coordinates $u = (x, y) = (x^i, y^i) \in TM$, where $i = 1, \dots, n$, $y = y^i \left(\frac{\partial}{\partial x^i} \right)_x$.

If $\gamma : [0, 1] \to M$ is a piecewise C^∞ curve on M, then its **Finslerian length** is defined as

$$\mathcal{L}_F(\gamma) := \int_0^1 F(\gamma(t), \dot{\gamma}(t)) \mathrm{d}t. \tag{2.1}$$

and the **Finslerian distance** function $d_F : M \times M \to [0, \infty)$ is defined by $d_F(p, q) = \inf_\gamma \mathcal{L}_F(\gamma)$, where the infimum is taken over all piceweise C^∞ curves γ on M joining the points $p, q \in M$. In general, this is not symmetric.

A curve $\gamma : [0, 1] \to M$ is called a **geodesic** of (M, F) if it minimizes the Finslerian length for all piecewise C^∞ curves that keep their end points fixed.

If we denote by $\overline{F}$ the **reverse Finsler** metric of F, that is $\overline{F} : TM \to (0, \infty)$, $\overline{F}(x, y) := F(x, -y)$, then $\overline{F}$ is also a Finsler metric. Moreover, we have:

LEMMA 2.1. *If $\gamma(t)$ is a geodesic of the Finsler space (M, F), then $\overline{\gamma}(t) := \gamma(1 - t)$ is a geodesic of $\overline{F}$, but not necessarily a geodesic of F in general.*

Recall the following definition ([7], [11] and other sources).

Definition 2.2. The Finsler metric F is called **with reversible geodesics** if and only if for any geodesic $\gamma(t)$ of F, the reverse curve $\bar{\gamma}(t) := \gamma(1-t)$ is also a geodesic of F.

We point out that in general $d_F(p,q) = d_{\overline{F}}(q,p)$, $\forall p, q \in M$, but even though a Finsler metric is with reversible geodesics it does not mean that it has symmetric distance function, except for the absolute homogeneous case. Indeed, we have

Proposition 2.3. *Let (M,F) be a connected, complete Finsler manifold with associated distance function $d_F : M \times M \to [0,\infty)$.*

Then d_F is symmetric distance function on $M \times M$ if and only if F is absolute homogeneous, i.e., $F(x,y) = \bar{F}(x,y) = F(x,-y)$.

Proof. If F is absolute homogeneous, then for any curve $c : [0,1] \to M$, $c(0) = p$, $c(1) = q$, we have

$$d_F(p,q) = \inf_c \int_0^1 F(c(t), c'(t))\mathrm{d}t = \tag{2.2}$$

$$= \inf_{\bar{c}} \Big[-\int_0^1 \bar{F}(\bar{c}(s), \bar{c}'(s))(-\mathrm{d}s) \Big] = d_F(q,p),$$

i.e., the induced distance is symmetric.

Conversely, we assume $d_F(p,q) = d_F(q,p)$ for any $p, q \in M$.

Recall that if $\gamma_v(t)$ is a Finslerian geodesic from p with initial velocity v, then the fundamental Finsler function can be recovered by the following Busemann-Meyer formula $F(p,v) = \lim_{t\searrow 0} \frac{d_F(p,\gamma_v(t))}{t}$ (see [2] for details).

Using these notations, the symmetry of the distance function and the Lemma 2.1 we have

$$F(p,v) = \lim_{t\searrow 0} \frac{d_F(p,\gamma_v(t))}{t} = \lim_{t\searrow 0} \frac{d_F(\gamma_v(t),p)}{t} = \lim_{t\searrow 0} \frac{d_{\bar{F}}(p,\bar{\gamma}_v(t))}{t} = F(p,-v)$$

and the Proposition is proved. □

Locally, a smooth curve $\gamma : [0,1] \to M$ is a **constant Finslerian speed geodesic** of (M,F) if and only if it satisfies $\ddot{\gamma}^i(t) + 2G^i(\gamma(t), \dot{\gamma}(t)) = 0$, $i = 1, \ldots, n$, where the functions $G^i : \widetilde{TM} \to \mathbb{R}$ are given by

$$G^i(x,y) = \Gamma^i_{jk}(x,y) y^j y^k, \tag{2.3}$$

with $\Gamma^i_{jk}(x,y) := \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right)$.

The vector field $\Gamma := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$, is a well defined vector field on TM, whose integral lines are the canonical lifts $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ of the geodesics of γ. Because of this, the vector field Γ is called the **canonical geodesic**

spray of the Finsler space (M, F) and G^i are called the coefficients of the geodesic spray Γ.

Recall ([7], [11]) that the Euler-Lagrange equation of (M, F) can be written in terms of the geodesic spray as $\Gamma\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0$.

If F and $\widetilde{F}$ are two different fundamental Finsler functions on the same manifold M, then they are called **projectively equivalent** if their geodesics coincide as set points. Using geodesics sprays, this is equivalent to $\widetilde{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0$, where $\widetilde{\Gamma}$ is the geodesic spray of $(M, \widetilde{F})$ (see [7], [11]).

A Finsler structure (M, F) is with **reversible geodesics** if and only if F and its reverse function $\overline{F}$ are projectively equivalent, i.e., the geodesics of F and $\overline{F}$ coincides as set of points. If we denote by $\overline{\Gamma}$ the reverse geodesic spray, then F is with reversible geodesics if and only if $\overline{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0$.

The following result is known ([7], [11]).

THEOREM 2.4. *If (M, F_0) is an absolute homogeneous Finsler structure and $\beta = b_i(x) \cdot y^i$ is a linear form in TM, then the Randers change $F := F_0 + \varepsilon \cdot \beta$ is a Finsler structure on M with reversible geodesics if and only if $\hat{\beta} := b_i(x)dx^i$ is a closed 1-form on M, where $\varepsilon \neq 0$ is a constant.*

The *proof* is immediate. □

Remark 2.5. By a completely similar computations as in the above proof, one can easily verify that actually the Finslerian metrics F and F_0 are projectively equivalent if and only if $\hat{\beta}$ is closed (this was proved for the first time in [10]). This gives a new perspective on the geometrical reason that $F = F_0 + \varepsilon\beta$ is with reversible geodesics.

Concrete Finsler structures having reversible geodesics can be easily obtained in the class of (α, β)**-metrics**, namely Finsler metrics $F = F(\alpha, \beta)$, where F is a positive 1-homogeneous function of two arguments α and β. Here, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric on M, and $\beta = b_i(x)y^i$ the linear form on TM. We will consider in the following only positive definite (α, β)-metrics, i.e., we impose always the condition $b^2 := a_{ij}(x)b^i b^j < 1$.

Following Shen ([18]), we remark that it can be very useful to write $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$, where $\phi : I = [-r, r] \to [0, \infty)$ is a C^∞ function and the interval I can be chosen large enough such that $r \geq |\frac{\beta}{\alpha}|$, for all $x \in M$ and $y \in T_xM$. One has the following

LEMMA 2.6 (Shen's Lemma, [18]). *The function $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ is a Finsler metric for any $\alpha = \sqrt{a_{ij}y^i y^j}$ and any $\beta = b_i y^i$ with $\|\beta_x\|_\alpha < b_0$ if and only if $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0. \tag{2.4}$$

We remark that a first condition following from here is

$$\phi(s) - s\phi'(s) > 0, \quad |s| < b_0. \tag{2.5}$$

Moreover, Lemma 2.6 implies that if $F = \alpha \cdot \phi\Big(\frac{\beta}{\alpha}\Big)$ is an (α, β) Finsler metric, then ϕ cannot be an odd function (see [12], Lemma 2.3).

For later use we give

Lemma 2.7 ([13]). *Let $\big(M, F(\alpha, \beta)\big)$ be a Finsler space with (α, β)-metric. Then*

$$f(x, y) \cdot \frac{\partial \alpha}{\partial y^i} + g(x, y) \cdot b_i = 0, \quad \forall i = 1, \dots, n \tag{2.6}$$

implies $f = g = 0$, for any smooth functions f, g on TM.

From Theorem 2.4, we get

Corollary 2.8. *Any of the following (α, β)-metrics:*

(a) $F(\alpha, \beta) = a_0 \cdot \alpha + \sum_{k=1}^{p} a_{2k} \cdot \frac{\beta^{2k}}{\alpha^{2k-1}} + \varepsilon \cdot \beta$, *$a_0$, a_{2k}, ε constants $(\neq 0)$, $p \in \mathbb{N}^*$;*

(b) $F(\alpha, \beta) = \alpha + \beta$, *i.e., Randers metric;*

(c) $F(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha}$, *i.e., quadratic metric*

all are with reversible geodesics if and only if $\hat{\beta}$ is a closed 1-form on M.

Let us point out here that the Randers change given in Theorem 2.4 is actually almost the best we can expect for a Finsler metric with reversible geodesics, at least in the (α, β)-metrics case. Indeed, we can reformulate at least here our main results from [11], [12] as follows

Theorem 2.9. *Let (M, F) be a non-Riemannian n (≥ 2)-dimensional Finsler structure with (α, β)-metric, which is not absolute homogeneous.*

Then F is with reversible geodesics if and only if

$$F(\alpha, \beta) = F_0(\alpha, \beta) + \varepsilon\beta, \tag{2.7}$$

where F_0 is absolute homogeneous (α, β)-metric, ε is a non zero constant and $\hat{\beta}$ is a closed 1-form on M.

A special case is when we consider a flat Riemannian metric α on $M = \mathbb{R}^n$ and a 1-form β with constant coefficients. Then any (α, β)-metric F constructed with these α and β has reversible geodesics. One can easily see that this F is locally Minkowski and that its geodesics are in fact straight lines.

Finally, we discuss the case when the Randers change $F(\alpha, \beta) = F_0(\alpha, \beta) + \varepsilon\beta$ is projectively equivalent to the underlying Riemannian structure (M, α).

We will give here a more general result

THEOREM 2.10. 1. *Assume* $\big(M, F(\alpha,\beta)\big)$ *is an* (α,β)*-Finsler structure,* $\hat{\beta}$ *closed,* $\Gamma_\alpha(\beta) \neq 0$*, where* Γ_α *is the geodesic spray of* α*. Then* F *is projectively equivalent to the Riemannian structure* (M,α) *if and only if* F *is a Randers metric.*

2. *If* $F = \alpha + \beta$ *is a Randers metric, then* F *is projectively equivalent to* (M,α) *if and only if* $\hat{\beta}$ *is closed.*

Proof. We start with a computation

$$\Gamma_\alpha\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = \Gamma_\alpha\left(F_\alpha \cdot \frac{\partial \alpha}{\partial y^i} + F_\beta \cdot \frac{\partial \beta}{\partial y^i}\right) - F_\alpha \cdot \frac{\partial \alpha}{\partial x^i} - F_\beta \cdot \frac{\partial \beta}{\partial x^i} = \tag{2.8}$$

$$= \Gamma_\alpha(F_\alpha) \cdot \frac{\partial \alpha}{\partial y^i} + \Gamma_\alpha(F_\beta) \cdot b_i + 2F_\beta \cdot curl_{ij} \cdot y^j,$$

where $F_\alpha = \frac{\partial F(\alpha,\beta)}{\partial \alpha}$, $F_\beta = \frac{\partial F(\alpha,\beta)}{\partial \beta}$. Here we have used the Euler-Lagrange equations for α, i.e., $\Gamma_\alpha\Big(\frac{\partial \alpha}{\partial y^i}\Big) - \frac{\partial \alpha}{\partial x^i} = 0$ and $\Gamma_\alpha(b_i) = \frac{\partial b_i}{\partial x^k} \cdot y^k$ from definition. If $\hat{\beta}$ closed, i.e., $curl_{ij} = 0$ and F projectively equivalent to the Riemannian structure α, then $\Gamma_\alpha(F_\alpha) \cdot \frac{\partial \alpha}{\partial y^i} + \Gamma_\alpha(F_\beta) \cdot b_i = 0$ and from Lemma 2.7 we obtain

$$\Gamma_\alpha(F_\alpha) = 0, \quad \Gamma_\alpha(F_\beta) = 0. \tag{2.9}$$

On the other hand, by using $\Gamma_\alpha(\alpha) = 0$, we have $\Gamma_\alpha(F_\alpha) = F_{\alpha\alpha} \cdot \Gamma_\alpha(\alpha) + F_{\alpha\beta} \cdot \Gamma_\alpha(\beta) = F_{\alpha\beta} \cdot \Gamma_\alpha(\beta)$, where we have used that $\Gamma_\alpha(\alpha) = 0$. Here we denote $F_{\alpha\alpha} = \frac{\partial F_\alpha(\alpha,\beta)}{\partial \alpha}$ and so on.

Similarly, $\Gamma_\alpha(F_\beta) = F_{\beta\beta} \cdot \Gamma_\alpha(\beta)$ and therefore when $\hat{\beta}$ is closed and F projectively equivalent to α, relation (2.9) implies $F_{\alpha\beta} \cdot \Gamma_\alpha(\beta) = 0$, $F_{\beta\beta} \cdot \Gamma_\alpha(\beta) = 0$.

If $\Gamma_\alpha(\beta) \neq 0$, it follows $F_{\alpha\beta} = 0$, $F_{\beta\beta} = 0$ and therefore $F_\beta =$ constant. From the homogeneity of F with respect to α, β, it follows $F_\alpha =$ constant, i.e., F must be of Randers type.

Conversely, if $F = \alpha + \beta$ is Randers metric, then it is known that it is projectively equivalent to α if and only if $\hat{\beta}$ is closed. □

Remark 2.11. Compare with [10] for the proof of the statement 2 in Theorem 2.10.

3. PROPERTIES OF (α,β)-METRICS WITH REVERSIBLE GEODESICS

A Finsler structure (M,F) is called (locally) **projectively flat** if all of its geodesics are straightlines. An equivalent condition is that the spray coefficients G^i of F can be expressed as $G^i = P(x,y) \cdot y^i$, where $P(x,y) = \frac{1}{2F} \cdot \frac{\partial F}{\partial x^k} \cdot y^k$.

An equivalent characterization of projectively flatness is the **Hamel's relation** $\frac{\partial^2 F}{\partial x^m \partial y^k} \cdot y^m - \frac{\partial F}{\partial x^k} = 0$ ([9]). We obtain

THEOREM 3.1. *Let $F = F_0 + \varepsilon\beta$ be a Randers change, where F_0 is an absolute homogeneous (α, β)-metric. Then any two of the following properties imply the third one:*
(i) *F is projectively flat;*
(ii) *F_0 is projectively flat;*
(iii) *$\hat{\beta}$ is closed.*

Proof. We compute $\frac{\partial^2 F}{\partial x^m \partial y^k} \cdot y^m = \frac{\partial^2 F_0}{\partial x^m \partial y^k} \cdot y^m + \varepsilon \cdot \frac{\partial b_k}{\partial x^m} \cdot y^m$ and therefore Hamel's relation reads $\frac{\partial^2 F}{\partial x^m \partial y^k} \cdot y^m - \frac{\partial F}{\partial x^k} = \frac{\partial^2 F_0}{\partial x^m \partial y^k} \cdot y^m - \frac{\partial F_0}{\partial x^k} + 2\varepsilon \cdot curl_{km} \cdot y^m$ and the conclusion follows. $\square$

By taking $F_0 = \alpha$ it follows immediately

COROLLARY 3.2. *A Randers metric $F = \alpha + \beta$ with reversible geodesics is projectively flat if and only if α is projectively flat.*

Taking now into account Beltrami's theorem, i.e., a Riemannian structure is projectively flat if and only if it is a space form, we obtain

COROLLARY 3.3. *If the Randers structure $(M, F = \alpha + \beta)$ is with reversible geodesics and projectively flat, then, if we identify M with its universal covering, the base manifold M must be isometric to one of the model manifolds $\mathbb{S}^n$, $\mathbb{E}^n$, $\mathbb{H}^n$.*

In general, we have

THEOREM 3.4. *If F is projectively flat, then it is with reversible geodesics.*

Proof. First, let us remark that F is projectively flat if and only if $\overline{F}$ is projectively flat. Indeed, this can be immediately verified by Hamel's relation. F projectively flat means $\frac{\partial^2 F}{\partial x^m \partial y^k} \cdot y^m - \frac{\partial F}{\partial x^k} = 0$, at every $(x, y) \in \widetilde{TM}$. We can write this relation at $(x, -y)$ i.e., $\left.\frac{\partial^2 F}{\partial x^m \partial y^k}\right|_{(x,-y)} \cdot (-y^m) - \left.\frac{\partial F}{\partial x^k}\right|_{(x,-y)} = 0$. On the other hand, Hamel's relation for $\overline{F}$ gives $\frac{\partial^2 F(x,-y)}{\partial x^m \partial y^k} \cdot y^m - \frac{\partial F(x,-y)}{\partial x^k} = -\left.\frac{\partial^2 F}{\partial x^m \partial y^k}\right|_{(x,-y)} \cdot y^m - \left.\frac{\partial F}{\partial x^k}\right|_{(x,-y)}$ and the statement follows.

This means that F and $\overline{F}$ are both projectively equivalent to the standard Euclidean metric and therefore F must be projective to $\overline{F}$, i.e., F must be with reversible geodesics. Obviously, if the geodesics of F and $\overline{F}$ coincide as trajectories, there is no reason for these to be straight lines. $\square$

Remark 3.5. 1. The converse of Theorem 3.4 is not true. This can be easily seen taking into account the metrics constructed in [19], [20].

2. More generally ([14], [20]) gives a characterization of all (α,β)-metrics that are projectively flat in dimension greater than 2. Obviously, all of these are examples of (α,β)-metrics with reversible geodesics.

3. Based on these, a similar study can be done for projective flat (α,β)-metrics of constant flag curvature. It can be shown that there exist (α,β)-metrics with reversible geodesics, which are of constant flag curvature, but not necessarily projectively flat. Obviously, in the Randers case, constant flag curvature implies α projectively flat, but for more general (α,β)-metrics, this is not true any more (compare [5]).

Theorem 3.6. *Let* $F=\sum_{k=0}^{p} a_{2k}\cdot\frac{\beta^{2k}}{\alpha^{2k-1}}+\varepsilon\cdot\beta$ *be a polynomial* (α,β)*-metric on an* $n\geq 3$ *dimensional smooth manifold* M*. Suppose that* $\hat{\beta}$ *is closed, but not parallel everywhere,* F *is not Randers and neither* $db\neq 0$ *nor* $b=constant$*. Then* F *is projectively flat if and only if the followings hold*

1. $\phi(s)=a_0+\sum_{k=1}^{p}(-1)^{k-1}\frac{(2k-3)!!}{(2k)!}\prod_{j=0}^{k-1}\left(1-\frac{j}{p}\right)a_0c^ks^{2k}+\varepsilon s,$
2. $b_{i|j}=2\tau\left[a_{ij}+\left(b^2a_{ij}-\frac{2p+1}{2p}b_ib_j\right)c\right],$
3. $G^i_\alpha=\xi y^i-c\tau\alpha^2b^i,$

where $b_{i|j}$ *is the covariant derivation of* b_i *with respect to* α*. Here* $\tau(x)$*,* ξ *are scalar functions on* M*, and* a_0*,* c *are arbitrary real constants.*

All these metrics are with reversible geodesics.

Proof. The proof is based on Theorem 1.1 in [19]. Indeed, by computing $\phi'(s)$, $\phi''(s)$ for $\phi(s)=\sum_{k=0}^{p}a_{2k}s^{2k}+\varepsilon s$, and substituting in relation $\{1+(c_1+c_2s^2)s^2+c_3s^2\}\phi''(s)=(c_1+c_2s^2)\{\phi(s)-s\phi'(s)\}$ from Theorem 1.1 in [19], where c_1, c_2, c_3 are real constants, we get an equality of two polynomials or degree $2p+2$ in s with constant coefficients. By comparing first the 0-order terms we obtain relation $a_2=\frac{1}{2}c_1a_0$. Next, we compare the highest order terms, and imposing $a_{2p}\neq 0$ it follows $c_2=0$. This leads to a lot of simplifications in computations. Moreover, comparing the coefficients of the $2p$ order terms it follows $c_3=-\frac{2p+1}{2p}c_1$. By continuing this comparison procedure, we obtain the formulas

$$a_{2k}=-\frac{(2k-3)}{2k(2k-1)}\{c_1+(2k-2)(c_1+c_3)\}a_{2k-2},$$

for all $k=1,2,\ldots,p$.

Remark that this can be regarded as a linear system of p equations that allows to compute the coefficients a_{2k}, $k=1,2,\ldots,p$, depending on a_0 and c_1 only (if we take into account formulas above).

Finally, by induction, we get

$$a_{2k} = (-1)^{k-1} \frac{(2k-3)!!}{(2k)!} \prod_{j=0}^{k-1} \left(1 - \frac{j}{p}\right) a_0 c^k,$$

for all $k = 1, 2, \ldots, p$, where we put $c := c_1$ for simplicity, $(-1)!! := -1$, $1!! := 1$ and the conclusion follows immediately from [19]. □

Example 3.7. We can easily obtain now examples of such Finsler metrics with (α, β)-metrics that are projectively flat and with reversible geodesics using 1 in Theorem 3.6. Indeed, $\phi(s) = a_0 + \frac{1}{2}a_0cs^2 + \varepsilon s$, $\phi(s) = a_0 + \frac{1}{2}a_0cs^2 - \frac{1}{48}a_0c^2s^4 + \varepsilon s$, $\phi(s) = a_0 + \frac{1}{2}a_0cs^2 - \frac{1}{36}a_0c^2s^4 + \frac{1}{1080}a_0c^3s^6 + \varepsilon s$ are polynomials that together with conditions 2 and 3 in Theorem 3.6 give examples of (α, β) metrics which are projectively flat and with reversible geodesics ($\dim M \geq 3$).

4. WARPED PRODUCTS

Warped products are n-dimensional Riemannian structures on manifolds $M = I \times N$, where $I \subset \mathbb{R}$ is an open interval, and (N, h) is an $(n-1)$-dimensional Riemannian manifold.

Typical Riemannian metrics on warped products are written as $a^2 = dt^2 + \varphi^2(t) \cdot h^2$, where $\varphi : I \to (0, \infty)$ is a smooth function on I.

One can easily remark that on these manifolds there exists always a closed 1-form $\hat{\beta}$ given by $\hat{\beta} = f(t) \cdot dt$, for any function f defined on I.

Therefore, we have

THEOREM 4.1. *Let* $(M = I \times N,\ a^2 = dt^2 + \varphi^2(t) \cdot h^2)$ *be a warped product Riemannian metric and let* $\hat{\beta} := f(t) \cdot dt$, *where* $f : I \to \mathbb{R}$ *is a smooth function.*

Then for any absolute homogeneous Finsler metric F_0 *on* M, *and a linear 1-form* $\beta := f(t)y^0$ *on* TM, *the function* $F = F_0 + \varepsilon \cdot \beta$ *is a Finsler metric on* M *with reversible geodesics, where we denote* $(y^0, y^1, \ldots, y^{n-1}) \in T_xM$, $n - 1 = \dim N$, *the induced coordinates of a tangent plane in a point to* M, *and by* ε *any non vanishing constant.*

For the warped Riemannian manifold $(M = I \times N, a)$, and a linear 1-form β as defined as in Theorem 4.1, we obtain

COROLLARY 4.2. *The* (α, β)*-metrics given by* $\phi(s) = \sum_{k=0}^{p} a_{2k} \cdot s^{2k} + \varepsilon \cdot s$, $p \in \mathbb{N}^*$ *are with reversible geodesics, where* a_{2k}, $k = 0, 1, \ldots, p$, *and* ε *are some non vanishing constants.*

Remark 4.3. From the discussions in the precedent section, one can see that there are Finsler structures with (α, β)-metrics global defined on warp products, with reversible geodesics, which are not projectively flat (compare [20]).

Important warped products are the manifolds $I \times \mathbb{S}^{n-1}$ with Riemannian metric $\alpha^2 = dt^2 + \varphi^2(t) \cdot ds^2_{n-1}$, where $(\mathbb{S}^{n-1}, ds^2_{n-1})$ is the standard round sphere with its canonical Riemannian metric. These are called **rotationally symmetric metrics**.

Remarkably, the standard sphere $\mathbb{S}^n$ can be written as a rotationally symmetric metric in all dimensions. Indeed, by writing $\mathbb{S}^n = (0, \pi) \times \mathbb{S}^{n-1}$, with metric given above, where $\varphi(t)$ is the unique solution of the differential equation with initial solution $\varphi''(t) + \varphi(t) = 0$, $\varphi(0) = 0$, $\varphi'(0) = 1$, then the map $G : (0, \pi) \times S^{n-1} \to \mathbb{R} \times \mathbb{R}^n$, $(t, z) \mapsto G(t, z) = (\cos t, \sin t \cdot z)$ maps the unit sphere in $\mathbb{R}^{n+1}$. Indeed, it can be seen by direct computation that G is a Riemannian isometry ([15]).

We obtain immediately.

Proposition 4.4. *Let $(\mathbb{S}^n, a)$ be the standard unit sphere described above, and put $\hat{\beta} = f(t) \cdot dt$, for any $f : \mathbb{S}^n \to \mathbf{R}$. Then the (α, β)-metric family given by $\phi(s) = \sum_{k=0}^{p} a_{2k} \cdot s^{2k} + \varepsilon \cdot s$, a_{2k} $(k = 0, 1, \ldots, p)$, $\varepsilon \neq 0$ constants is a Finsler metric with reversible geodesics, globally defined on $\mathbb{S}^n$.*

An interesting metric can be obtained from Theorem 5.1 in [6] as follows.

Theorem 4.5. *Let (M, a) be a warped product with metric $a^2 = dt^2 + \big(\varphi'(t)\big)^2 \cdot h^2$ and consider $\hat{\beta} = \frac{1}{10} \cdot \varphi^{-\frac{3}{5}} \cdot \varphi' dt$, where $\varphi = \varphi(t)$ is a solution of $\varphi''(t) = 20 \cdot \varphi^{\frac{1}{5}} + \frac{2}{5} \cdot \varphi^{-1} \cdot (\varphi')^2$. Then $F = \frac{(\alpha+\beta)^2}{\alpha}$ is a non-Berwald Finslerian metric on M with the properties: it has reversible geodesics, it is projectively flat, and of constant flag curvature $K = 0$.*

More generally, taking into account that the Hessian of a smooth function φ defined on a Riemannian manifold (M, a) equals the half of the Lie derivative of a with respect to the gradient of φ, namely $\mathrm{Hess}_a \varphi := \frac{1}{2} \mathcal{L}_{\nabla \varphi} a$ (see [16]), we get.

Theorem 4.6. *If there are smooth functions φ and λ on a Riemannian manifold (M, α) such that $\mathrm{Hess}_\alpha \varphi = \lambda \alpha^2$, $d\varphi \neq 0$, then M is a warped product that can be endowed with a Finsler metric whose geodesics are reversible.*

Proof. From [16], or Lemma 3.1 in [6], it follows that under the assumptions in the hypothesis, (M, α) is a warped product structure. Indeed, in this case $M = \mathbb{R} \times N$, the function φ depends only on the parameter $t \in \mathbb{R}$, N is a level set of φ and $\alpha^2 = dt^2 + (\varphi'(t))^2 h^2$, where h is the Riemannian metric on N. From our Theorem 4.1 it follows that we can always construct on such manifold M a Finsler structure with reversible geodesics. □

5. SYMPLECTIC MANIFOLDS

We will show, in the following, that on symplectic manifolds (see [4], [8] for definitions) one can construct (α, β) Finsler structures with reversible geodesics.

It is clear that the space $\mathfrak{sp}(\Omega)$ of symplectic vector fields can be identified with $\mathcal{Z}^1(M)$, the space of closed 1-forms on M, by the isomorphism of $C^\infty(M)$-modules $\flat : \chi(M) \to \Lambda^1(M)$, $X \mapsto X \lrcorner \Omega$ with the inverse $\# : \Lambda^1(M) \to \chi(M)$. Therefore, we obtain

THEOREM 5.1. *Let $F = F_0 + \varepsilon \cdot \beta$ be a Finsler metric on a symplectic manifold (M, Ω), where F_0 is an absolute homogeneous Finsler metric on M and $\hat{\beta} = -X \lrcorner \Omega$, for any symplectic vector field X on M. Then F is with reversible geodesics.*

One can see easily that if $\Omega \in \Lambda^2(M^{2n})$ is a symplectic structure on a compact manifold M, then the de Rham cohomology class $[\Omega] \in H^2_{dR}(M, \mathbb{R})$ must be non-vanishing. This immediately eliminates the existence of symplectic structures on even-dimensional spheres, except for $\mathbb{S}^2$ (see for example [4]).

THEOREM 5.2. *Any orientable smooth surface S can be endowed with a Finsler metric with reversible geodesics.*

Proof. Indeed, it can be seen that if S is an orientable smooth surface, then it has a volume form, or area form, say μ on S. By definition, μ is a non-degenerate closed 2-form on S and therefore it defines a symplectic structure on S.

Using this, we can obtain Finsler metrics with reversible geodesics as above by taking $\hat{\beta} := -X \lrcorner \mu$, for any symplectic vector field and any Riemannian metric α on S. $\square$

For any $f \in C^\infty(M)$, the vector field $X_f := \#(df)$ is called the **Hamiltonian vector field** associated to f. The set of Hamiltonian vector fields is denoted with $\mathfrak{h}(\Omega)$ and we have $\mathfrak{h}(\Omega) = \#(\mathcal{B}^1(M))$, where $\mathcal{B}^1(M)$ is the subspace of exact 1-forms on M. It can be shown that $\mathfrak{h}(\Omega)$ is an ideal in the Lie algebra $\mathfrak{sp}(\Omega)$, i.e., $[\mathfrak{sp}(\Omega), \mathfrak{h}(\Omega)] \subset \mathfrak{h}(\Omega)$.

Obviously, the Hamiltonian vector fields generate closed one forms in the same way as symplectic vector fields and therefore, they always allow the construction of Finsler structures with reversible geodesics.

As an application, let us see how this construction works on $\mathbb{S}^2$. If $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is the 2-dimensional unit sphere, then one can identify the tangent space $T_p\mathbb{S}^2$ at $p \in \mathbb{S}^2$ with the set of vectors in $\mathbb{R}^3$ orthogonal to p with respect to the standard inner product $\langle \cdot\,, \cdot \rangle$. Then, $\Omega_p(v, w) = \langle p, v \times w \rangle$ defines a closed non-degenerate differential 2-form, such that $(\mathbb{S}^2, \Omega)$ becomes a symplectic manifold. Let us consider $\mathbb{S}^2$ endowed with cylindrical coordinates $(h, \theta) \in [-1, 1] \times (0, 2\pi]$, i.e., h is the "height" of the point p and θ is the "angle". In these coordinates, the area form of $\mathbb{S}^2$ reads $\Omega = d\theta \wedge dh$. If we consider the Hamiltonian function $H = h$, the height function on $\mathbb{S}^2$, then the north and south poles are critical points.

With the notations above, we define the vector field X_h such that $dh = X_h \lrcorner\, d\theta \wedge dh$. It follows $X_h = \frac{\partial}{\partial \theta}$. This Hamiltonian vector field generates a periodic $\mathbb{R}$-action, i.e., an $\mathbb{S}^1$-action on $\mathbb{S}^2$ by rotation that preserves the level sets, i.e., the parallel curves, of the Hamiltonian function h. Indeed, $X_h \lrcorner dh = \Omega(X_h, X_h) = 0$ (see for example [8]).

Then, denoting by α a Riemannian metric on $\mathbb{S}^2$ and $\hat{\beta} := dh$, we get

THEOREM 5.3. *The Finsler metric* $\phi(s) = \sum_{k=0}^{n} a_{2k} \cdot s^{2k} + \varepsilon \cdot s$ *on* $\mathbb{S}^2$, *constructed with* α *and* β *above is with reversible geodesics.*

Remark 5.4. There is no need to consider the standard Riemannian metric on $\mathbb{S}^2$ for α. Any other metric can be used as well.

It can be seen that if one consider α to be the standard Riemannian metric on $\mathbb{S}^2$, then X_h is a Killing vector field on $\mathbb{S}^2$ and therefore the induced Randers metric $F = \alpha + \beta$ must be of constant flag curvature $K = 1$. In this case, F is with reversible geodesics and projectively flat in the same time (see [5] for a more general case).

Another interesting case is when we consider a Zoll metric α on $\mathbb{S}^2$ (see [3] for details on Zoll metrics) and using it we construct Finsler metrics with reversible geodesics by taking $\hat{\beta}$ as before. Therefore we obtain

THEOREM 5.5. *If* $(\mathbb{S}^2, a)$ *is a round sphere with a Zoll metric and* $\hat{\beta}$ *is a closed form on* $\mathbb{S}^2$ *defined as before, then the induced Randers metric* $F = \alpha + \beta$ *on* $\mathbb{S}^2$ *has the following properties: it is with reversible geodesics and all its geodesics are closed and periodic.*

The *proof* is trivial taking into account that in this case F is actually projectively equivalent to the Zoll metric a. This Randers structure has the remarkable property that its set of geodesics is a differentiable manifold in fact diffeomorphic to $\mathbb{S}^2$.

Further relations between symplectic geometry and the geometry of a Finsler space with reversible geodesics can be subject of a future research.

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