DIRAC STRUCTURES ON GENERALIZED Riemannian MANIFOLDS

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We characterize the Dirac structures that are parallel with respect to Gualtieri’s canonical connection of a generalized Riemannian metric. On the other hand, we discuss Dirac structures that are images of generalized tangent structures. These structures turn out to be Dirac structures that, if seen as Lie algebroids, have a symplectic structure. In particular, if compatibility with a generalized Riemannian metric is required, the symplectic structure is of the Kähler type.

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1. INTRODUCTION

The concept of generalized geometry is due to Hitchin [8] and is of interest in the physical theory of supersymmetry (e.g., [21]). In generalized geometry the tangent bundle $TM$ of the $m$-dimensional, differentiable manifold $M$ is replaced by the big tangent bundle $T^G M = TM \oplus T^* M$. The latter has the non degenerate, neutral metric $g((X, \alpha), (Y, \mu)) = \alpha(Y) + \mu(X)$ [in many papers on generalized geometry $g$ is defined by $(1/2)(\alpha(Y) + \mu(X))]$, and the Courant bracket

$$[(X, \alpha), (Y, \mu)] = ([X, Y], L_X \mu - L_Y \alpha + \frac{1}{2} d(\alpha(Y) - \mu(X)))$$

where $X, Y \in \chi^1(M)$, $\alpha, \mu \in \Omega^1(M)$ $(\chi^k(M)$ is the space of $k$-vector fields and $\Omega^k(M)$ is the space of differential $k$-forms on $M$; we will also use calligraphic characters for pairs: $\mathcal{X} = (X, \alpha)$, $\mathcal{Y} = (Y, \mu)$, etc.). Thus, the structure group of $T^G M$ is $O(m, m)$ and, by definition, the generalized geometric structures are reductions of this structure group to various subgroups.

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The almost Dirac structures, which are maximal $g$-isotropic subbundles $E$ of $TM$ and are important in mechanics and physics [2], are generalized structures where $O(m, m)$ is reduced to the subgroup that preserves a maximal isotropic subbundle of $\mathbb{R}^{2m}$ endowed with the standard neutral metric. The structure $E$ is a Dirac structure if it is integrable, i.e., closed under the Courant bracket.

Hitchin’s work and the subsequent work of Gualtieri [5, 7] started a stream of research and publications on generalized complex structures. A generalized complex structure is a $g$-skew-symmetric endomorphism $J \in \text{End}(TM)$ with $J^2 = -\text{Id}$ and a vanishing Courant-Nijenhuis torsion (see Section 3). Generalized complex structures may be combined with generalized Riemannian structures, which are reductions of the structure group of $TM$ to $O(n) \times O(n)$, thus leading to generalized Kähler manifolds [5]. In [6], it was proven that the generalized Riemannian metric produces a canonical connection $\nabla$ on $TM$ and the generalized Kähler structures are characterized by $\nabla J = 0$ plus a certain torsion condition. The theory of generalized complex structures also motivated some work on related generalized structures: paracomplex, contact, $F$, CRF, Sasakian, etc., [12, 16, 18, 19, 20].

In the present paper, we will discuss the relationship between a Dirac structure and a generalized Riemannian metric. In Section 2 we give a straightforward definition of the canonical connection of a generalized Riemannian metric and compute its torsion. In Section 3, passing through a discussion of metric, generalized product structures, we show that, on a generalized Riemannian manifold, a Dirac structure $E$ may be represented by a tensor field $F_E \in \text{Iso}(TM)$ and we get the conditions for $E$ to be preserved by the canonical connection. In Section 4, we study Dirac structures $E$ that are images of a generalized tangent structure and show that these are characterized by the existence of a symplectic structure on the Lie algebroid $E$ with the Courant bracket. In Section 5, we show that, on a generalized Riemannian manifold, a symplectic form on the Dirac structure $E$ is equivalent with a Kähler type form.

2. GENERALIZED RIEMANNIAN MANIFOLDS

A generalized, Riemannian structure on $M$ is a reduction of the structure group of $(TM, g)$ from $O(m, m)$ to $O(m) \times O(m)$, i.e., a decomposition

$$TM = V_+ \oplus V_-,$$

where $V_\pm$ are maximal positive, respectively negative, subbundles of $g$. Obviously, $\text{rank } V_+ = m$ and $V_+ \perp_g V_-$, which shows that, in fact, the reduction is defined by one of these subbundles.
Equivalently [5], the structure may be seen as a positive definite metric $G$ together with a $G$-orthogonal decomposition (1) such that $G|_{V_\pm} = \pm g$. We may define $G$ by the endomorphism $\phi$ of $TM$ given by $\phi|_{V_\pm} = \pm \text{Id}$, equivalently, by

$G((X, \alpha), (Y, \mu)) = g(\phi(X, \alpha), (Y, \mu))$.

(2)

The endomorphisms $\phi$ that produce generalized Riemannian metrics are characterized by the conditions

$\phi^2 = \text{Id}, \quad g(\phi(X, \alpha), \phi(Y, \mu)) = g((X, \alpha), (Y, \mu))$

(3)

and the requirement that $G$ given by (2) is positive definite (the second condition (3) comes from the symmetry of $G$ and ensures that the $\pm 1$-eigebundles $V_\pm$ of $\phi$ are $G$-orthogonal).

In [5], it was shown that $G$ is equivalent with a pair $(\gamma, \psi)$, where $\gamma$ is a usual Riemannian metric on $M$ and $\psi \in \Omega^2(M)$. This equivalence is realized by putting

$V_\pm = \{(X, b_{\psi \pm \gamma}X) / X \in TM\}$.

(4)

Formula (4) also shows the existence of isomorphisms

$\tau_\pm : V_\pm \rightarrow TM, \quad \tau_\pm(X, b_{\psi \pm \gamma}X) = X,$

(5)

which may be used to transfer structures between $V_\pm$ and $TM$. In particular, the two metrics $G|_{V_\pm}$ transfer to $2\gamma$.

On $TM$, it is natural to consider connections $\nabla$ that are compatible with the neutral metric $g$, i.e., such that

$X(g(\mathcal{Y}, \mathcal{Z})) = g(\nabla_X \mathcal{Y}, \mathcal{Z}) + g(\mathcal{Y}, \nabla_X \mathcal{Z}), \quad \forall X \in TM, \mathcal{Y}, \mathcal{Z} \in \Gamma TM$;

we call them big connections.

Furthermore, on a generalized Riemannian manifold $(M, G)$, a connection $\nabla$ on $TM$ is a $G$-metric connection if it is compatible with $G$, i.e., (6) with $g$ replaced by $G$ holds. If $\nabla$ is a big connection, condition (6) for $G$ is equivalent with

$\nabla_X(\phi \mathcal{Y}) - \phi(\nabla_X \mathcal{Y}) = 0$,

which, furthermore, is equivalent with the commutation between $\nabla$ and the projections $(1/2)(\text{Id} \pm \phi)$. Hence, $\nabla$ is $G$-metric iff it preserves the subbundles $V_\pm$. By using the transport to $TM$ via $\tau_\pm$, we see that there exists a bijective correspondence between $G$-metric big connections $\nabla$ and pairs $D_\pm$ of $\gamma$-metric connections on $M$, which is realized by

$\nabla_X(Y, b_{\psi \pm \gamma}Y) = (D_X^\pm Y, b_{\psi \pm \gamma}D_X^\pm Y)$.

For any Riemannian metric $\gamma$, there exists a unique $\gamma$-metric connection with a prescribed torsion. In particular, a generalized Riemannian metric $G \Leftrightarrow$
(γ,ψ) produces two connections on TM, which are the γ-metric connections $D^\pm$ with the torsion $T^{D^\pm}$ defined by

$$\gamma(T^{D^\pm}(X,Y),Z) = \pm d\psi(X,Y,Z).$$

The connections $D^\pm$ are given by

$$(8)\quad D^\pm_X Y = D_X Y \pm \frac{1}{2}\gamma^{-1}(i(Y)i(X)d\psi),$$

where $D$ is the Levi-Civita connection of $\gamma$.

The G-metric big connection $\nabla$ defined by the connections (8) is called the canonical big connection of $G$; one can see that $\nabla$ coincides with the connection defined by Gualtieri [6] and Ellwood [4].

If we define the Courant torsion

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

we get an object that is not $C^\infty(M)$-bilinear. This is corrected in the Gualtieri torsion [6]

$$(9)\quad T^\nabla(X,Y,Z) = g(T^{\nabla}X,Y,Z) + \frac{1}{2}[g(\nabla_Z X,Y) - g(\nabla_Y X,Z)].$$

The tensorial character and, also, the total skew symmetry of $T^\nabla$ follow from the properties of the Courant bracket [10].

We compute the Gualtieri torsion of the canonical big connection; the results will agree with those of [6]). For any $X,Y \in \chi^1(M)$, computations give

$$(10)\quad [(X, b_\psi X), (Y, b_\psi Y)] = ([X,Y], b_\psi [X,Y] + i(Y)i(X)d\psi),$$

$$(11)\quad [(X, b_{\psi\pm\gamma} X), (Y, b_{\psi\pm\gamma} Y)] = ([X,Y], b_{\psi\pm\gamma}[X,Y] + i(Y)i(X)d\psi + (L_X i(Y)\gamma - i(X)L_Y \gamma),$$

$$(12)\quad [(X, b_{\psi+\gamma} X), (Y, b_{\psi-\gamma} Y)] = ([X,Y], b_{\psi}[X,Y] + (0, i(Y)i(X)d\psi - L_X (b_\gamma Y) - L_Y (b_\gamma X) + d(\gamma(X,Y))).$$

Now, insert (12) in the expression of the mixed Courant torsion

$$T^\nabla((X, b_{\psi+\gamma} X), (Y, b_{\psi-\gamma} Y)) = (D_X Y, b_{\psi-\gamma} D_X Y) - (D_Y X, b_{\psi+\gamma} D_Y X) - [((X, b_{\psi+\gamma} X), (Y, b_{\psi-\gamma} Y)],$$

where $D^\pm$ are given by (8). After some technical calculations, we shall obtain

$$T^\nabla((X, b_{\psi+\gamma} X), (Y, b_{\psi-\gamma} Y)) = 0.$$

As a matter of fact, the previous equality is equivalent to (7). The annulation of the mixed Courant torsion implies

$$T^\nabla(X_+, Y_+, Z_-) = 0, \quad T^\nabla(X_-, Y_-, Z_+) = 0, \quad \forall X_\pm, Y_\pm, Z_\pm \in V_\pm.$$
Furthermore, we have
\[
pr_{TM}T^\nabla((X, b_{\psi\pm\gamma}X), (Y, b_{\psi\pm\gamma}Y)) = T^{D^\pm}(X, Y)
\]
and then, with (11),
\[
pr_{T^*M}T^\nabla((X, b_{\psi\pm\gamma}X), (Y, b_{\psi\pm\gamma}Y)) = b_{\psi\pm\gamma}T^{D^\pm}(X, Y) - \left[-i(Y)i(X)d\psi \pm (L_Xi(Y)\gamma - i(X)L_Y\gamma)\right].
\]
From (13), (14) and (7), we get
\[
T^\nabla((X, b_{\psi\pm\gamma}X), (Y, b_{\psi\pm\gamma}Y), (Z, b_{\psi\pm\gamma}Z)) = 2d\psi(X, Y, Z).
\]

**Remark 2.1.** The curvature of the canonical big connection is equivalent with the pair of curvature tensors \(R^{D^\pm}\) of the connections \(D^\pm\). Indeed, it follows easily that one has
\[
R^\nabla(X, Y)(Z, b_{\psi\pm\gamma}Z) = (R^{D^\pm}(X, Y)Z, b_{\psi\pm\gamma}R^{D^\pm}(X, Y)Z).
\]

We end this section by a presentation of the alternative notion of a generalized connection [6], although we will not use it in the paper. Assume that the pair \((A, A^*)\) is a Lie bialgebroid [11] and \(V\) is a vector bundle on \(M\). Consider a pair \((\nabla, \nabla^*)\) where \(\nabla, \nabla^*\) are an \(A\)-connection, respectively an \(A^*\)-connection on \(V\). The operator
\[
D_{(a, a^*)}v = (\nabla_a v + \nabla^*_{a^*}v),
\]
where \(a \in \Gamma A, a^* \in \Gamma A^*\) is called an \((A, A^*)\)-generalized connection or covariant derivative. \(D\) is \(\mathbb{R}\)-bilinear and has the properties
\[
D_{(f a, f a^*)}v = f D_{(a, a^*)}v,
\]
\[
D_{(a, a^*)}(f v) = f D_{(a, a^*)}v + (\sharp A a + \sharp A^* a^*)(f)v,
\]
where \(\sharp A, \sharp A^*\) are the anchors of \(A, A^*\). \(D\) is said to preserve \(g \in \Gamma \otimes^2 V^*\) if
\[
g(\sharp A a + \sharp A^* a^*)(g(v_1, v_2) = g(D_{(a, a^*)}v_1, v_2) + g(v_1, D_{(a, a^*)}v_2),
\]
which means that both \(\nabla\) and \(\nabla^*\) preserve \(g\).

In the particular case \(A = TM\) with the Lie bracket, \(A^* = T^*M\) with zero anchor and zero bracket we simply speak of a generalized connection and \(\nabla^*\) is a tensor. Furthermore, if \(V = TM\) and if we are interested in generalized connections that preserve \(g\) and \(G\), \(\nabla\) must be a \(G\)-metric big connection and the tensor \(\nabla^*\) must satisfy the conditions
\[
g(\nabla^*_a X, Y) + g(X, \nabla^*_a Y) = 0, \quad G(\nabla^*_a X, Y) + G(X, \nabla^*_a Y) = 0.
\]
It follows that $\nabla^*$ commutes with $\phi$ and preserves the subbundles $V_{\pm}$, therefore, $\nabla^*$ is equivalent with a pair $\Lambda^\pm$ of $\gamma$-skew-symmetric tensor fields of the type $(2,1)$ such that

$$\nabla^*_\alpha(Y, b_{\psi \pm \gamma} Y) = (\Lambda^\pm_\alpha Y, b_{\psi \pm \gamma} \Lambda^\pm_\alpha Y).$$

If we denote

$$\Xi^\pm(X, Y, Z) = \gamma(\Lambda^\pm_\alpha Y, b_{\psi \pm \gamma} \Lambda^\pm_\alpha Y),$$

the skew-symmetry condition becomes

$$\Xi^\pm(X, Y, Z) + \Xi^\pm(Y, X, Z) = 0.$$  

(Notice that the non-degeneracy of $\gamma$ implies the non-degeneracy of $\psi \pm \gamma$.)

For instance, we may take $\nabla = \nabla^{LC}$ to be the big Levi-Civita connection defined by taking both $D^\pm$ equal to the Levi-Civita connection $D$ and $\Xi^\pm = \pm d\psi$. The corresponding $D^{LC}$ is the generalized Levi-Civita connection. It codifies the same data as the canonical big connection, but in a different way.

For generalized connections, the Courant torsion and the (totally skew symmetric) Gualtieri torsion may be defined as for big connections, using the operator $D$ instead of $\nabla$, and we have

$$T^D((X, \alpha), (Y, \mu)) = T^{\nabla}((X, \alpha), (Y, \mu)) + \nabla^*_\alpha(Y, \mu) - \nabla^*_\mu(X, \alpha).$$

In particular, technical calculations give

$$T^{D^{LC}}(X, Y, Z) = T^{\nabla^{LC}}(X, Y, Z) \pm 3d\psi(X, Y, Z),$$

and

$$T^{D^{LC}}(X, Y, Z) = T^{\nabla^{LC}}(X, Y, Z) \pm d\psi(X, Y, \pm \psi \pm \gamma Z),$$

$\forall X, Y, Z \in V_{\pm}$. Further calculations that use the bracket formulas (11), (12) yield

$$T^{\nabla^{LC}}(X, Y, Z) = -d\psi(pr_{TM} X, pr_{TM} Y, pr_{TM} Z),$$

$\forall X, Y, Z \in \Gamma T(M)$.

Notice that the correction that led to the Gualtieri torsion consists in the use of a modified Courant bracket $[\mathcal{X}, \mathcal{Y}]^D$ defined by

$$g([\mathcal{X}, \mathcal{Y}]^D, Z) = g([\mathcal{X}, \mathcal{Y}], Z) - \frac{1}{2}g(DZ \mathcal{X}, \mathcal{Y}) + \frac{1}{2}g(D \mathcal{X} \mathcal{Y}, Z).$$

Then, the formula

$$\mathcal{R}^D(\mathcal{X}, \mathcal{Y})Z = D\mathcal{X}D\mathcal{Y}Z - D\mathcal{Y}D\mathcal{X}Z - D[\mathcal{X}, \mathcal{Y}]^D Z$$

defines a tensor that may be called the generalized curvature tensor.

For the modified Courant bracket, we see that $[\mathcal{X}_+, \mathcal{Y}_-]^D = [\mathcal{X}_+^D, \mathcal{Y}_-^D] \forall \mathcal{X}_+ \in \Gamma V_+, \mathcal{Y}_- \in \Gamma V_-$. Furthermore, we get

$$g([\mathcal{X}_+, \mathcal{Y}_-]^D, Z) = g([\mathcal{X}_+, \mathcal{Y}_-], Z) \pm \frac{1}{2}(\gamma(X, D^\pm Y) - \gamma(D^\pm X, Y)), $$
and, if we denote by $\kappa^\pm(X,Y)$ the 1-form defined by the last term of the previous formula (which evaluates $\kappa^\pm(X,Y)$ on $Z$), we have

$$[X_\pm, Y_\pm]^D = [X_\pm, Y_\pm] \pm (0, \kappa^\pm(X,Y)) .$$

(In the previous formulas, $X, Y, Z$ are the projections of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ on $TM$.)

Now, if the previous result is used and if we act by $[X,Y] = \text{pr}_{TM}[X,Y]$ on $g(Z,U)$ and $G(Z,U)$, we get the skew symmetry conditions

$$g(R^D(X,Y)Z,U) + g(Z,R^D(X,Y)U) = 0,$$

$$G(R^D(X,Y)Z,U) + G(Z,R^D(X,Y)U) = 0.$$

3. PARALLEL DIRAC STRUCTURES

Before referring to a single Dirac structure, we look at pairs of transversal structures. Consider an endomorphism $\Psi \in \text{End}(TM)$ such that

$$\Psi^2 = \epsilon \text{Id}, \quad g(\mathcal{X}, \Psi \mathcal{Y}) + g(\Psi \mathcal{X}, \mathcal{Y}) = 0, \quad \epsilon = \pm 1. \quad (16)$$

Then, the expression

$$N_\Psi(\mathcal{X}, \mathcal{Y}) = [\mathcal{X}, \mathcal{Y}] - \Psi[\mathcal{X}, \mathcal{Y}] - \Psi[\Psi \mathcal{X}, \mathcal{Y}] + \Psi^2[\mathcal{X}, \mathcal{Y}] = 0 \quad (17)$$

is $C^\infty(M)$-bilinear and it is called the Courant-Nijenhuis torsion of $\Psi$. If $\epsilon = -1$, $\Psi$ is a generalized, almost complex structure $J$. If $\epsilon = 1$, $\Psi$ is a generalized, almost product structure, equivalently (because (16) implies that the two eigenbundles of $\Psi$ must be $g$-isotropic, hence, maximally isotropic, subbundles), a generalized, almost paracomplex structure. In both cases, if $N_\Psi = 0$, the structure is integrable and “almost” is omitted.

We refer to [5, 16] for the basics. In the complex case $J$ may be identified with the pair $E, \bar{E}$ of complex conjugate, transversal, almost Dirac structures defined by its $\pm \sqrt{-1}$-eigenbundles. In the paracomplex case $\Psi$ may be identified with the pair $E, E'$ of real, transversal, almost Dirac structures defined by its $\pm 1$-eigenbundles. In both cases, integrability is equivalent with the property that the eigenbundles are integrable, i.e., closed under Courant brackets. $\Psi$ has a representation by classical tensor fields

$$\Psi \left( \begin{array}{c} X \\ \alpha \end{array} \right) = \left( \begin{array}{cc} A & \#_\pi \\ b_\sigma & -tA \end{array} \right) \left( \begin{array}{c} X \\ \alpha \end{array} \right), \quad (18)$$

where $A \in \text{End}(TM), \pi \in \chi^2(M), \sigma \in \Omega^2(M), t$ denotes transposition and $A^2 = \epsilon \text{Id} - \#_\pi \circ b_\sigma, \quad \pi(\alpha \circ A, \beta) = \pi(\alpha, \beta \circ A), \quad \sigma(AX,Y) = \sigma(X,AY)$. 

$$\text{pr}_{TM}[X,Y]$$
The expression of the integrability condition in terms of \((A, \pi, \sigma)\) is known and it includes the fact that \(\pi\) is a Poisson bivector field.

We are interested in structures \(\Psi\) on a generalized Riemannian manifold \((M, G)\). Then, \(\Psi\) is compatible with \(G\) if
\[
G(\Psi X, \Psi Y) = G(X, Y),
\]
equivalently,
\[
\phi \circ \Psi = -\epsilon \Psi \circ \phi,
\]
where \(\phi\) is defined by (2). A compatible pair \((G, \Psi)\) with \(\epsilon = -1\), respectively \(\epsilon = 1\), is an almost generalized Hermitian, respectively metric, generalized almost product, structure and “almost” is omitted in the integrable case. In the Hermitian case, condition (19) is equivalent with the fact that the complex subbundles \(E, \bar{E}\) are \(G\)-isotropic. In the metric product case, condition (19) is equivalent with the fact that the eigenbundles \(E, E'\) are \(G\)-orthogonal.

In the Hermitian case, (20) shows that \(J = \Psi\) preserves the eigenbundles \(V_{\pm}\), hence, it corresponds bijectively with a pair of \(\gamma\)-compatible, almost complex structures \(J_{\pm}\) of \(M\) obtained by the transfer of \(J|_{V_{\pm}}\) to \(TM\) via the isomorphisms \(\tau_{\pm}\) of (5). In other words, \(J\) is expressed by
\[
J(X, \flat \psi_{\pm} X) = (J_{\pm} X, \flat \psi_{\pm} J_{\pm} X).
\]

In the metric product case, (20) shows that \(\Psi\) interchanges the eigenbundles \(V_{\pm}\) and \(\Psi\) bijectively corresponds to a bundle isomorphism \(F\) of \(TM\) such that
\[
\Psi(X, \flat \psi_{+\gamma} X) = (FX, \flat \psi_{-\gamma} FX)
\]
and \(F\) satisfies the condition
\[
\gamma(FX, FY) = \gamma(X, Y).
\]
By replacing \(X\) with \(F^{-1} X\), we get
\[
\Psi(X, \flat \psi_{-\gamma} X) = (F^{-1} X, \flat \psi_{+\gamma} F^{-1} X), \quad \gamma(F^{-1} X, F^{-1} Y) = \gamma(X, Y).
\]
If we express \(\Psi\) of (21) by (18), we get
\[
F = A + \sharp_{\pi} \circ b_{\psi_{+\gamma}}, \quad b_{\psi_{-\gamma}} \circ F = b_{\sigma} - t A \circ b_{\psi_{+\gamma}},
\]
\[
F^{-1} = A + \sharp_{\pi} \circ b_{\psi_{-\gamma}}, \quad b_{\psi_{+\gamma}} \circ F^{-1} = b_{\sigma} - t A \circ b_{\psi_{-\gamma}}.
\]
Then, by addition and subtraction
\[
\sharp_{\pi} = \frac{1}{2} (F - F^{-1}) \circ \sharp_{\gamma}, \quad A = \frac{1}{2} (F + F^{-1}) - \sharp_{\pi} \flat \psi,
\]
\[
b_{\sigma} = \flat b_{\psi} \circ (F + F^{-1}) - t A \circ b_{\psi}.
\]
Furthermore, since the projections \((1/2)(\text{Id} \pm \Psi)\) restrict to isomorphisms \(V_+ \to E, V_- \to E'\), we have

\[
E = \{(X, b_{\psi+\gamma}X) + (FX, b_{\psi-\gamma}FX)\}, \\
E' = \{(X, b_{\psi+\gamma}X) - (FX, b_{\psi-\gamma}FX)\},
\]

where the representation of the elements of \(E, E'\) is unique.

Now, we shall address the question of integrability and we start with the following result.

**Proposition 3.1.** On any generalized almost Hermitian or almost metric product manifold \((M, \Psi)\) there are compatible, \(G\)-metric, big connections.

**Proof.** For a big connection \(\nabla\), compatibility means \(\nabla \circ \Psi = \Psi \circ \nabla\). Using the expressions of \(\nabla\) and \(\Psi\) on \(V_\pm\), we see that, in the Hermitian case, the compatibility condition is equivalent with

\[
D^\pm \circ J_\pm = J_\pm \circ D^\pm.
\]

Since there exist many \(\gamma\)-metric connections that satisfy (25) (connections on the unitary principal bundles of frames associated with \((\gamma, J_\pm)\)), the required existence result holds.

In the metric product case, the compatibility condition reduces to

\[
D^- \circ F = F \circ D^+,
\]

which implies the second required condition \(D^+ \circ F^{-1} = F^{-1} \circ D^-\) because \(F\) is an isomorphism. Locally, pairs of connections satisfying (26) exist (take a local basis \(\{e_i\}\) of \(TM\) and put \(D^+ e_i = 0, D^- (F e_i) = 0\)). Then, corresponding global pairs can be constructed by the usual gluing procedure using a partition of unity. \(\Box\)

**Remark 3.1.** Compatibility of \(\Psi\) with a generalized connection is a more complicated condition since it adds the requirement \(\nabla^*_\alpha \circ \Psi = \Psi \circ \nabla^*_\alpha\). The compatibility of the generalized Levi-Civita connection with \(\Psi\) requires the conditions

\[
d\psi(J_\pm X, Y, Z) = -d\psi(X, J_\pm Y, Z), \\
d\psi(FX, FY, Z) = -d\psi(X, Y, b_{\psi+\gamma}b_{\psi-\gamma}Z),
\]

respectively for \(\epsilon = \pm 1\). The first condition holds iff \(d\psi = 0\) (check on arguments in the eigenspaces of \(J_\pm\) and use the skew-symmetry of \(d\psi\)). The second condition follows by using

\[
\alpha = b_{\psi+\gamma}b_{\psi+\gamma}\alpha, \quad Z = b_{\psi-\gamma}\alpha
\]

and replacing \(Y\) by \(FY\).

The next result that we need is
Proposition 3.2. If $\nabla$ is a big connection that commutes with the integrable, generalized almost (para-)complex structure $\Psi$, the Gualtieri torsion of $\nabla$ satisfies the condition
\begin{equation}
\epsilon T \nabla(X,Y,Z) + T \nabla(X,\Psi Y,\Psi Z) + T \nabla(\Psi X,Y,\Psi Z) + \epsilon T \nabla(\Psi X,\Psi Y,Z) = 0.
\end{equation}
Conversely, if there exists a big connection that commutes with $\Psi$ and satisfies (27), $\Psi$ is integrable.

Proof. A straightforward calculation [6] shows that the Courant-Nijenhuis torsion of $\Psi$ and the Gualtieri torsion of a $\Psi$-compatible big connection $\nabla$ are related by the following formula
\begin{equation}
g(N_{\Psi}(X,Y),Z) + \epsilon T \nabla(X,Y,Z) + T \nabla(X,\Psi Y,\Psi Z) + \\
+ T \nabla(\Psi X,Y,\Psi Z) = 0.
\end{equation}

Remark 3.2. Proposition 27 also holds for generalized connections $D$ and it holds if the Courant brackets are twisted by a 3-form $\Theta$ [13], in which case one has to use the form $d\psi + \Theta$ instead of $d\psi$.

Proposition 3.2 implies the fact that a generalized Kähler structure $(G,J)$ (see [5] for the definition) is characterized by the following couple of properties [6]: (a) the canonical big connection $\nabla$ commutes with the generalized almost complex structure $J$, (b) the Gualtieri torsion of the canonical connection is a sum of components of $J$-type $(2,1)$ and $(1,2)$. Indeed, property (a) is equivalent with $D^{-}J_{\pm} = 0$ and, if the notion of $J$-type is defined like for usual complex structures, property (b) is equivalent with (27) and, further, with the fact that $d\psi$ is a sum of components of $J_{\pm}$-type $(2,1)$ and $(1,2)$. These two properties characterize the generalized Kähler structures [5]. By Remark 3.2 and with the results of [5], a similar characterization holds for twisted generalized Kähler structures.

Definition 3.1. A generalized metric product structure $(G,\Psi)$ is said to be parallel if the (integrable) structure $\Psi$ commutes with the canonical big connection $\nabla$ of $G$.

Proposition 3.3. The generalized metric product structure $(G,\Psi)$ is parallel iff $d\psi = 0$ and the $\gamma$-isometry $F$ that defines $\Psi$ is parallel with respect to the Levi-Civita connection $D$ of $\gamma$.

Proof. Since the canonical big connection has no mixed torsion, by looking at three arguments in the same subbundle $V_{\pm}$, we see that (27) is equivalent with $d\psi = 0$. Then, $D^{-} = D$ and the commutation condition (26) becomes $DF = 0$. □
Remark 3.3. From (23), it follows that a parallel structure has an associated, Levi-Civita parallel, Poisson bivector field $\pi$. Hence, by a result of Lichnerowicz (e.g., see [15], Proposition 3.12) $\gamma$ is a decomposable metric with a Kählerian component.

Example 3.1. Let $(M, \gamma, J)$ be a Kähler manifold. Then, $F = J$ is parallel and, for any generalized Riemannian metric $G$ defined by $\gamma$ and by a closed 2-form $\psi$, we get a parallel structure $\Psi$, namely,

$$\Psi(X, b_{\psi \pm \gamma} X) = \pm (JX, b_{\psi \mp \gamma} JX).$$

For the structure (29), formulas (24) yield

$$E = \text{graph } b_{\psi - \omega}, E' = \text{graph } b_{\psi + \omega},$$

where $\omega(X, Y) = \gamma(JX, Y)$ is the Kähler form of $(\gamma, J)$.

Now, let us consider a single almost Dirac structure $E$ on a generalized Riemannian manifold $(M, G)$.

Proposition 3.4. The generalized Riemannian metric $G \leftrightarrow (\gamma, \psi)$ produces a bijective correspondence between the almost Dirac structures $E$ on $M$ and the tensor fields $F \in \text{End}(TM)$ that are fields of isometries of $\gamma$.

Proof. Obviously, $E$ may be identified with the unique, $G$-compatible, generalized, paracomplex structure $\Psi_E$ of $+1$-eigenbundle $E$ and $-1$-eigenbundle $E' = E^{\perp_G} = \phi(E)$ (the last equality follows from (2) and (3)). Therefore, $E$ may be identified with the $\gamma$-isometry $F_E$ of the bundle $(TM, \gamma)$ that corresponds to $\Psi_E$. □

We give a few more details about the bijective correspondence $E \leftrightarrow F_E$ of Proposition 3.4. If $F_E$ is given, the subbundle $E \subset TM$ is defined by the first formula (24), which may also be written as

$$E = \{(F_1 X, b_{\psi} F_1 X + b_{\gamma} F_2 X) / X \in TM\},$$

where $F_1 = \text{Id} + F_E$, $F_2 = \text{Id} - F_E$. The fact that $F_E$ is a $\gamma$-isometry is equivalent with

$$\gamma(F_1 X, F_2 Y) = -\gamma(F_2 X, F_1 Y).$$

We call (30) the $G$-representation of $E$ and, because $F_1 + F_2 = 2\text{Id}$, the vector $X$ that corresponds to an element of $E$ is uniquely defined; in other words, (30) is a vector bundle isomorphism $TM \rightarrow E$. Formula (30) gives the characteristic distribution of $E$ and its 2-form, which are the well known $TM$-equivalent of an almost Dirac structure [2]:

$$\pi|_{TM} E = \text{im } F_1, \quad \varpi(F_1 X, F_1 Y) = \psi(F_1 X, F_1 Y) + \gamma(F_2 X, F_1 Y).$$
Conversely, we claim that if $E$ is given by the pair $S = pr_T M E$, $\varpi \in \wedge^2 S^*$, the $G$-representation of $E$ is given by the tensor fields

$$F_1 X = 2 \sharp_{\gamma + \varpi - \psi} \varrho \varpi \gamma pr_S \gamma X, \quad F_2 = 2 \text{Id} - F_1, \quad F = \frac{1}{2}(F_1 - F_2).$$

Formula (33) is to be understood pointwisely and at each point $x \in M$ we identify $\varpi$ with a 2-form on $M$ by extending $\varrho \varpi$ to $\varrho \varpi \perp \gamma$ by 0. Furthermore, we have used the general notation $\sharp = \varrho - 1$ and the fact that $\gamma$ positive definite implies the non-degeneration of any tensor obtained by adding a 2-form to $\gamma$.

(33) follows either by a comparison between (30) and the known formula

$$E = \{(X, \varrho \varpi X + \nu) / X \in S, \nu \in \text{ann}(S)\},$$

or, straightforwardly, as follows. Given $E \hookrightarrow (S, \varpi)$, use the right hand side of (30) with $F_1$ given by (33) to pointwisely define some $E' \subset T M$. We will show that $E' = E$. The second condition (33) shows that the involved mapping $TM \rightarrow E'$ is injective and $\dim E' = m$. Then, since $\sharp_{\gamma + \varpi - \psi}, \varrho \gamma$ are isomorphisms, $F_1$ is surjective on $S$ and its kernel is $S_{\perp \gamma}$. Furthermore, with (34), we see that $E' \subset E$ since

$$b_\psi F_1 X + b_\gamma (2 \text{Id} - F_1) X - b_\varpi F_1 X = 2 b_\gamma X - 2 b_{\gamma + \varpi - \psi} F_1 X$$

$$= 2 b_\gamma pr_S \gamma \perp, X \in \text{ann}(S).$$

The conclusion is $E' = E$.

**Remark 3.4.** The projections $pr_{E}, pr_{E_{\perp}}$ may not be differentiable since, generally, the dimension of $E_{\perp}$ is upper semi-continuous instead of lower semi-continuous. However, $F_1, F_2, F$ are differentiable. This follows from the justification of their existence using $\Psi_E$, and may also be seen by using $F_1 + F_2 = 2 \text{Id}$.

A first expression of the integrability condition of $E$ is

$$g([X, Y], Z) = 0, \quad X, Y, Z \in \Gamma E.$$

By (30), we have

$$X = (F_1 X, b_\psi F_1 X + b_\gamma F_2 X), \quad Y = (F_1 Y, b_\psi F_1 Y + b_\gamma F_2 Y),$$

$$Z = (F_1 Z, b_\psi F_1 Z + b_\gamma F_2 Z),$$

where $X, Y, Z$ are vector fields on $M$. Then, using formula (10) and making the required technical computations, the integrability condition of $E$ becomes

$$\sum_{Cycl(X, Y, Z)} \{ (F_1 X) \gamma (F_1 Y, F_2 Z) - \gamma ([F_1 X, F_1 Y], F_2 Z) \} = d\psi(F_1 X, F_1 Y, F_1 Z).$$
Below, we show another way to express the integrability of $E$. For any generalized paracomplex structure $\Psi$, we may define the Courant-Ehresmann curvature of $E$ with respect to $E'$ by

$$E_{(E'; E)}(\mathcal{X}, \mathcal{Y}) = \langle \text{Id} - \Psi \rangle \langle \text{Id} + \Psi \rangle \mathcal{X}, \langle \text{Id} + \Psi \rangle \mathcal{Y} \rangle.$$

Then, we get

$$E_{(E'; E)} + E_{(E'; E)} = N_\Psi, \quad E_{(E'; E)} - E_{(E'; E)} = \Psi N_\Psi,$$

therefore,

$$(35) \quad E_{(E'; E)} = \frac{1}{2} \langle \text{Id} - \Psi \rangle N_\Psi. \quad \text{Obviously, } E \text{ is integrable iff } E_{(E'; E)} = 0.$$

Our next remark is that a $G$-metric big connection preserves the almost Dirac structure (i.e., $\nabla_X Y \in \Gamma E, \forall Y \in \Gamma E$) iff $\nabla$ commutes with $\Psi L$ and by Proposition 3.1, such connections exist for every $E$. Then, we get

**Proposition 3.5.** If there exists a big connection $\nabla$ that preserves $E$ and is such that the Gualtieri torsion satisfies the condition

$$(36) \quad T^\nabla(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) + T^\nabla(\mathcal{X}, \Psi E \mathcal{Y}, \mathcal{Z}) + T^\nabla(\Psi E \mathcal{X}, \mathcal{Y}, \mathcal{Z}) + T^\nabla(\Psi E \mathcal{X}, \Psi L \mathcal{Y}, \mathcal{Z}) = 0, \quad \forall \mathcal{X}, \mathcal{Y} \in \Gamma TM, \mathcal{Z} \in \Gamma E,$$

then, the almost Dirac structure $E$ is integrable. Conversely, if $E$ is integrable, (36) holds for any big connection $\nabla$ that preserves $E$.

**Proof.** Use (35) and insert $N_\Psi$ as given by (28) in the integrability condition $E_{(E'; E')} = 0$. Then, use $E = \text{im}(\text{Id} + \Psi)$ and $\Psi|_E = \text{Id}$.

**Definition 3.2.** The Dirac structure $E$ is parallel on $(M, G)$ if the canonical big connection $\nabla$ of $G$ preserves $E$.

**Proposition 3.6.** The almost Dirac structure $E$ is a parallel Dirac structure iff the following two conditions hold

$$(37) \quad \gamma(F_E Z, D_X F_E Y)) = \frac{1}{2} [d\psi(X, Y, Z) + d\psi(X, F_E Y, F_E Z)],$$

$$(38) \quad d\psi(X, Y, Z) + d\psi(F_E X, F_E Y, F_E Z) = 0,$$

where $D$ is the Levi-Civita connection of $\gamma$.

**Proof.** With a few simple technicalities, (37) follows from the expression of the commutation condition (26) for the connections (8). Then, if we replace $Z$ by $(\text{Id} + \Psi_E)Z$ in (36), and consider the result for all possible combinations of arguments in $V_L$ while remembering that the canonical big connection has no mixed torsion and satisfies (15), we see that the only condition required for the integrability of $E$ is (38).
**Example 3.2.** Take $E = \text{graph} \sharp P$, $P \in \chi^2(M)$. If we express $(\sharp P \alpha, \alpha)$ by the first formula (24), we get

\[ X + F_E X = \sharp P \alpha, \quad b_\psi (X + F_E X) + b_\gamma (X - F_E X) = \alpha, \]

which leads to

\[ X = \frac{1}{2} \sharp \gamma (\alpha - b_\psi \gamma \sharp P \alpha), \quad F_E X = -\frac{1}{2} \sharp \gamma (\alpha - b_\psi \gamma \sharp P \alpha). \]

Thus, $\text{Id} - b_\psi \gamma \sharp P$ must be an isomorphism, which we may also see as follows. \forall U \in TM we have

\[ \langle (\text{Id} - b_\psi \gamma \sharp P) b_\psi \gamma + U, U \rangle = (\psi + \gamma)(U, U) + \langle \sharp P \gamma b_\psi \gamma U, b_\psi \gamma U \rangle = \gamma(U, U), \]

which vanishes only for $U = 0$. Then, since $\psi + \gamma$ is non degenerate, $\langle (\text{Id} - b_\psi \gamma \sharp P) b_\psi \gamma + U \rangle = 0$ iff $U = 0$, and we are done. Similarly, $\text{Id} - b_\psi \gamma \sharp P$ is an isomorphism.

In the previous expressions of $X, F_E X$ it is preferable to replace $\alpha$ by $b_\gamma \sharp \gamma \alpha$. Accordingly, the isometry $F_E$ gets the form

\[ F_E X = (Q^+ - \text{Id})(Q^- + \text{Id})^{-1} X \quad (Q^\pm = \pm \sharp \gamma b_\psi \gamma \sharp P b_\gamma). \]

Then, we may check that $F_E$ satisfies condition (22) by writing down the latter for $(Q^- + \text{Id}) X, (Q^- + \text{Id}) Y$ instead of $X, Y$ and taking into account the skew symmetry of $\psi$ and $P$.

Now, if we replace the arguments $Y, Z$ by $(Q^- + \text{Id}) Y, (Q^- + \text{Id}) Z$ in (37) and $X, Y, Z$ by $(Q^- + \text{Id}) X, (Q^- + \text{Id}) Y, (Q^- + \text{Id}) Z$ in (38), we get the following characteristic conditions for $\text{graph} \sharp P$ to be parallel on $(M, G)$

\[ \gamma((Q^+ - \text{Id}) Z, (D_X F_E)(Q^+ - \text{Id}) Y) = \]

\[ = \frac{1}{2} [d\psi(X, (Q^- + \text{Id}) Y, (Q^- + \text{Id}) Z) + d\psi(X, (Q^+ - \text{Id}) Y, (Q^+ - \text{Id}) Z)], \]

\[ d\psi((Q^- + \text{Id}) X, (Q^- + \text{Id}) Y, (Q^- + \text{Id}) Z) + \]

\[ + d\psi((Q^+ - \text{Id}) X, (Q^+ - \text{Id}) Y, (Q^+ - \text{Id}) Z) = 0. \]

If $\psi = 0$, then, $Q^+ = Q_+ = Q$ and (39) reduce to $DF_E = 0$, equivalently,

\[ DQ = (Q - \text{Id})(Q + \text{Id})^{-1} DQ. \]

Putting $(Q + \text{Id})^{-1} DQ = S$, the previous condition becomes

\[ (Q + \text{Id}) S = (Q - \text{Id}) S, \]

i.e., $S = 0$. But, $S = 0$ iff $D^2 P = 0$. Thus, in the classical case ($\psi = 0$), the graph of $P$ is parallel iff $P$ is a $\gamma$-parallel Poisson structure.
4. SYMPLECTIC DIRAC STRUCTURES

In this section we shall discuss a special kind of Dirac structures that appear in connection with endomorphisms \( \tau \in \text{End}(TM) \) such that (16) with \( \Psi = \tau \) and \( \epsilon = 0 \) holds, i.e.,

\[
\tau^2 = 0, \quad g(\mathcal{X}, \tau \mathcal{Y}) + g(\tau \mathcal{X}, \mathcal{Y}) = 0.
\]

In [16] such endomorphisms were called generalized subtangent structures. In the present paper, a generalized subtangent structure of constant rank will be called a generalized 2-nilpotent structure. If \( \text{rank } \tau = \dim M \), we stick with the terminology of [16] and call \( \tau \) a generalized almost tangent structure.

By (40), the image \( E = \text{im } \tau \) of a generalized, 2-nilpotent structure is a \( g \)-isotropic subbundle, i.e., a big-isotropic structure in the sense of [17]. We denote by \( E^\perp g \) the \( g \)-orthogonal subbundle of \( E \) and notice that (40) implies \( E^\perp g \subseteq \ker \tau \). Moreover, since these subbundles have the same rank, we have \( E^\perp g = \ker \tau \). This remark leads to the existence of a well defined, non-degenerate 2-form \( \omega \in \Gamma \wedge^2 E^* \) (\( E^* \) is the dual bundle of \( E \)) given by

\[
\omega(e_1, e_2) = g(e_1, \mathcal{X}_2), \quad e_1, e_2 \in \Gamma E, \quad \tau \mathcal{X}_2 = e_2
\]

(independent of the choice of \( \mathcal{X}_2 \)). The converse is also true, i.e., if \( E \) is a big-isotropic structure and \( \omega \in \Gamma \wedge^2 E^* \) is non degenerate, (41) uniquely defines an element \( e_2 = \tau \mathcal{X}_2 \in E \) and we see that there exists a unique generalized 2-nilpotent structure \( \tau \) with \( E = \text{im } \tau \) and with the given form \( \omega \). The non-degeneracy of \( \omega \) implies the fact that a generalized 2-nilpotent structure \( \tau \) has an even rank. It also follows that a big-isotropic structure \( E \) is the image of a generalized 2-nilpotent structure \( \tau \) iff the structure group of \( E \) is reducible to a symplectic group.

Put \( \tilde{E} = TM/E^\perp g \). Since \( E^\perp g = \ker \tau \), \( \tau \) induces an isomorphism \( \tau' : \tilde{E} \to E \) given by \( \tau' \mathcal{X} \mod E^\perp g = \tau \mathcal{X} \). On \( \tilde{E} \) we have the skew-symmetric, non-degenerate 2-form \( \Lambda \) defined by

\[
\Lambda(\mathcal{X} \mod E^\perp g, \mathcal{Y} \mod E^\perp g) = g(\tau \mathcal{X}, \mathcal{Y}).
\]

The quotient bundle \( \tilde{E} \) is canonically isomorphic to the dual bundle \( E^* \) by means of the pairing

\[
(\mathcal{X} \mod E^\perp g, \mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}) \quad (\mathcal{X} \in \Gamma TM, \mathcal{Y} \in \Gamma E).
\]

Thus, \( \Lambda \) may be seen as a bivector field of \( E \). Recall the musical isomorphisms \( \flat_\omega : E \to E^* \), \( \flat_\omega e = i(e)\omega \), and \( \sharp_\Lambda : E^* \to E \), \( \sharp_\Lambda \mathcal{X} \mod E^\perp g = i(\mathcal{X} \mod E^\perp g)\Lambda \).

From the given definitions we see that \( \flat_\omega e \in \Gamma E^* \) identifies with \( \mathcal{Y} \mod E^\perp g \) such that \( \tau \mathcal{Y} = -e \). On the other hand, we have \( \sharp_\Lambda \mathcal{X} \mod E^\perp g = \tau' \mathcal{X} \mod E^\perp g = \tau \mathcal{X} \), therefore, \( \sharp_\Lambda \circ \flat_\omega = -\text{Id} \) and \( \Lambda(\flat_\omega e_1, \flat_\omega e_2) = \omega(e_1, e_2) \).
Definition 4.1. The generalized 2-nilpotent structure $\tau$ is weakly integrable if the big-isotropic structure $E = \text{im} \tau$ is integrable, i.e., closed under Courant brackets. The generalized 2-nilpotent structure $\tau$ is integrable if its Courant-Nijenhuis torsion is $N_\tau = 0$. If $\tau$ is integrable and $\text{rank} \tau = \dim M$, then $\tau$ is a generalized tangent structure.

Thus, $\tau$ is weakly integrable iff, $\forall X, Y \in \Gamma TM$, one has $[\tau X, \tau Y] \in \Gamma E$, equivalently, the formula
\[
(42) \quad [X_{\text{mod } E^\perp}, Y_{\text{mod } E^\perp}]_{\tilde{E}} = \tau' - 1 [\tau X, \tau Y]
\]
yields a well defined new bracket on $\Gamma \tilde{E}$. Then, $(\tilde{E}, pr_{TM} \circ \tau', [\cdot, \cdot]_{\tilde{E}})$ is a Lie algebroid.

Proposition 4.1. The generalized, 2-nilpotent structure $\tau$ is integrable iff it is weakly integrable and $\omega$ is a symplectic form of the Lie algebroid $E$.

Proof. If $d_E$ denotes the exterior differential of the Lie algebroid $E$, we have
\[
(43) \quad d_E\omega(\tau X, \tau Y, \tau Z) = pr_{TM} \tau X(g(\tau Y, Z)) - pr_{TM} \tau Y(g(\tau X, Z)) +
+ pr_{TM} \tau Z(g(\tau X, Y)) - g([\tau X, \tau Y], Z) + g([\tau X, \tau Z], Y) - g([\tau Y, \tau Z], X).
\]
If the general property $[10]
\[
pr_{TM} \tau X(f) = g([X, Y] + \partial g(X, Y), Z) + g(Y, [X, Z] + \partial g(X, Z)),
\]
where $\partial$ is defined by
\[
(44) \quad pr_{TM} \tau X(f) = 2g(X, \partial f), \quad f \in C^\infty(M),
\]
is applied to the second and third term of the right hand side of (43), reductions lead to the formula
\[
d_E\omega(\tau X, \tau Y, \tau Z) = -g(X, N_\tau(Y, Z)),
\]
which proves the conclusion of the proposition. $\square$

The proposition characterizes the big-isotropic and Dirac structures that are images of an integrable, generalized, 2-nilpotent structure and we will call them symplectic big-isotropic and Dirac structures.

Remark 4.1. The symplectic structure of the Lie algebroid $E$ defines a Poisson structure on $M$, which is given by
\[
\{f, h\} = \Lambda(d_E f, d_E h), \quad f, h \in C^\infty(M).
\]
Using (44), it follows that $d_E f, d_E h$ are represented by $2[\partial f]_{\text{mod } E^\perp g}$, $2[\partial h]_{\text{mod } E^\perp g}$ in $\tilde{E}$ and that $\partial f = (1/2)(0, df)$. Thus, with the definition of $\Lambda$, we get

$$\{f, h\} = g(\tau(0, df), (0, dh)) = \pi(df, dh),$$

where $\pi$ is the bivector field of the matrix representation (18).

\textit{Example 4.1.} For any closed 2-form $\theta$, \textit{graph} $\♭ \theta = \{(X, \♭ \theta X) / X \in TM\}$ is a Dirac structure. Since one has $(\{X, \♭ \theta X\}, \{Y, \♭ \theta Y\}) = ([X, Y], \♭ \theta [X, Y])$ (see (10)), $\{X, \♭ \theta X\} \mapsto X$ is an isomorphism between the Lie algebroids $E_\theta$ and $TM$, which identifies 2-$E$-forms with differential 2-forms on $M$ and $d_E$ with $d$. Proposition 4.1 gives a bijection between the generalized tangent structures $\tau$ on $M$ with image graph $\theta$ and symplectic forms $\mu$ on $M$. With (41), we get the expression of this correspondence

$$\tau(X, \alpha) = (U, \♭ \theta U), \quad U = \sharp_\mu (\alpha - \♭ \theta X) (\sharp_\mu \♭ \mu = - \text{Id}).$$

If the manifold $M$ has no symplectic forms, the graph of a presymplectic form is not the image of a generalized tangent structure.

\textit{Example 4.2.} If $P$ is a Poisson bivector field, \textit{graph} $P = \{\sharp_P \alpha, \alpha\} / \alpha \in T^* M$ is a Dirac structure on $M$. The Courant bracket within graph $P$ is

$$[(\sharp_P \alpha, \alpha), (\sharp_P \beta, \beta)] = (\sharp_P \{\alpha, \beta\} P, \{\alpha, \beta\} P),$$

where the bracket of 1-forms is that of the Lie algebroid structure of $T^* M$ defined by $P$ (e.g., [15]). Therefore, the mapping $(\sharp_P \alpha, \alpha) \mapsto \alpha$ is an isomorphism between the Lie algebroids graph $P$ and $T^* M$, and the generalized tangent structures $\tau$ on $M$ with image graph $P$ are in a one-to-one correspondence with the non degenerate 2-cocycles of the Lie algebroid $T^* M$, i.e., the bivector fields $W$ on $M$ that satisfy the condition $[P, W] = 0$ (Schouten-Nijenhuis bracket). Explicitly, the correspondence is given by

$$\tau(X, \alpha) = (\sharp_P \lambda, \lambda), \quad \lambda = \♭_W (X - \sharp_P \alpha), \quad \♭_W \sharp_W = - \text{Id}.$$

In order to give another expression of the relation between integrability and weak integrability we define the bracket

$$[X, Y]_\tau = [\tau X, \mathcal{Y}] + [X, \tau \mathcal{Y}],$$

which puts the integrability condition $N_\tau = 0$ under the form

$$\tau[X, Y]_\tau = [\tau X, \tau Y].$$
Straightforward computations that use (40) and the Courant algebroid axioms [10] for $TM$ give the following properties of the new bracket

$\{X, fY\}_\tau = f\{X, Y\}_\tau + pr_{TM} \tau X (f) Y + pr_{TM} \tau (f) Y\tau$, \hspace{1cm} (46)

$\sum_{\text{cycl}(X,Y,Z)} [\{X, Y\}_\tau, Z]_\tau = \sum_{\text{cycl}(X,Y,Z)} [Z, N_\tau (X, Y)] + \frac{1}{3} \theta \sum_{\text{cycl}(X,Y,Z)} g(Z, N_\tau (X, Y))$, \hspace{1cm} (47)

Let us assume that $\tau$ is weakly integrable. Since the closure of $E$ under Courant brackets is equivalent to $[\Gamma E, \Gamma E^\perp] \subseteq \Gamma E^\perp$ [17], we get $[X, Y]_\tau \in E^\perp$, $\forall X \in \Gamma TM$, $\forall Y \in \Gamma E^\perp$, and

$\{X_{mod E^\perp}, Y_{mod E^\perp}\}_\tau = [X, Y]_\tau, mod E^\perp$ is a well defined bracket on the quotient bundle $\tilde{E}$, which we call the induced $\tau$-bracket.

**Proposition 4.2.** If the generalized 2-nilpotent structure $\tau$ is integrable, then $\tilde{E}$ with the induced $\tau$-bracket and the anchor $pr_{TM} \circ \tau'$ is a Lie algebroid and $\tau'$ is an isomorphism of Lie algebroids. Furthermore, the weakly integrable, generalized, 2-nilpotent structure $\tau$ is integrable iff the $\tau$-induced bracket of $\tilde{E}$ is equal to the bracket $[\cdot, \cdot]_{\tilde{E}}$.

**Proof.** For the first part of the proposition check the axioms of a Lie algebroid using (46), (47) and $N_\tau = 0$ (the Lie algebroid $E$ has the usual Courant bracket of $TM$). In the second part of the proposition, $[\cdot, \cdot]_{\tilde{E}}$ is the bracket defined by (42) and the conclusion follows from the integrability condition (45). \boxend

**Remark 4.2.** We may transfer the previous Lie algebroid structure of $\tilde{E}$ to $E^*$. Thus, Proposition 4.2 may be reformulated in terms of $E^*$. On the other hand, we may transfer the Lie algebroid structure to any subbundle $Q$ such that $TM = E^\perp \oplus Q$; then, $\tau'$ yields an isomorphism $\tau'_Q : Q \rightarrow E$.

Furthermore, for any weakly integrable, generalized, 2-nilpotent structure $\tau$ with 2-form $\omega$ and the corresponding inverse $\Lambda$, we have the Gelfand-Dorfman dual bracket [3]

$\{X_{mod E^\perp}, Y_{mod E^\perp}\}_\Lambda = L_\Lambda X_{mod E^\perp} Y_{mod E^\perp} - L_\Lambda Y_{mod E^\perp} X_{mod E^\perp} - d_E (\Lambda (X_{mod E^\perp}, Y_{mod E^\perp}))$, \hspace{1cm} (48)

where the Lie derivative and the differential are those of the Lie algebroid $E$. We continue to use the identification of $E^*$ with $TM/E^\perp$ by the $g$-pairing...
and the identification of $\sharp_\Lambda X_{mod E^\perp g}$ with $\tau X$. Then, the evaluation of the bracket (48) on $\tau Z \in \Gamma E$ yields
\[
\langle \{X_{mod E^\perp g}, Y_{mod E^\perp g} \}_\Lambda, \tau Z \rangle = -d_E \omega(\tau X, \tau Y, \tau Z) + \omega(\tau Z, [\tau X, \tau Y]),
\]
which is equivalent to
\[
\langle \{X_{mod E^\perp g}, Y_{mod E^\perp g} \}_\Lambda, \tau Z \rangle = -\omega(\tau Z, [\tau X, \tau Y]) - d_E \omega.
\]

**Proposition 4.3.** The weakly integrable, generalized, 2-nilpotent structure $\tau$ is integrable iff $(E^\ast, \{ , \}_\Lambda, \sharp_\Lambda)$ is a Lie algebroid.

**Proof.** If $\tau$ is integrable, $\omega_E$ is symplectic, $\Lambda$ is a Poisson structure and $(E^\ast, \{ , \}_\Lambda, \sharp_\Lambda)$ is the corresponding, dual Lie algebroid. Conversely, if $(E^\ast, \{ , \}_\Lambda, \sharp_\Lambda)$ is a Lie algebroid and we apply its anchor to (49), we get
\[
\sharp_\Lambda((\sharp_\Lambda X_{mod E^\perp g}) \wedge (\sharp_\Lambda Y_{mod E^\perp g})) = 0,
\]
which is equivalent with $d_E \omega = 0$. □

**Remark 4.3.** Proposition 4.3 is just the known fact that $\Lambda$ is a Poisson bivector of $E$ iff $\omega_E$ is a symplectic form. Essentially, the proposition tells that the weakly integrable, generalized, 2-nilpotent structure $\tau$ is integrable iff $(E, E^\ast)$ has a natural structure of a triangular Lie bialgebroid.

## 5. METRICS AND SYMPLECTIC DIRAC STRUCTURES

In this section we discuss symplectic big-isotropic and Dirac structures $E = \text{im} \tau$ on a generalized Riemannian manifold $(M, G)$ and we shall use again the notation of Sections 2 and 4. We start with the following remarks. The endomorphism $\phi \in \text{End}(T)$ associated with $G$ is both a $g$-isometry and a $G$-isometry. This implies that $\phi(E)$ is again a $g$-isotropic subbundle of $TM$ and $\phi(E^\perp g) = (\phi(E))^\perp g$, and that the following relations hold
\[
(E^\perp g)^\perp g = (E^\perp g)^\perp g = \phi(E).
\]
Thus, if the subbundle $S$ is such that $E^\perp g = E \oplus_{\perp_G} S$, we have a decomposition
\[
TM = (E \oplus_{\perp_G} \phi(E)) \oplus_{\perp_G} S,
\]
where the subbundles $E \oplus_{\perp_G} \phi(E)$ and $S$ are invariant by $\phi$.

By (50), since $E^\perp g = \ker \tau$, the mapping $\tau_{\phi(E)}^\phi : \phi(E) \to E$ defined by $\tau_{\phi(E)}^\phi$ is an isomorphism.

**Definition 5.1.** The structures $G, \tau$ are compatible and the pair $(G, \tau)$ is a generalized, metric, 2-nilpotent structure if $\tau_{\phi(E)}^\phi$ is a $G$-isometry, i.e.,
\[
G(\tau X, \tau Y) = G(X, Y), \quad \forall X, Y \in \phi(E).
\]
The following proposition gives several equivalent conditions.

**Proposition 5.1.** The structures $G$ and $\tau$ are compatible iff one of the following conditions holds:

1) for any $X, Y \in T_xM$ ($x \in M$) one has

\[ G(\tau \phi \tau X, \tau \phi \tau Y) = G(\tau X, \tau Y); \]

2) the form $\omega_E$ associated with $\tau$ satisfies the condition

\[ \omega_E(\lambda \tau X, \lambda \tau Y) = \omega_E(\tau X, \tau Y), \]

where $\lambda = \tau \circ \phi : E \to E$;

3) the morphism $\lambda = \tau \circ \phi : E \to E$ is a complex structure on $E$ (i.e., $\lambda^2 = -\text{Id}$);

4) the morphism $\lambda' = \phi \circ \tau : \phi(E) \to \phi(E)$ is a complex structure on $\phi(E)$ ($\lambda'^2 = -\text{Id}$);

5) the morphism $\lambda' = \tau \circ \phi : TM \to TM$ satisfies the condition $\lambda'^3 + \lambda' = 0$;

6) the morphism $\lambda' = \phi \circ \tau : TM \to TM$ satisfies the condition $\lambda'^3 + \lambda' = 0$.

**Proof.** Condition 1) is equivalent to (51) because the general expression of elements of $\phi(E)$ is $\phi \tau X, \phi \tau Y$ and $\phi$ is a $G$-isometry.

Furthermore, rewrite (52) as

\[ g(\phi \lambda \tau X, \lambda \tau Y) = g(\phi \tau X, \tau Y). \]

The definition of $\omega_E$ transforms the latter into the equality

\[ \omega_E(\lambda^2 \tau X, \lambda \tau Y) = \omega_E(\lambda \tau X, \tau Y), \]

which, therefore, also is equivalent with the compatibility between $G$ and $\tau$. Since $\lambda$ is an isomorphism of $E$, we may take $\lambda \tau X = \tau \mathcal{U}$ and we see that (54) is equivalent to (53).

On the other hand, using the definition of $\omega_E$ and the $g$-skew-symmetry of $\tau$, we get

\[ \omega_E(\lambda \tau X, \tau Y) = -\omega_E(\tau X, \lambda \tau Y). \]

This implies the equivalence of (54) with

\[ \omega_E(\lambda^2 \tau X, \lambda \tau Y) = -\omega_E(\tau X, \lambda \tau Y), \]

which is equivalent with $\lambda^2 = -\text{Id}$ because $\omega_E$ is non degenerate. Thus, we have proven conditions 2) and 3).

Then, $\lambda^2 = -\text{Id}$ is equivalent to

\[ g(\tau \phi \tau \phi X, \phi Y) = -g(\tau X, \phi Y), \quad \forall X, Y \in \Gamma TM, \]

which transforms into

\[ g(\tau X, \lambda^2 \phi Y) = -g(\tau X, \phi Y). \]

It follows that $\lambda^2 = -\text{Id}$ is equivalent with $\lambda'^2 = -\text{Id}$, which is condition 4).
Finally, since $\phi$ preserves the subbundle $S$, $\tilde{\lambda}$ vanishes on $S \oplus \phi(E)$, which, together with $\lambda^2 = -\text{Id}$ implies $\tilde{\lambda}^3 + \lambda = 0$ and conversely. Similarly, $\lambda^2 = -\text{Id}$ is equivalent to $\tilde{\lambda}'^3 + \lambda' = 0$. This proves conditions 5) and 6). □

Using (2) we see that

\begin{equation}
G(\tau X, \tau Y) = \omega_E(\tau X, \lambda \tau Y).
\end{equation}

Then, if we use for Lie algebroids the same terminology as for manifolds, we have

**Proposition 5.2.** If $(M, G)$ is a generalized Riemannian manifold and $E$ is a $g$-isotropic subbundle of $TM$, there exist a bijection between the set of generalized, metric, 2-nilpotent structures $\tau$ with $\text{im}\tau = E$ and the set of complex structures $\lambda$ on $E$ that are compatible with $G|_E$. The structure $\tau$ is integrable iff $E$ is closed by Courant brackets and $(G|_E, \lambda)$ is an almost Kähler structure on the Lie algebroid $E$.

**Proof.** For a given $\lambda$ that satisfies the hypotheses, formula (55) yields $\omega_E$, which, then, produces the following structure $\tau$

\begin{equation}
\tau|_{E_{+g}} = 0, \quad \tau|_{\phi(E)} = \lambda \circ \phi. \quad \Box
\end{equation}

**Definition 5.2.** An integrable, generalized, metric, 2-nilpotent structure $(G, \tau)$ is of the Kähler type if the associated complex structure $\lambda$ is integrable in the sense that it has a vanishing $E$-Nijenhuis tensor; the latter is defined like the usual Nijenhuis tensor but with brackets in $\Gamma E$.

The Riemannian Lie algebroid $(E, G|_E)$ has a Levi-Civita $E$-connection $D^E$ [1] and we have

**Proposition 5.3.** An integrable, generalized, metric, 2-nilpotent structure $(G, \tau)$ is of the Kähler type iff $D^E_X \lambda = 0$, $\forall X \in \Gamma E$.

**Proof.** The same calculations like in the proof of Proposition IX.4.2 of [9] (with different factor conventions) give the formula

\[ G((D^E_X \lambda)(Y), Z) = \frac{1}{2}[d_E\omega_E(X, Y, Z) - d_E\omega_E(X, \lambda Y, \lambda Z) + G(N_\lambda(X, Y), \lambda Z), \forall X, Y, Z \in \Gamma E. \]

The latter proves the required conclusion. □

The operators $\tilde{\lambda}, \tilde{\lambda}'$ are not generalized F-structures [18] because they are not $g$-skew-symmetric. However, we have

**Proposition 5.4.** A generalized, metric, 2-nilpotent structure $(G, \tau)$ has a canonically associated generalized, metric F-structure.

**Proof.** See [18] for the definition of generalized, metric F-structures. The required structure is defined by

\begin{equation}
\Phi = \tilde{\lambda} + \tilde{\lambda}' = \tau \phi + \phi \tau.
\end{equation}
The properties of $\tilde{\lambda}, \tilde{\lambda}'$ proven in Proposition 5.1 imply $\Phi^3 + \Phi = 0$. The metric compatibility conditions

$$g(\Phi X, \Phi Y) + g(X, \Phi Y) = 0,$$

$$G(\Phi X, Y) + G(X, \Phi Y) = 0$$

easily check for all possible combinations of arguments in $E, \phi(E), S$. (We have to use the facts that $\phi$ is a $g$-isometry and that $S \perp g E, S \perp g E$.) □

Let us restrict ourselves to the almost tangent case. Then, $TM = E \oplus \phi(E)$ and the structure $(G, \Phi)$ associated to $(G, \tau)$ is a generalized almost Hermitian structure $(\Phi^2 = -\Id)$. On the other hand, the pair $(G, \tau)$ has the associated, generalized, almost paracomplex structure $\Psi = \Psi_E$ defined in Section 3, i.e.,

$$\Psi|_E = \Id, \quad \Psi|_{\phi(E)} = -\Id,$$

which is $G$-compatible. Furthermore, by checking separately on $E, \phi(E)$, we get

$$\Phi \circ \Psi = \Psi \circ \Phi = \tau \phi - \phi \tau = \tilde{\lambda} - \tilde{\lambda}' .$$

Together with the expression (56) of $\Phi$ this leads to the equality

$$\tau = \frac{1}{2} \Phi \circ (\Id + \Psi) \circ \phi .$$

**Proposition 5.5.** On a generalized Riemannian manifold $(M, G)$, there exists a canonical bijection between the $G$-compatible, generalized, almost tangent structures $\tau$ and the set of commuting pairs $(\Phi, \Psi)$ where $\Phi$ is a $G$-compatible, generalized, almost complex structure and $\Psi$ is a $G$-compatible, generalized, almost paracomplex structure on $M$.

**Proof.** We have seen how to construct the pair $(\Phi, \Psi)$ from $\tau$. Conversely, for a given pair $(\Phi, \Psi)$, let us define $\tau \in \text{End}(TM)$ by formula (57). Since $\phi$ and $\Phi$ are isomorphisms, we see that $\text{im} \tau = \text{im}(\Id + \Psi)$, which is the $(+1)$-eigenbundle of $\Psi$ and has rank $m$. Thus, we will define this subbundle as $E$ and, necessarily, the $(-1)$-eigenbundle of $\Psi$ will be $\phi(E)$. It is easy to check that $\tau^2 = 0$ on both $E$ and $\phi(E)$. For this structure $\tau$, we have

$$\lambda = \tau \circ \phi = \frac{1}{2} \Phi \circ (\Id + \Psi)|_E$$

and the commutation between $\Phi$ and $\Psi$ yields $\lambda^2 = -\Id$. Thus, by 3) of Proposition 5.1 $\tau$ is $G$-compatible. □

**Proposition 5.6.** On a generalized Riemannian manifold $(M, G)$, there exists a canonical bijection between the $G$-compatible, generalized, almost tangent structures $\tau$ and the set of pairs $(E, \Phi)$ where $E$ is an almost Dirac structure and $\Phi$ is a $G$-compatible, generalized, almost complex structure such
that $\Phi(E) \subseteq E$. Furthermore, the structure $\tau$ is integrable iff $E$ is integrable and $\Phi$ satisfies the following condition

$$[\phi X, \phi Y] + [\phi Y, \phi X] + \phi [\phi X, \phi Y] \in \Gamma E, \quad \forall X, Y \in \Gamma E. \quad (58)$$

Proof. The structure $\Phi$ associated with $\tau$ is (56) again. In the converse direction, take $\Psi = \Psi_E$ in Proposition 5.5, alternatively, take $\tau|_E = 0$, $\tau|_{\phi(E)} = (\Phi \circ \phi)|_E$.

Furthermore, notice that if $E$ is integrable, then, $N_\tau$ vanishes if at least one of the arguments is in $\Gamma E$. Furthermore, since $\Phi|_E = \tau \circ \phi$, the remaining part of the $\tau$-integrability condition reduces to

$$[\phi X, \phi Y] - \tau([\phi X, \phi Y] + [\phi Y, \phi X]) = 0, \quad \forall X, Y \in \Gamma E.$$

Since $E$ is $\Phi$-invariant, we may replace $[\phi X, \phi Y] = -\Phi^2[\phi X, \phi Y] = -\tau \phi [\phi X, \phi Y]$ and we get the integrability condition (58). □

Now, in analogy to Section 3, we prove

**Proposition 5.7.** For any generalized 2-nilpotent structure $\tau$, there exist big connections $\nabla$ that commute with $\tau$.

Proof. First, we notice that a big connection $\nabla$ commutes with $\tau$ iff $\nabla$ preserves the subbundle $E = \text{im } \tau$ and the induced connection $\nabla'$ of $E$ preserves the corresponding 2-form $\omega_E$. The preservation of $E$ obviously is a necessary condition for $\nabla \tau = \tau \nabla$. Thus, $\nabla'$ exists and the definition of $\omega_E$ shows that

$$\nabla' \omega_E(e, \tau Y) = \nabla_X g(e, Y) = 0, \quad e \in \Gamma E, \quad Y \in \Gamma TM.$$

Conversely, by subtracting $\nabla_X g(e, Y) = 0$ from $\nabla'_X \omega_E(e, \tau Y) = 0$ we get

$$g(e, \nabla_X Y) = \omega_E(e, \tau \nabla_X Y) = \omega_E(e, \nabla_X \tau Y),$$

therefore, $\nabla \tau Y = \tau \nabla Y$.

Now, in order to get the required big connection $\nabla$ we first construct an $\omega_E$-preserving connection $\nabla'$ on $E$. $\nabla'$ is given by the known formulas of almost symplectic geometry (e.g., [14]), for instance

$$\nabla' X e = \nabla^0_X e + \Theta(X, e), \quad \omega_E(\Theta(X, e), e') = \frac{1}{2} \nabla^0_X \omega_E(e, e'),$$

where $\nabla^0$ is an arbitrary connection on the vector bundle $E$ and $\Theta : \Gamma TM \times \Gamma E$ is a “tensor”. Then, we take a metric connection $\nabla^S$ on the pseudo-Euclidean subbundle $(S, g|_S)$ of (50). Finally, we define the connection $\nabla''$ on $\phi(E)$ such that $\nabla' + \nabla''$ preserves $g|_{E \oplus \phi(E)}$ (in the identification of $\phi(E)$ with $E^*$, $\nabla''$ is $\nabla'$ acting on $E^*$). With these choices, $\nabla = \nabla' + \nabla^S + \nabla''$ is a big connection.
that preserves $E$ and induces the $\omega_E$-preserving connection $\nabla'$ on $E$, hence, $\nabla$ commutes with $\tau$. □

Furthermore, in analogy with Proposition 3.2, we have

**Proposition 5.8.** If $\nabla$ is a big connection that commutes with the integrable, generalized 2-nilpotent structure $\tau$, the Gualtieri torsion of $\nabla$ satisfies the conditions

\begin{equation}
T^\nabla(X, Y, Z) = 0,
\end{equation}

if $X, Y, Z \in \Gamma E^1_g$,

\begin{equation}
T^\nabla(\tau X, \tau Y, Z) + T^\nabla(\tau X, Y, \tau Z) T^\nabla(X, Y, \tau Z),
\end{equation}

if none of the arguments $X, Y, Z$ belongs to $E^1_g$. Conversely, if there exists a big connection that commutes with $\tau$ and satisfies (59), (60), $\tau$ is integrable.

**Proof.** Formula (28) with $\epsilon = 0$ shows that the torsion condition

\[ T^\nabla(\tau X, \tau Y, Z) + T^\nabla(\tau X, Y, \tau Z) T^\nabla(X, Y, \tau Z) = 0 \]

for arbitrary arguments makes the assertions of the proposition hold. This torsion condition is equivalent to the couple (59), (60) (recall that $E^1_g = \ker \tau$). □

**Proposition 5.9.** Let $\tau$ be an integrable $G$-compatible 2-nilpotent structure on the generalized Riemannian manifold $(M, G)$. If $\nabla$ is a $G$-metric big connection that commutes with $\tau$, then, the $E$-connection induced by $\nabla$ on $E$ is the $E$-Levi-Civita connection and the structure $\tau$ is of the Kähler type.

**Proof.** Let $\nabla'$ be the usual connection induced by $\nabla$ on $E$, which is known to preserve the 2-form $\omega_E$. Under the hypotheses of the corollary, it also preserves the metric $G|_E$ and the complex structure $\lambda$. The integrability condition (59) implies $T^\nabla(e_1, e_2) = 0$. The induced $E$-connection is defined by $\nabla^E_{e_1} e_2 = \nabla'_{\text{pr}_TM} e_2$ and, by the previous remarks, it follows that $\nabla^E$ is torsionless and preserves $G|_E$ and $\lambda$, which means that $\nabla^E$ is the $E$-Levi-Civita connection and $\tau$ is of the Kähler type. □

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