Dedicated to Professor Radu Miron
on the occasion of his 85^{th} birthday

GENERALIZED BERWALD SPACES AS AFFINE DEFORMATIONS OF MINKOWSKI SPACES

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Generalized Berwald spaces are the Finsler spaces which admit metric linear connections in the tangent bundle of their base manifold. They reduce to Berwald spaces if the torsion of this connection vanishes. We show that generalized Berwald spaces with base manifold $\mathbb{R}^n$ are exactly the affine deformations of Minkowski spaces. They can also be represented as a pair of a Riemannian and a Minkowski space, and they coincide with the Finsler spaces of 1-form metric. We investigate their reduction to Minkowski, Riemannian or Euclidean spaces, as well as their conformal relations.

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1. INTRODUCTION

One of the most important notions in the theory of metric structures over a manifold is a linear connection in the tangent bundle of the base manifold which is metric in the sense that the associated parallel translations preserve the metric. Riemannian and locally Minkowski spaces admit such connections. However, a Finsler structure determines a metric linear connection only in the vertical subbundle of the second tangent bundle of the base manifold in general. The Finsler spaces which still admit metric linear connections in the tangent bundle of the base manifold enjoy a special interest, and are called *generalized Berwald spaces*. If such a metric connection is torsion-free, then the space is a *Berwald space*. Berwald spaces are well understood [8, 9], while we hardly have any deep results concerning generalized Berwald spaces. In this paper we continue and complete our investigations presented on the FERT-2011 conference [13], putting them into a somewhat different setting.

A *Finsler space* is a manifold M together with a Finsler function $\mathcal{F}$ on the tangent manifold TM. The $\mathcal{F}$-unit vectors in TM constitute the *indicatrix bundle* I of TM. Conversely, an indicatrix bundle $I \subset TM$, i.e., a smooth family of strongly convex 'unit spheres' around the origin in every tangent space, determines a Finsler function $\mathcal{F}$. So we may equivalently consider a Finsler space F^n as a pair (M, I), where M is a manifold and $I \subset TM$ is an indicatrix bundle. This interpretation fits our considerations much better. An *affine deformation* is an alteration of a given Finsler structure by regular linear (called *affine*) distortion of the indicatrices.

The structure of the paper is the following. In Section 2 we derive the metric linear connection induced by an affine deformation in $\mathbb{R}^n$. For simplicity, *we work over* $\mathbb{R}^n$ *as base manifold throughout the paper.* Our method and the results so obtained may be extended to more general classes of manifolds (e.g., to parallelizable manifolds), however such an extension still requires certain topological restrictions. In Section 3 we show that generalized Berwald spaces (denoted by $\mathcal{B}^n$ in this paper) are exactly the affine deformations of Minkowski spaces. It turns out in Section 4 that generalized Berwald spaces coincide with the Finsler spaces of 1-form metric, and they can uniquely be represented as a pair of a Riemannian and a Minkowski space. In this section we give a geometric condition for the reduction of a Finsler space to a $\mathcal{B}^n$ space, and for the reduction of a $\mathcal{B}^n$ space to a Minkowski, a Riemannian or a Euclidean space. Also, we mention a possible extension of these investigations to the more general class of *Lagrange spaces.* In Section 5 conformal relations are studied. We show that if a Finsler space is non-homothetically conformal to a Minkowski space, then it is a proper (i.e., non-Berwaldian) generalized Berwald space.

2. AFFINE DEFORMATIONS IN $\mathbb{R}^n$

As we have remarked above, our considerations will be of purely local character, so we assume that our base manifold is the Euclidean n-space $\mathbb{R}^n$ ($n \geq 2$), although the results obtained in this setting may be extended to some more general classes of manifolds. We also assume that $\mathbb{R}^n$ is equipped with the canonical scalar product $\langle\ ,\ \rangle$, which allows us to talk of the Euclidean unit sphere S^{n-1}, ellipsoids, etc. The tangent space $T_p\mathbb{R}^n$ of $\mathbb{R}^n$ at p is just $\{p\} \times \mathbb{R}^n$, so the tangent bundle of $\mathbb{R}^n$ is $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. If $(e_i)_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$, then the mappings

$$E_i : \mathbb{R}^n \to T\mathbb{R}^n \ p \mapsto E_i(p) := (p, e_i), \quad i \in \{1, \dots, n\}$$

are (smooth) vector fields on $\mathbb{R}^n$. We say that the family $(E_i)_{i=1}^n$ is the *natural frame field* on $\mathbb{R}^n$.

For any $p \in \mathbb{R}^n$, the tangent space $T_p\mathbb{R}^n$ may be identified with $\mathbb{R}^n$ by the canonical isomorphism $\mathfrak{i}_p : T_p\mathbb{R}^n \to \mathbb{R}^n,\ (p, v) \mapsto v$. Whenever it is convenient, we use this identification without any comment. Given two points $p, q \in \mathbb{R}^n$, the composite mapping

$$\mathfrak{i}_{p,q} := \mathfrak{i}_q^{-1} \circ \mathfrak{i}_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^n$$

yields a canonical isomorphism between $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^n$, called the *natural parallel translation* between $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^n$. We have

$$\text{(1)} \qquad \mathfrak{i}_{a,q} \circ \mathfrak{i}_{p,a} = \mathfrak{i}_{p,q}, \quad \mathfrak{i}_{p,p} = 1_{T_p\mathbb{R}^n}, \quad a, p, q \in \mathbb{R}^n,$$

so the family of the isomorphisms $\mathfrak{i}_{p,q}$ $(p, q \in \mathbb{R}^n)$ is a *parallelism on* $\mathbb{R}^n$, which we also call *natural*. (For the general concept of *parallelism on a manifold* we refer to [3], vol. I, p. 174 and vol. II, p. 361.)

Definition. By an *affine deformation* in $\mathbb{R}^n$ we mean a family

$$\mathfrak{a}_p : T_p\mathbb{R}^n \to T_p\mathbb{R}^n, \quad p \in \mathbb{R}^n$$

of regular linear transformations, depending smoothly on p, i.e., a type $(1, 1)$ tensor field

$$\text{(2)} \qquad \mathfrak{a} : p \in \mathbb{R}^n \mapsto \mathfrak{a}_p \in \mathsf{T}^1_1(T_p\mathbb{R}^n) \cong \mathrm{End}(T_p\mathbb{R}^n)$$

such that $\mathfrak{a}_p \in \mathrm{GL}(T_p\mathbb{R}^n)$ for all $p \in \mathbb{R}^n$.

Now, with the help of an affine deformation (2), we introduce a further parallelism on $\mathbb{R}^n$, which is no longer natural. Let for any two points p, q in $\mathbb{R}^n$

$$\text{(3)} \qquad P_{p,q} := \mathfrak{a}_q \circ \mathfrak{i}_{p,q} \circ \mathfrak{a}_p^{-1} : T_p\mathbb{R}^n \to T_q\mathbb{R}^n.$$

Then $P_{p,q}$ is a linear isomorphism and we have

$$\text{(4)} \qquad P_{a,q} \circ P_{p,a} = P_{p,q}, \quad P_{p,p} = 1_{T_p\mathbb{R}^n}, \quad a, p, q \in \mathbb{R}^n,$$

so the family $\mathbb{P} = (P_{p,q})_{(p,q)\in\mathbb{R}^n\times\mathbb{R}^n}$ is a parallelism on $\mathbb{R}^n$, called the *parallelism induced by the affine deformation* $\mathfrak{a}$. A vector field X on M is said to be *parallel* with respect to $\mathbb{P}$, if $P_{p,q}(X_p) = X_q$ for all $p, q \in \mathbb{R}^n$. Define the vector fields $X_i \in \mathfrak{X}(\mathbb{R}^n)$ by

$$\text{(5)} \qquad (X_i)_p := P_{o,p}(E_i(o)), \quad p \in \mathbb{R}^n, \ i \in \{1, \dots, n\},$$

where $o \in \mathbb{R}^n$ is the origin. Then the X_i's are parallel with respect to $\mathbb{P}$, since for each $p, q \in \mathbb{R}^n$,

$$\begin{aligned} P_{p,q}((X_i)_p) &\overset{(3),(5)}{=} \mathfrak{a}_q \circ \mathfrak{i}_{p,q} \circ \mathfrak{a}_p^{-1} \circ \mathfrak{a}_p \circ \mathfrak{i}_{o,p} \circ \mathfrak{a}_o^{-1}(E_i(o)) \overset{(1)}{=} \mathfrak{a}_q \circ \mathfrak{i}_{o,q} \circ \mathfrak{a}_o^{-1}(E_i(o)) \\ &= P_{o,q}(E_i(o)) =: (X_i)_q. \end{aligned}$$

Let the components of the tensor field $\mathfrak{a}$ with respect to the natural frame $(E_i)_{i=1}^n$ be the functions $a^i_j \colon \mathbb{R}^n \to \mathbb{R}$. Then

$$\text{(6)} \qquad \mathfrak{a}(E_j) = a^i_j E_i, \quad j \in \{1, \dots, n\},$$

where the matrix (a^i_j) is invertible. Let $(b^i_j) := (a^i_j)^{-1}$. An immediate calculation shows that

$$X_i = a^k_j b^j_i(o) E_k, \quad i \in \{1, \dots, n\}, \tag{7}$$

i.e., the components of X_i with respect to the natural frame $(E_i)_{i=1}^n$ form the invertible matrix

$$(P^k_i) = (a^k_j b^j_i(o)). \tag{8}$$

The inverse of (P^k_i) is

$$(\widetilde{P}^k_i) := (P^k_i)^{-1} = (a^k_j(o) b^j_i). \tag{9}$$

There exists a unique linear connection $\nabla^{\mathfrak{a}}$ in $T\mathbb{R}^n$ such that

$$\nabla^{\mathfrak{a}}_{X_i} X_j = 0, \quad i, j \in \{1, \dots, n\}. \tag{10}$$

$\nabla^{\mathfrak{a}}$ is called the *connection induced by* $\mathfrak{a}$. Since the components of the curvature tensor $\mathcal{R}^{\mathfrak{a}}$ of $\nabla^{\mathfrak{a}}$ with respect to the frame $(X_i)_{i=1}^n$ are given by

$$\mathcal{R}^{\mathfrak{a}}(X_i, X_j)X_k = \nabla^{\mathfrak{a}}_{X_i} \nabla^{\mathfrak{a}}_{X_j} X_k - \nabla^{\mathfrak{a}}_{X_j} \nabla^{\mathfrak{a}}_{X_i} X_k - \nabla^{\mathfrak{a}}_{[X_i, X_j]} X_k,$$

it follows that $\mathcal{R}^{\mathfrak{a}} = 0$, i.e., $\nabla^{\mathfrak{a}}$ is a flat connection. We determine the Christoffel symbols of $\nabla^{\mathfrak{a}}$ with respect to the natural frame $(E_i)_{i=1}^n$, i.e., the functions Γ^k_{ij} defined by

$$\nabla^{\mathfrak{a}}_{E_i} E_j = \Gamma^k_{ij} E_k, \quad i, j \in \{1, \dots, n\}.$$

Since

$$\nabla^{\mathfrak{a}}_{E_i} E_j = \nabla^{\mathfrak{a}}_{\widetilde{P}^k_i X_k} \widetilde{P}^l_j X_l = \widetilde{P}^k_i (X_k \widetilde{P}^l_j) X_l + \widetilde{P}^k_i \widetilde{P}^l_j \nabla^{\mathfrak{a}}_{X_k} X_l \overset{(10)}{=} \widetilde{P}^k_i (X_k \widetilde{P}^l_j) X_l,$$

applying (7) and (9) we find that

$$\Gamma^k_{ij} = a^k_l (D_i b^l_j) = -b^l_j (D_i a^k_l), \tag{11}$$

where D_i stands for the standard ith partial derivative operator (which may be identified with E_i).

Remark 1. Following an idea of N. Hicks [4], the linear connection $\nabla^{\mathfrak{a}}$ may also be constructed more directly as follows.

Let ∇ be the canonical flat connection in $T\mathbb{R}^n$, and let $\widetilde{\nabla}^{\mathfrak{a}}$ be defined by

$$\widetilde{\nabla}^{\mathfrak{a}}_X Y := \mathfrak{a} \nabla_X \mathfrak{a}^{-1} Y, \quad X, Y \in \mathfrak{X}(\mathbb{R}^n).$$

It is easy to check that $\widetilde{\nabla}^{\mathfrak{a}}$ satisfies the conditions of a covariant derivative operator. Since

$$\widetilde{\nabla}^{\mathfrak{a}}_{E_i} E_j := \mathfrak{a} \nabla_{E_i} \mathfrak{a}^{-1} E_j = \mathfrak{a} \nabla_{E_i} b^l_j E_l = (E_i b^l_j) \mathfrak{a}(E_l) = a^k_l (D_i b^l_j) E_k \overset{(11)}{=} \Gamma^k_{ij} E_k,$$

it follows that $\widetilde{\nabla}^{\mathfrak{a}} = \nabla^{\mathfrak{a}}$.

3. AFFINE DEFORMATIONS OF FINSLER SPACES

Let $\mathcal{U}$ be a nonempty domain of $\mathbb{R}^n$. By a *Finsler function on $\mathcal{U}$* we mean a continuous function

$$\mathcal{F}\colon T\mathcal{U} = \mathcal{U} \times \mathbb{R}^n \to [0, \infty[$$

such that

$(\mathcal{F}_1)$ $\mathcal{F}$ is smooth on $\mathcal{U} \times (\mathbb{R}^n \setminus \{0\})$;

$(\mathcal{F}_2)$ $\mathcal{F}_p := \mathcal{F} \restriction T_p\mathbb{R}^n$ is a norm on the vector space $T_p\mathbb{R}^n \cong \mathbb{R}^n$ for all $p \in \mathcal{U}$;

$(\mathcal{F}_3)$ $\mathcal{F}^2(p, \cdot\,)$ has positive definite Hessian on $\mathbb{R}^n \setminus \{0\}$ for every $p \in \mathcal{U}$.

In what follows, unless otherwise stated, by an n-dimensional *Finsler space* we mean a pair $F^n := (\mathbb{R}^n, \mathcal{F})$, where $\mathcal{F}$ is a Finsler function on the *whole* $\mathbb{R}^n$, although the case of (nonempty) convex subsets instead of $\mathbb{R}^n$ requires only obvious modifications. This remark goes also for most of the special Finsler spaces which will be introduced below.

The *indicatrix* of a Finsler space F^n at a point $p \in \mathbb{R}^n$ is

$$I_p := \{v \in \mathbb{R}^n \mid \mathcal{F}_p(v) := \mathcal{F}(p, v) = 1\} \subset T_p\mathbb{R}^n;$$

their union $I_{\mathbb{R}^n} := \bigcup_{p \in \mathbb{R}^n} I_p \subset T\mathbb{R}^n$ is called the *indicatrix bundle* of F^n. $I_{\mathbb{R}^n}$ and the Finsler function $\mathcal{F}$ determine each other mutually, so in place of $F^n = (\mathbb{R}^n, \mathcal{F})$ we also write $F^n = (\mathbb{R}^n, I_{\mathbb{R}^n})$. The indicatrices I_p prove to be more appropriate for our geometric considerations than the Finsler function $\mathcal{F}$.

Observe that if $\mathfrak{a}$ is an affine deformation in $\mathbb{R}^n$ and

$$\bar{I}_{\mathbb{R}^n} := \mathfrak{a} I_{\mathbb{R}^n} := \bigcup_{p \in \mathbb{R}^n} \mathfrak{a}_p I_p,$$

then $\mathfrak{a}F^n := (\mathbb{R}^n, \bar{I}_{\mathbb{R}^n})$ is also a Finsler space, called an *affine deformation of F^n*.

Example. (a) Consider the Euclidean n-space as the very special Finsler space $E^n = (\mathbb{R}^n, \mathcal{E})$, where

$$\mathcal{E}(p, v) := \langle v, v\rangle^{\frac{1}{2}} =: \|v\| \quad \text{for all } (p, v) \in T\mathbb{R}^n.$$

If $S^{n-1} := \{v \in \mathbb{R}^n \mid \|v\| = 1\}$ is the unit sphere in $\mathbb{R}^n$, then the indicatrix of E^n at a point p is

$$S_p = \mathfrak{i}_p^{-1}(S^{n-1}) = \{(p, v) \in T_p\mathbb{R}^n \mid \|v\| = 1\},$$

and the indicatrix bundle of E^n is $S = \bigcup_{p \in \mathbb{R}^n} S_p$. So we may write $E^n = (\mathbb{R}^n, S)$.

If $\mathfrak{a}$ is an affine deformation in $\mathbb{R}^n$, then

$$Q_p := \mathfrak{a}_p S_p = \mathfrak{a}_p \circ \mathfrak{i}_p^{-1}(S^{n-1}) \subset T_p\mathbb{R}^n \tag{12}$$

is an ellipsoid, and

$$\mathfrak{a}E^n := (\mathbb{R}^n, Q) := \left(\mathbb{R}^n, \bigcup_{p\in\mathbb{R}^n} Q_p\right) =: V^n \tag{13}$$

is a Riemannian space. Conversely, any Riemannian space V^n whose underlying manifold is $\mathbb{R}^n$ can be obtained in this way. From this it follows that given two Riemannian spaces $V_1^n = (\mathbb{R}^n, Q_1)$ and $V_2^n = (\mathbb{R}^n, Q_2)$, each of them can be obtained by an affine deformation from the other. This is not true, however, for two generic Finsler spaces!

(b) A Finsler space $(\mathbb{R}^n, \mathcal{F}) = (\mathbb{R}^n, I_{\mathbb{R}^n})$ is said to be a *Minkowski space*, denoted by $\mathcal{M}^n$, if the Finsler function 'does not depend on the position', i.e., if

$$\mathcal{F}_q = \mathcal{F}_p \circ \mathfrak{i}_{q,p} \quad \text{for all } p, q \in \mathbb{R}^n. \tag{14}$$

Then $\mathcal{F}$ may simply be considered as a continuous norm on $\mathbb{R}^n$, satisfying the smoothness and strong convexity requirements according to $(\mathcal{F}_1)$ and $(\mathcal{F}_3)$. In terms of the indicatrices, condition (14) takes the form

$$I_q = \mathfrak{i}_{p,q}(I_p), \quad p, q \in \mathbb{R}^n. \tag{15}$$

Thus for a Minkowski space one can unambiguously use the notation $\mathcal{M}^n = (\mathbb{R}^n, I_0)$, where $I_0 := \mathfrak{i}_p(I_p) \subset \mathbb{R}^n$, with an arbitrarily chosen point $p \in \mathbb{R}^n$. Applying an affine deformation $\mathfrak{a}$, we obtain from $\mathcal{M}^n$ the Finsler space $\mathfrak{a}\mathcal{M}^n = (\mathbb{R}^n, \bar{I}_{\mathbb{R}^n}) = (\mathbb{R}^n, \bigcup_{p\in\mathbb{R}^n} \bar{I}_p)$, where

$$\bar{I}_p := \mathfrak{a}_p(I_p) = \mathfrak{a}_p \circ \mathfrak{i}_p^{-1}(I_0), \quad p \in \mathbb{R}^n. \tag{16}$$

Then $\mathfrak{a}\mathcal{M}^n$ is not a Minkowski space any longer. An important exception is the case when the deforming tensor $\mathfrak{a}$ does not depend on the position, i.e., there is a regular linear transformation $\varphi \in \mathrm{GL}(\mathbb{R}^n)$ such that

$$\mathfrak{a}_p = \mathfrak{i}_p^{-1} \circ \varphi \circ \mathfrak{i}_p \quad \text{for any point } p \in \mathbb{R}^n.$$

For more information, see Theorem 2 below.

(c) A Finsler space $(\mathbb{R}^n, \mathcal{F})$ is said to be a *generalized Berwald space* if there is a linear connection ∇ in $T\mathbb{R}^n$ which is metric in the sense that the parallel translations

$$P_{p,q}^{\nabla}(\gamma) : T_p\mathbb{R}^n \to T_q\mathbb{R}^n, \quad p, q \in \mathbb{R}^n$$

along any curve segment γ connecting p with q, with respect to ∇ preserve the $\mathcal{F}$-norms of the tangent vectors, or, equivalently, if

$$I_q = P_{p,q}^{\nabla}(\gamma)(I_p) \quad \text{for all } p, q \in \mathbb{R}^n.$$

If, in addition, ∇ is torsion-free, then $(\mathbb{R}^n, \mathcal{F})$ is called a *Berwald space*. A generalized Berwald space will be called *proper*, if it is not a Berwald space.

For an analytic treatment of generalized Berwald spaces in an abstract setting we refer to [10] and [11]. Berwald spaces are treated in most books on Finsler spaces, and in a number of papers. For a recent survey on the different approaches to Berwald manifolds see [12].

Simplifying the notation, in what follows we write $P^{\nabla}_{p,q}$ for a parallel translation with respect to ∇, if it does not depend on the curve segment connecting p with q.

Now, we are in a position to formulate and prove the following important result (see also [14], and cf. [5]):

THEOREM 1. *An affine deformation of a Minkowski space is a generalized Berwald space, and any generalized Berwald space can be obtained in this way.*

Proof. Let $\mathcal{M}^n = (\mathbb{R}^n, I_0)$ be a Minkowski space, and $\mathfrak{a}$ an affine deformation in $\mathbb{R}^n$. Consider the affine deformation $\mathfrak{a}\mathcal{M}^n = (\mathbb{R}^n, \bigcup_{p\in\mathbb{R}^n} \bar{I}_p)$ of $\mathcal{M}^n$, whose indicatrices $\bar{I}_p$ are given by (16). Let $\nabla^{\mathfrak{a}}$ be the linear connection induced by $\mathfrak{a}$. Then the parallel translation with respect to $\nabla^{\mathfrak{a}}$ does not depend on the curves connecting the points (since $\mathcal{R}^{\mathfrak{a}} = 0$), and are just the mappings given by (3). So for any two points p, q in $\mathbb{R}^n$ we have

$$\begin{aligned} P_{p,q}(\bar{I}_p) &\overset{(3),(16)}{=} \mathfrak{a}_q \circ \mathfrak{i}_{p,q} \circ \mathfrak{a}_p^{-1} \circ \mathfrak{a}_p \circ \mathfrak{i}_p^{-1}(I_0) \\ &= \quad \mathfrak{a}_q \circ \mathfrak{i}_{p,q} \circ \mathfrak{i}_p^{-1}(I_0) = \mathfrak{a}_q \circ \mathfrak{i}_q^{-1}(I_0) \overset{(16)}{=} \bar{I}_q. \end{aligned}$$

Thus, the parallel translations preserve the indicatrices of $\mathfrak{a}\mathcal{M}^n$, therefore $\mathfrak{a}\mathcal{M}^n$ is a generalized Berwald space.

Conversely, let a generalized Berwald space $\mathcal{B}^n = (\mathbb{R}^n, \widetilde{I}_{\mathbb{R}^n})$ be given, and let ∇ be a metric linear connection which preserves the indicatrices of $\mathcal{B}^n$. Choose a point $p_0 \in \mathbb{R}^n$, and define a Minkowski space $\mathcal{M}^n = (\mathbb{R}^n, \bigcup_{p\in\mathbb{R}^n} I_p)$ by

$$I_{p_0} := \widetilde{I}_{p_0}, \ I_p := \mathfrak{i}_{p_0,p}(\widetilde{I}_{p_0}). \tag{17}$$

Let an affine deformation $\mathfrak{a}$ in $\mathbb{R}^n$ be given by

$$p \in \mathbb{R}^n \mapsto \mathfrak{a}_p := P^{\nabla}_{p_0,p}(\gamma) \circ \mathfrak{i}_{p,p_0} \in \mathrm{GL}(T_p\mathbb{R}^n),$$

where $P^{\nabla}_{p_0,p}(\gamma) : T_{p_0}\mathbb{R}^n \to T_p\mathbb{R}^n$ is the parallel translation with respect to ∇ along the parametrized line segment $\gamma\colon [0,1] \to \mathbb{R}^n,\ t \mapsto \gamma(t) := (1-t)p_0 + tp$. Displaying by a diagram

$$\begin{array}{ccc} T_p\mathbb{R}^n & \xrightarrow{\mathfrak{i}_{p,p_0}} & T_{p_0}\mathbb{R}^n \\ \mathfrak{a}_p \searrow & & \swarrow P^{\nabla}_{p_0,p}(\gamma) \\ & T_p\mathbb{R}^n & \end{array}$$

We claim that $\mathfrak{a}\mathcal{M}^n = \mathcal{B}^n$. Indeed, for every point $p \in \mathbb{R}^n$, we find that

$$\mathfrak{a}_p(I_p) = P^{\nabla}_{p_0,p}(\gamma) \circ \mathfrak{i}_{p,p_0} \circ \mathfrak{i}_{p_0,p}(\widetilde{I}_{p_0}) = P^{\nabla}_{p_0,p}(\gamma)(\widetilde{I}_{p_0}) = \widetilde{I}_p,$$

since $P^{\nabla}_{p_0,p}(\gamma)$ carries indicatrix to indicatrix. □

4. SOME PROPERTIES OF GENERALIZED BERWALD SPACES

In 1978 M. Matsumoto and H. Shimada [6] introduced and investigated an important class of special Finsler spaces, called *Finsler spaces with* 1*-form metric*. It turns out in our approach that this class is constituted by the affine deformations of the Minkowski spaces. So, in terms of the Finsler functions, we may formulate the following

Definition. Let $\mathcal{M}^n = (\mathbb{R}^n, \mathcal{G})$ be a Minkowski space, and $\mathfrak{a}$ an affine deformation on $\mathbb{R}^n$. If for each point $p \in \mathbb{R}^n$,

$$\mathcal{F}_p(v) := \mathcal{G}_p(\mathfrak{a}_p(v)), \quad v \in \mathbb{R}^n, \tag{18}$$

then

$$\mathcal{F} : T\mathbb{R}^n \to \mathbb{R}, \quad (p, v) \mapsto \mathcal{F}(p, v) := \mathcal{F}_p(v)$$

is a Finsler function, called a 1*-form Finsler function*. Then, we also say that $(\mathbb{R}^n, \mathcal{F})$ is a *Finsler space with* 1*-form Finsler function* (or with 1-form metric).

To justify the terminology, let $(E^i)_{i=1}^n$ be the dual frame of the natural frame field $(E_i)_{i=1}^n$. If for each $i \in \{1, \dots, n\}$ and $p \in \mathbb{R}^n$

$$\alpha^i(p) := E^i(p) \circ \mathfrak{a}_p, \tag{19}$$

then the α^i's are 1-forms on $\mathbb{R}^n$, and $\mathcal{F}$ may be written in the form

$$\mathcal{F} = \mathcal{G} \circ (\alpha^1, \dots, \alpha^n) \tag{20}$$

(cf. [11], Section 5).

If $(\mathbb{R}^n, \mathcal{F})$ is a Finsler space with 1-form Finsler function given by (18), then $(\mathbb{R}^n, \mathcal{F})$ is the affine deformation of $\mathcal{M}^n = (\mathbb{R}^n, \mathcal{G})$ by $\mathfrak{a}^{-1}$, i.e., $(\mathbb{R}^n, \mathcal{F}) = \mathfrak{a}^{-1}\mathcal{M}^n$. Indeed, let I_p and $\widetilde{I}_p$ be the indicatrices of $\mathcal{M}^n$ and $(\mathbb{R}^n, \mathcal{F})$ at the point p. Then

$$v \in \widetilde{I}_p \Leftrightarrow \mathcal{F}_p(v) = 1 \Leftrightarrow \mathcal{G}_p(\mathfrak{a}_p(v)) = 1 \Leftrightarrow \mathfrak{a}_p(v) \in I_p \Leftrightarrow v \in \mathfrak{a}_p^{-1}(I_p),$$

hence $\widetilde{I}_p = \mathfrak{a}_p^{-1}(I_p)$.

Since an affine deformation of a Minkowski space is clearly a Finsler space of 1-form metric, Theorem 1 leads immediately to

Corollary 1. *A Finsler space $(\mathbb{R}^n, \mathcal{F})$ is a generalized Berwald space if, and only if, $\mathcal{F}$ is a* 1*-form Finsler function.* □

THEOREM 2. *Let* $\mathcal{B}^n = \mathfrak{a}\mathcal{M}^n$ *be a generalized Berwald space, and let the* $(1,1)$ *tensor* $\mathfrak{a} \in \mathsf{T}^1_1(\mathbb{R}^n)$ *be represented as the sequence* $(\alpha^1, \dots, \alpha^n)$ *of* 1*-forms given by* (19)*. Then* $\mathcal{B}^n$ *is the same Minkowski space* $\mathcal{M}^n$ *if, and only if, the* 1*-forms* α^i *are closed.*

Proof. If $\mathcal{M}^n = (\mathbb{R}^n, \mathcal{G})$ and $\mathcal{B}^n = \mathfrak{a}\mathcal{M}^n = (\mathbb{R}^n, \mathcal{F})$, then

$$\mathcal{F}_p(v) = \mathcal{G}_p \circ \mathfrak{a}_p^{-1}(v) \quad \text{for all } (p,v) \in T\mathbb{R}^n. \tag{21}$$

Suppose first that $\mathcal{B}^n$ is a Minkowski space. Then $\mathfrak{a}$ does not depend on the position, i.e., as we have already remarked, there exists a linear isomorphism $\varphi \in \mathrm{GL}(\mathbb{R}^n)$ such that

$$\mathfrak{a}_p = \mathfrak{i}_p^{-1} \circ \varphi \circ \mathfrak{i}_p \quad \text{for all } p \in \mathbb{R}^n. \tag{22}$$

If $\varphi(e_j) = a^i_j e_i$, then $(a^i_j) \in \mathrm{GL}_n(\mathbb{R})$, and

$$\alpha^i_p(e_j)_p \overset{(19)}{=} E^i(p)(\mathfrak{i}_p^{-1} \circ \varphi(e_j)) = E^i(p)(p, a^k_j e_k) = a^k_j E^i(p)(E_k(p)) = a^i_j,$$

therefore $\alpha^i = a^i_j E^j$, and hence $d\alpha^i = (da^i_j)E^j = 0$, $i \in \{1, \dots, n\}$, since a^i_j do not depend on p. Thus the α^i's are closed forms.

Let, conversely, the 1-forms α^i be closed. Then they are exact as well, so there exist smooth functions $\varphi^i : \mathbb{R}^n \to \mathbb{R}$ such that $\alpha^i = d\varphi^i$. Since $\mathfrak{a}_p \in \mathrm{GL}(T_p\mathbb{R}^n)$ $(p \in \mathbb{R}^n)$, it follows that the mapping

$$\varphi := (\varphi^1, \dots, \varphi^n) : \mathbb{R}^n \to \mathbb{R}^n, \quad p \mapsto (\varphi^1(p), \dots, \varphi^n(p))$$

has invertible derivative at each point $p \in \mathbb{R}^n$, and $\mathfrak{a}_p$ may be identified with the mapping

$$\mathfrak{i}_{\varphi(p),p} \circ (\varphi_*)_p : (p,v) \in T_p\mathbb{R}^n \mapsto (p, \varphi'(p)(v)) \in T_p\mathbb{R}^n. \tag{23}$$

By the inverse mapping theorem, every point $p \in \mathbb{R}^n$ has an open neighbourhood $\mathcal{U}$ such that φ is a diffeomorphism of $\mathcal{U}$ onto some neighbourhood of $\varphi(p)$. Given such an open subset $\mathcal{U}$, consider the push-forward vector fields

$$X_i := \varphi_\# E_i = \varphi_* \circ E_i \circ \varphi^{-1}, \quad i \in \{1, \dots, n\}$$

over $\mathcal{U}$. Then (cf. (5)), (X_i) is a frame field on $\mathcal{U}$, and at each point $q \in \mathcal{U}$ we have

$$\begin{aligned}
\mathcal{F}(X_i(q)) = \mathcal{F}(\varphi_*((E_i)_{\varphi^{-1}(q)})) &\overset{(21)}{=} \mathcal{G} \circ \mathfrak{a}_q^{-1}(\varphi_*((E_i)_{\varphi^{-1}(q)})) \\
&\overset{(23)}{=} \mathcal{G} \circ (\varphi_*)_q^{-1} \circ \mathfrak{i}_{q,\varphi(q)}((\varphi_*)(\varphi^{-1}(q), e_i)) \\
&\overset{(22)}{=} \mathcal{G} \circ (\varphi_*)_q^{-1}(\varphi(q), \varphi'(q)(e_i)) = \mathcal{G}(q, e_i) = \mathcal{G}((E_i)_q),
\end{aligned}$$

therefore

$$\mathcal{F} \circ X_i = \mathcal{G} \circ E_i, \quad i \in \{1, \dots, n\}.$$

This means that $\mathcal{F}$ acts along the 'new' local coordinate vector fields X_i in the same way as $\mathcal{G}$ acts along the natural coordinate vector fields (restricted to $\mathcal{U}$), hence $(\mathcal{U}, \mathcal{F} \restriction \mathcal{U})$ is also a Minkowski space. Since this is true in a suitable neighbourhood of every point of $\mathbb{R}^n$, it follows that $(\mathbb{R}^n, \mathcal{F})$ itself is also a Minkowski space. $\square$

Now let us consider a Riemannian space $V^n = (\mathbb{R}^n, Q)$ where the indicatrix bundle Q is defined by (12) and (13). Given an ellipsoid Q_p and the sphere $S_p = \mathfrak{i}_p^{-1}(S^{n-1})$ in $T_p\mathbb{R}^n$, the linear isomorphism $\mathfrak{a}_p \in \mathrm{GL}(T_p\mathbb{R}^n)$ which carries S_p into Q_p, is unique up to a rotation of S_p, i.e., up to an element of the special orthogonal group $SO(T_p\mathbb{R}^n)$. This rotation may be specified by the requirement that the canonical basis $(p, (e_i)_{i=1}^n)$ of $T_p\mathbb{R}^n$ is mapped by $\mathfrak{a}_p$ into the basis $(p, (v_i)_{i=1}^n)$ formed by the principal axes of Q_p. Conversely, given an affine deformation $\mathfrak{a} \in \mathbb{R}^n$, the image $Q_p = \mathfrak{a}_p(S_p)$ is unique at each point $p \in \mathbb{R}^n$. Thus, we have a bijective correspondence

$$\mathfrak{a} \in \mathsf{T}_1^1(\mathbb{R}^n) \rightleftarrows V^n = (\mathbb{R}^n, Q),$$

from which the next result may be immediately concluded.

Theorem 3. *Any generalized Berwald space $\mathcal{B}^n = \mathfrak{a}\mathcal{M}^n$ is determined by a pair $(V^n, \mathcal{M}^n)$, where $V^n = (\mathbb{R}^n, Q)$ is a Riemannian space. Conversely, each pair $(V^n, \mathcal{M}^n)$ determines a unique generalized Berwald space.* $\square$

By Theorem 3, any generalized Berwald space $\mathcal{B}^n$ may be written in the form $(V^n, \mathcal{M}^n)$, which we call a *Riemann–Minkowski representation* of $\mathcal{B}^n$.

Corollary 2. *If $(V_0^n, \mathcal{M}^n)$ is a Riemann–Minkowski representation of a generalized Berwald space $\mathcal{B}^n$, then $\mathcal{B}^n$ has a metric linear connection which is determined alone by the Riemannian space V_0^n.*

Proof. The statement is an immediate consequence of the fact that the metric linear connection $\nabla^{\mathfrak{a}}$ constructed for $\mathcal{B}^n$ in the proof of Theorem 1 depends only on the affine deformation $\mathfrak{a}$, and $\mathfrak{a}$ is determined by V_0^n in our case. $\square$

Theorem 4. *A Riemann–Minkowski representation $(V^n, \mathcal{M}^n)$ of a generalized Berwald space is the Riemannian space V^n if, and only if, $\mathcal{M}^n = E^n$.*

Proof. With the notation introduced in the Examples in Section 3,

$$V^n = \mathfrak{a}E^n = \Big(\mathbb{R}^n, \bigcup_{p \in \mathbb{R}^n} Q_p\Big), \quad Q_p = \mathfrak{a}_p \circ \mathfrak{i}_p^{-1}(S^{n-1}),$$

$$\mathcal{M}^n = \Big(\mathbb{R}^n, \bigcup_{p \in \mathbb{R}^n} I_p\Big), \quad \mathcal{B}^n = \mathfrak{a}\mathcal{M}^n = \Big(\mathbb{R}^n, \bigcup_{p \in \mathbb{R}^n} \mathfrak{a}_p(I_p)\Big),$$

so $\mathcal{B}^n = V^n$ if, and only if, for each $p \in \mathbb{R}^n$ we have

$$\mathfrak{a}_p \circ \mathfrak{i}_p^{-1}(S^{n-1}) = \mathfrak{a}_p(I_p) \Leftrightarrow I_p = \mathfrak{i}_p^{-1}(S^{n-1}),$$

i.e., if $\mathcal{M}^n = E^n$. □

THEOREM 5. *A Riemann–Minkowski representation* $(V^n, \mathcal{M}^n)$ *of a generalized Berwald space* $\mathcal{B}^n = \mathfrak{a}\mathcal{M}^n$ *is the Euclidean space* E^n *if, and only if,* $\mathcal{M}^n = E^n$ *and* $\mathfrak{a} = (d\varphi^1, \dots, d\varphi^n)$*, where* $\varphi^i \in C^\infty(\mathbb{R}^n)$*,* $i \in \{1, \dots, n\}$*.*

Proof. By the previous theorem, $\mathcal{B}^n$ reduces to V^n if, and only if, $\mathcal{M}^n = E^n$. By the arguments applied in the proof of Theorem 2 it follows that an affine deformation $\mathfrak{a} = (\alpha^1, \dots, \alpha^n)$ of E^n leads to E^n itself if, and only if, the 1-forms α^i are closed. □

Finally, we give a simple characterization of generalized Berwald spaces among the Finsler spaces.

THEOREM 6. *A Finsler space* $(\mathbb{R}^n, I_{\mathbb{R}^n})$ *is a generalized Berwald space if, and only if, all of its indicatrices* I_p *are of the form* $a \circ \mathfrak{i}_{p_0,p}(I_{p_0})$*, where* p_0 *is a fixed point and* $a \in \mathrm{GL}(T_p\mathbb{R}^n)$

Proof. The necessity of the condition is obvious. To prove the sufficiency, consider the sets $G_p := \{A \in \mathrm{GL}(T_p\mathbb{R}^n) \mid A \circ \mathfrak{i}_{p_0,p}(I_{p_0}) = I_p\}$ for all $p \in \mathbb{R}^n$. Since the I_p's depend smoothly on p, it follows that $\bigcup_{p\in\mathbb{R}^n} G_p$ is a smooth fibre bundle over $\mathbb{R}^n$ with typical fibre G_{p_0}. Since $\mathbb{R}^n$ is contractible, this fibre bundle is trivial (see [1], Supplement 3.4B), and hence it has a global section $\mathfrak{a}$ which yields the desired affine deformation. □

Remark 2. We may see in the same way that a $\mathcal{B}^n$ space is a V^n space if, and only if, at least one of the indicatrices is an ellipsoid.

Remark 3. Our above considerations can also be extended to the more general class of Lagrange spaces. We recall that a *Lagrange space* with base manifold $\mathbb{R}^n$ is a pair $L^n = (\mathbb{R}^n, \mathcal{L})$, where the *Lagrangian*

$$\mathcal{L}\colon T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (p, v) \mapsto \mathcal{L}(p, v)$$

satisfies the following conditions:

($\mathcal{L}_1$) $\mathcal{L}$ is continuous on $T\mathbb{R}^n$, smooth on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$;

($\mathcal{L}_2$) at each point $p \in \mathbb{R}^n$, the Hessian of $\mathcal{L}(p, \cdot)$ is of rank n and has a constant signature over $\mathbb{R}^n \setminus \{0\}$.

Observe that the homogeneity of the Lagrangian is not required at all! Lagrange spaces, their geometry and wide-ranging applications have been extensively studied by R. Miron and his collaborators, we refer here only to the recent monographs [2] and [7].

In this paper we have no possibility for an effective generalization of our results in a Lagrangian setting, we mention only a possible transition from special Finslerian structures to corresponding special Lagrangian structures:

Minkowski space $\mathcal{M}^n = (\mathbb{R}^n, \mathcal{F}) \to$ *Lagrange-Minkowski space* $L\mathcal{M}^n = (\mathbb{R}^n, \mathcal{L}_{\mathcal{M}})$, where $\mathcal{L}_{\mathcal{M}}$ does not depend on the position.

Generalized Berwald space $\mathcal{B}^n = (\mathbb{R}^n, \mathcal{F}) \to$ *generalized Lagrange-Berwald space* $L\mathcal{B}^n = (\mathbb{R}^n, \mathcal{L}_{\mathcal{B}})$, admitting $\mathcal{L}_{\mathcal{B}}$-norm preserving linear connection in $T\mathbb{R}^n$.

Then we also have $L\mathcal{B}^n = \mathfrak{a}L\mathcal{M}^n$ in the sense that $\mathcal{L}_{\mathcal{B}}(p,v) = \mathcal{L}_{\mathcal{M}}(\mathfrak{a}_p^{-1}(p,v))$ for all $(p,v) \in T\mathbb{R}^n$, where $\mathfrak{a}$ is an affine deformation in $\mathbb{R}^n$.

5. CONFORMAL RELATIONS

Two Finsler functions $\mathcal{F}_1, \mathcal{F}_2$ over $\mathbb{R}^n$ are said to be *conformally related* if there exists a positive smooth function $\sigma\colon \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{F}_2 = (\sigma \circ \mathrm{pr}_1)\mathcal{F}_1$ where $\mathrm{pr}_1\colon T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the first projection. Then, we also say that the Finsler manifolds $(\mathbb{R}^n, \mathcal{F}_1)$ and $(\mathbb{R}^n, \mathcal{F}_2)$ are conformally related. The conformal relation is called *homothetic* if the function σ is constant. By a *proper* conformal relation we mean a non-homothetic conformal relation.

Claim. *Minkowski spaces do not admit proper conformal relations.*

Proof. Suppose that $(\mathbb{R}^n, \mathcal{G}_1)$ and $(\mathbb{R}^n, \mathcal{G}_2)$ are conformally related Minkowski spaces, i.e., $\mathcal{G}_2 = (\sigma \circ \mathrm{pr}_1)\mathcal{G}_1$, where $\sigma\colon \mathbb{R}^n \to \mathbb{R}$ is a positive smooth function. Then for each $(p,v) \in T\mathbb{R}^n$ we have $\mathcal{G}_2(p,v) = \sigma(p)\mathcal{G}_1(p,v)$. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ do not depend on the position, this implies that σ is constant. □

Theorem 7. *If a Finsler space is properly conformal to a Minkowski space, then it is a proper generalized Berwald space.*

Proof. Suppose that $F^n = (\mathbb{R}^n, \mathcal{F})$ is properly conformal to the Minkowski space $\mathcal{M} = (\mathbb{R}^n, \mathcal{G})$. Then, by definition, $\mathcal{F} = (\sigma \circ \mathrm{pr}_1)\mathcal{G}$, where $\sigma\colon \mathbb{R}^n \to \mathbb{R}$ is a non constant positive smooth function. Consider the affine deformation

$$\mathfrak{a}\colon p \in \mathbb{R}^n \mapsto \mathfrak{a}_p := \sigma(p)1_{T_p\mathbb{R}^n} \in \mathrm{GL}(T_p\mathbb{R}^n).$$

In terms of tensor components, $\mathfrak{a}_p = (a_i^j(p)) = \sigma(p)(\delta_j^i)$. The inverse of $\mathfrak{a}_p$ is

$$\mathfrak{a}_p^{-1} =: (b_i^j(p)) = \frac{1}{\sigma(p)}(\delta_i^j).$$

We have $F^n = \mathfrak{a}\mathcal{M}^n$, so F^n is indeed a generalized Berwald space. We show that this space does not reduce to a Berwald space. We have only to check that the torsion $\mathcal{T}^{\mathfrak{a}}$ of the metric linear connection $\nabla^{\mathfrak{a}}$ of $F^n = \mathfrak{a}\mathcal{M}^n$ does not vanish. Applying (11), the Christoffel symbols of $\nabla^{\mathfrak{a}}$ with respect to the canonical frame field $(E_i)_{i=1}^n$ are

$$\Gamma_{ij}^k = -b_j^l(D_i a_l^k) = -\frac{1}{\sigma}\delta_j^l(D_i(\sigma\delta_l^k)) = -\frac{1}{\sigma}\delta_j^k D_i\sigma.$$

Hence, the components of the torsion $\mathcal{T}^{\mathfrak{a}}$ are

$$\mathcal{T}_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k = \frac{1}{\sigma}(\delta_i^k D_j\sigma - \delta_j^k D_i\sigma),$$

therefore $\mathcal{T}^{\mathfrak{a}}$ vanishes if, and only if,

$$\delta_i^k D_j\sigma - \delta_j^k D_i\sigma = 0, \quad i,j,k \in \{1,\dots,n\}.$$

With the choice $i = k$ and $j \neq k$ we find that

$$D_j\sigma = 0, \quad j \in \{1,\dots,n\} \setminus \{k\}.$$

Since k may be chosen arbitrarily, from this it follows that σ is a constant function. But this is impossible, since the conformal relation is proper. $\square$

Theorem 8. *Two generalized Berwald spaces $\mathcal{B}_1^n = (V_1^n, \mathcal{M}_1^n)$ and $\mathcal{B}_2^n = (V_2^n, \mathcal{M}_2^n)$ are in conformal relation if, and only if, $\mathcal{M}_1^n$ and $\mathcal{M}_2^n$ are isometric, and V_1^n is conformal to V_2^n.*

Proof. Let $\mathcal{B}_1^n = \mathfrak{a}_1\mathcal{M}_1^n$, $\mathcal{B}_2^n = \mathfrak{a}_2\mathcal{M}_2^n$, where, with the notation of Section 3 (b), $\mathcal{M}_1^n = (\mathbb{R}^n, I_1)$, $\mathcal{M}_2^n = (\mathbb{R}^n, I_2)$. The associated Riemannian spaces V_1^n and V_2^n are the affine deformations of the Euclidean space $E^n = (\mathbb{R}^n, S)$, so $V_i^n = (\mathbb{R}^n, Q_i)$, where $Q_i = \mathfrak{a}_i S$, $i \in \{1,2\}$ (see Theorem 3). In terms of the indicatrix bundles,

$$\mathcal{B}_i^n = (\mathbb{R}^n, \bar{I}_i), \quad \bar{I}_i = \mathfrak{a}_i I_i, \quad i \in \{1,2\}.$$

If $\mathcal{B}_1^n$ and $\mathcal{B}_2^n$ are conformally related, there exists a positive smooth function $\sigma\colon \mathbb{R}^n \to \mathbb{R}$ such that $\bar{I}_1 = \sigma\bar{I}_2$ (i.e., $\bar{I}_1(p) = \sigma(p)\bar{I}_2(p)$ for all $p \in \mathbb{R}^n$). Then

$$\mathfrak{a}_1 I_1 = \bar{I}_1 = \sigma\bar{I}_2 = \sigma\mathfrak{a}_2 I_2,$$

whence

$$I_1 = \sigma\mathfrak{a}_1^{-1} \circ \mathfrak{a}_2 I_2,$$

which implies that $\sigma\mathfrak{a}_1^{-1} \circ \mathfrak{a}_2$ does not depend on the position. Hence, after a rescaling, $I_1 = I_2$, therefore $\mathcal{M}_1^n$ and $\mathcal{M}_2^n$ are isometric. Relation

$$\sigma\mathfrak{a}_1^{-1} \circ \mathfrak{a}_2 = \text{identity}$$

implies $\mathfrak{a}_1 = \sigma\mathfrak{a}_2$, whence

$$Q_1 = \mathfrak{a}_1 S = \mathfrak{a}_1 \circ \mathfrak{a}_2^{-1} Q_2 = \sigma Q_2,$$

which proves that V_1^n and V_2^n are conformally related.

Conversely, if $\mathcal{M}_1^n = \mathcal{M}_2^n$, and $V_1^n = (\mathbb{R}^n, Q_1)$ is conformal to $V_2^n = (\mathbb{R}^n, Q_2)$, then $I_1 = I_2$ and $Q_2 = \sigma Q_1$ imply that $\mathfrak{a}_2 S = \sigma\mathfrak{a}_1 S$. Hence $\mathfrak{a}_2 = \sigma\mathfrak{a}_1$, therefore $\mathfrak{a}_2 I_2 = \sigma \mathfrak{a}_1 I_1$ and so we have $\bar{I}_2 = \sigma \bar{I}_1$. This means that $\mathcal{B}_1^n$ is conformal to $\mathcal{B}_2^n$. $\square$

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