BOUNDEDNESS AND STABILITY OF A RATIONAL DIFFERENCE EQUATION WITH DELAY

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In this paper we investigate the global stability, persistence, boundedness of solutions of the recursive sequence

$$x_{n+1} = \frac{ax_n^q + \sum_{r=1}^{p} x_{n-r}^q}{bx_n^q + \sum_{r=1}^{p} x_{n-r}^q}, \tag{0.0.1}$$

where $a, b \in (0, \infty)$, $p, q \geq 1$ with the initial conditions $x_0, x_{-1}, \ldots$ and $x_{-p} \in (0, \infty)$.

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1. INTRODUCTION

Difference equations or discrete dynamical systems are a diverse field which impacts almost every branch of pure and applied mathematics. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vice versa. Recently, there has been a great interest in studying difference equations. One of the reasons for this is a necessity for some techniques that can be used in investigating equations arising in mathematical models describing real life situations in population biology, economy, probability theory, genetics, psychology, etc. Recently, there has been a lot of interest in studying the boundedness character and the periodic nature of non-linear difference equations. For some results in this area, see for example [17–21]. Difference equations have been studied in various branches of mathematics for a long time. First results in qualitative theory of such systems were obtained by Poincar and Perron in the end of the nineteenth and the beginning of the twentieth century. Many researchers have investigated the behavior of the solution of difference equations, for example: Camouzis et al. [3] investigated the behaviour of solutions of the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^2}{1 + x_{n-1}^2}.$$

Elabbasy et al. [7] investigated the global stability, boundedness, periodicity character and gave the solution for some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Grove, Kulenovic and Ladas [10] presented a summary of a recent work and many opened problems and conjectures on the third order rational recursive sequence of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}.$$

In [22] Kulenovic, Ladas and Sizer studied the global stability character and the periodic nature of the recursive sequence

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\gamma x_n + \delta x_{n-1}}.$$

Kulenovic and Ladas [21] studied the second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}.$$

Ibrahim in [11] studied the third order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(\alpha + \beta x_n x_{n-2})}.$$

For other important references, we refer the reader to [1], [2], [4], [5], [6], [8], [9], [12], [13], [14], [15], [16], [22], [23], [24], [25], [26].

Our goal in this paper is to investigate the global stability character and boundedness of solutions of the recursive sequence

$$x_{n+1} = \frac{a x_n^q + \sum_{r=1}^{p} x_{n-r}^q}{b x_n^q + \sum_{r=1}^{p} x_{n-r}^q},$$

where $a, b \in (0, \infty)$, $p, q \geq 1$ with the initial conditions $x_0, x_{-1}, \ldots$ and $x_{-p} \in (0, \infty)$.

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I$$

be a continuously differentiable function. Then, for every set of initial conditions $x_0, x_{-1}, \ldots$ and $x_{-k} \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{1.0.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [18].

Definition 1.0.1. A point $\overline{x} \in I$ is called an equilibrium point of equation (1.0.2) if

$$\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of equation (1.0.2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 1.0.2. The difference equation (1.0.2) is said to be persistence if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_0, x_{-1}, \ldots$ and $x_{-k} \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m \leq x_n \leq M$$

for all $n \geq N$.

Definition 1.0.3. Let I be some interval of real numbers.

(i) The equilibrium point $\overline{x}$ of equation (1.0.2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

we have $|x_n - \overline{x}| < \epsilon$ for all $n \geq -k$.

(ii) The equilibrium point $\overline{x}$ of equation (1.0.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of equation (1.0.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have $\lim\limits_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of equation (1.0.2) is global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$, we have $\lim\limits_{n\to\infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ of equation (1.0.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of equation (1.0.2).

(v) The equilibrium point $\overline{x}$ of equation (1.0.2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of equation (1.0.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} (\partial F(\overline{x}, \overline{x}, \ldots, \overline{x}))/(\partial x_{n-i}) y_{n-i}. \tag{1.0.3}$$

THEOREM 1.0.4. *Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$. Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + p x_n + q x_{n-k} = 0, \quad n = 0, 1, \ldots.$$

Remark 1.0.5. Theorem 1.0.4 can be easily extended to a general linear equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \quad n = 0, 1, \ldots, \tag{1.0.4}$$

where $p_1, p_2, \ldots p_k \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Then equation (1.0.4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT

In this section we study the local stability character of the solutions of equation (0.0.1)

LEMMA 2.0.6. *Equation* (0.0.1) *has a unique positive equilibrium point and is given by*

$$\overline{x} = \frac{a+p}{b+p}.$$

Remark 2.0.7. Let $f : (0,\infty)^{p+1} \to (0,\infty)$ be a continuous function defined by

$$f(u_0, u_1, u_2, \ldots u_p) = \frac{au_0^q + u_1^q + u_2^q + \cdots + u_p^q}{bu_0^q + u_1^q + u_2^q + \cdots + u_p^q}.$$

Therefore, it follows that

$$\frac{\partial f(u_0, u_1, u_2, \ldots u_p)}{\partial u_0} =$$

$$= \frac{aqu_0^{q-1}(bu_0^q + u_1^q + u_2^q + \cdots + u_p^q) - bqu_0^{q-1}(au_0^q + u_1^q + u_2^q + \cdots + u_p^q)}{(bu_0^q + u_1^q + u_2^q + \cdots + u_p^q)^2} =$$

$$= \frac{qu_0^{q-1}(au_1^q + au_2^q + \cdots + au_p^q - bu_1^q - bu_2^q - \cdots - bu_p^q)}{(bu_0^q + u_1^q + u_2^q + \cdots + u_p^q)^2} =$$

$$= \frac{qu_0^{q-1}[u_1^q + u_2^q + \cdots + u_p^q][a-b]}{(bu_0^q + u_1^q + u_2^q + \cdots + u_p^q)^2},$$

$$\frac{\partial f(u_0, u_1, u_2, \ldots, u_p)}{\partial u_1} =$$

$$= \frac{qu_1^{q-1}(bu_0^q + u_1^q + u_2^q + \cdots + u_p^q) - qu_1^{q-1}(au_0^q + u_1^q + u_2^q + \cdots + u_p^q)}{(bu_0^q + u_1^q + u_2^q + \cdots + u_p^q)^2} =$$

$$= \frac{qu_1^{q-1}[u_0^q][b-a]}{(bu_0^q + u_1^q + u_2^q + \cdots + u_p^q)^2},$$

$$\frac{\partial f(u_0,u_1,u_2,\dots,u_p)}{\partial u_2}=$$

$$=\frac{qu_2^{q-1}(bu_0^q+u_1^q+u_2^q+\cdots+u_p^q)-qu_2^{q-1}(au_0^q+u_1^q+u_2^q+\cdots+u_p^q)}{(bu_0^q+u_1^q+u_2^q+\cdots+u_p^q)^2}=$$

$$=\frac{qu_2^{q-1}[u_0^q][b-a]}{(bu_0^q+u_1^q+u_2^q+\cdots+u_p^q)^2},$$

$$\frac{\partial f(u_0,u_1,u_2,\dots,u_p)}{\partial u_p}=\frac{qu_p^{q-1}[u_0^q][b-a]}{(bu_0^q+u_1^q+u_2^q+\cdots+u_p^q)^2}.$$

At the equilibrium point, we have that

$$\begin{aligned}\frac{\partial f(\overline{x},\overline{x},\dots,\overline{x})}{\partial u_0}&=\frac{q\overline{x}^{q-1}[\overline{x}^q+\overline{x}^q+\cdots+\overline{x}^q][a-b]}{(b\overline{x}^q+\overline{x}^q+\overline{x}^q+\cdots+\overline{x}^q)^2}=\frac{q\overline{x}^{q-1}[p\overline{x}^q][a-b]}{(b\overline{x}^q+p\overline{x}^q)^2}\\&=\frac{pq[a-b]}{\overline{x}(b+p)^2}=\frac{pq[a-b]}{(\frac{a+p}{b+p})(b+p)^2}=\frac{pq[a-b]}{(a+p)(b+p)},\\\frac{\partial f(\overline{x},\overline{x},\overline{x},\dots,\overline{x})}{\partial u_1}&=\frac{q\overline{x}^{q-1}[\overline{x}^q][b-a]}{(b\overline{x}^q+p\overline{x}^q)^2}=\frac{q[b-a]}{\overline{x}(b+p)^2}=\frac{q[b-a]}{(a+p)(b+p)},\\\frac{\partial f(\overline{x},\overline{x},\overline{x},\dots,\overline{x})}{\partial u_2}&=\frac{q[b-a]}{(a+p)(b+p)},\\&\vdots\\\frac{\partial f(\overline{x},\overline{x},\overline{x},\dots,\overline{x})}{\partial u_p}&=\frac{q[b-a]}{(a+p)(b+p)}.\end{aligned}$$

Then the linearized equation of equation (0.0.1) about $\overline{x}$ is

$$y_{n+1}+\sum_{i=0}^{p}d_iy_{n-i}=0,$$

where $d_i=-f_{u_i}(\overline{x},\overline{x},\overline{x},\dots,\overline{x})$ for $i=0,1,\dots p$ whose characteristic equation is

$$\lambda^{p+1}+\sum_{i=0}^{p}d_i\lambda^i=0. \tag{2.0.5}$$

THEOREM 2.0.8. *Assume that*

$$|a-b|<\frac{(a+p)(b+p)}{2pq}.$$

Then the positive equilibrium point of equation (0.0.1) *is locally asymptotically stable.*

Proof. It follows that equation (1.0.4) is symptotically stable if all roots of equation (2.0.5) lie in the open disc $|\lambda| < 1$ that is if

$$\left|\frac{pq[a-b]}{(a+p)(b+p)}\right| + \left|\frac{(q)[b-a]}{(a+p)(b+p)}\right| + \left|\frac{(q)[b-a]}{(a+p)(b+p)}\right| + \cdots + \left|\frac{(q)[b-a]}{(a+p)(b+p)}\right| < 1,$$

$$\left|\frac{pq[a-b]}{(a+p)(b+p)}\right| + p\left|\frac{(q)[b-a]}{(a+p)(b+p)}\right| < 1$$

or,

$$2\left|\frac{pq[a-b]}{(a+p)(b+p)}\right| < 1, \qquad \frac{|a-b|}{(a+p)(b+p)} < \frac{1}{2pq}.$$

This completes the proof. □

3. BOUNDEDNESS OF SOLUTIONS

Here, we study the permanence of equation (0.0.1).

THEOREM 3.0.9. *Every solution of equation* (0.0.1) *is bounded from above.*

Proof. Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of equation (0.0.1). It follows from equation (0.0.1) that

$$x_{n+1} = \frac{ax_n^q + \sum_{r=1}^{p} x_{n-r}^q}{bx_n^q + \sum_{r=1}^{p} x_{n-r}^q} =$$

$$= \frac{ax_n^q}{bx_n^q + \sum_{r=1}^{p} x_{n-r}^q} + \frac{\sum_{r=1}^{p} x_{n-r}^q}{bx_n^q + \sum_{r=1}^{p} x_{n-r}^q} \le \frac{ax_n^q}{bx_n^q} + \frac{\sum_{r=1}^{p} x_{n-r}^q}{\sum_{r=1}^{p} x_{n-r}^q} = \frac{a}{b} + 1.$$

Then $x_n \le \frac{a}{b} + 1 = M$ for all $n \ge 1$.

Therefore, every solution of equation (0.0.1) is bounded from above. □

4. SOME SPECIAL CASES

4.1. Case 1: $p = 1$, $q = 1$ (see [18, Theorem 6.9.1], [22])

In this case, we have the following difference equation

$$x_{n+1} = \frac{ax_n + x_{n-1}}{bx_n + x_{n-1}}. \tag{4.1.1}$$

LEMMA 4.1.1. *Equation* (4.1.1) *has a unique positive equilibrium point and is given by*

$$\overline{x} = \frac{a+1}{b+1}.$$

THEOREM 4.1.2. *Assume that*

$$|a-b| < \frac{(a+1)(b+1)}{2}.$$

Then the positive equilibrium point of equation (4.1.1) *is locally asymptotically stable.*

4.2. **Case 2:** $p = 2$, $q = 3$

In this case, we have the following difference equation

$$x_{n+1} = \frac{ax_n^3 + x_{n-1}^3 + x_{n-2}^3}{bx_n^3 + x_{n-1}^3 + x_{n-2}^3}. \tag{4.2.1}$$

LEMMA 4.2.1. *Equation* (4.2.1) *has a unique positive equilibrium point and is given by*

$$\overline{x} = \frac{a+2}{b+2}.$$

THEOREM 4.2.2. *Assume that*

$$|a-b| < \frac{(a+2)(b+2)}{12}.$$

Then the positive equilibrium point of equation (4.2.1) *is locally asymptotically stable.*

4.3. **Case 3:** $q = 1$, $p \geq 2$

In this case, we have the following difference equation

$$x_{n+1} = \frac{ax_n + x_{n-1} + x_{n-2} + \cdots + x_{n-p}}{bx_n + x_{n-1} + x_{n-2} + \cdots + x_{n-p}}. \tag{4.3.1}$$

LEMMA 4.3.1. *Equation* (4.3.1) *has a unique positive equilibrium point and is given by*

$$\overline{x} = \frac{a+p}{b+p}.$$

THEOREM 4.3.2. *Assume that*

$$|a-b| < \frac{(a+p)(b+p)}{2p}.$$

Then the positive equilibrium point of equation (4.3.1) *is locally asymptotically stable.*

4.4. **Case 4:** $p = 2$, $q = 2$

In this case, we have the following difference equation

$$x_{n+1} = \frac{ax_n^2 + x_{n-1}^2 + x_{n-2}^2}{bx_n^2 + x_{n-1}^2 + x_{n-2}^2}. \tag{4.4.1}$$

LEMMA 4.4.1. *Equation* (4.4.1) *has a unique positive equilibrium point and is given by*

$$\overline{x} = \frac{a+2}{b+2}.$$

THEOREM 4.4.2. *Assume that*

$$|a-b| < \frac{(a+2)(b+2)}{8}.$$

Then the positive equilibrium point of equation (4.4.1) *is locally asymptotically stable.*

4.5. **Case 5:** $p = 1$, $q \geq 2$

In this case, we have the following difference equation

$$x_{n+1} = \frac{ax_n^q + x_{n-1}^q}{bx_n^q + x_{n-1}^q}. \tag{4.5.1}$$

LEMMA 4.5.1. *Equation* (4.5.1) *has a unique positive equilibrium point and is given by*

$$\overline{x} = \frac{a+1}{b+1}.$$

THEOREM 4.5.2. *Assume that*

$$|a-b| < \frac{(a+1)(b+1)}{2q}.$$

Then the positive equilibrium point of equation (4.5.1) *is locally asymptotically stable.*

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