SPATIAL CONDITIONAL QUANTILE REGRESSION: WEAK CONSISTENCY OF A KERNEL ESTIMATE

SOPHIE DABO-NIANG, ZOULIKHA KAID and ALI LAKSACI

We consider a conditional quantile regression model for spatial data. More precisely, given a strictly stationary random field $Z_i = (X_i, Y_i)_{i \in \mathbb{N}}$, we investigate a kernel estimate of the conditional quantile regression function of the univariate response variable $Y_i$ given the functional variable $X_i$. The main purpose of the paper is to prove the convergence (with rate) in $L^p$ norm and the asymptotic normality of the estimator.

AMS 2010 Subject Classification: 62M30, 62M20, 62G08, 62G05.

Key words: Kernel conditional quantile estimation, Kernel regression estimation, spatial process.

1. INTRODUCTION

Conditional quantile estimation is an important field in statistics which dates back to Stone (1977) and has been widely studied in the non-spatial case. It is useful in all domain of statistics, such as time series, survival analysis and growth charts, among others, see Koenker ([20], [21]) for a review. There exist an extensive literature and various nonparametric approaches in conditional quantile estimation in the non spatial case for independent samples and dependent non-functional or functional observations. Among the many papers dealing with conditional quantile estimation in finite dimension, one can refer, for example, to key works of Portnoy [27], Koul and Mukherjee [23], Honda [19].

Potential applications of quantile regression to spatial data are number less. Indeed, there is an increasing number of situations coming from different fields of applied sciences (soil science, geology, oceanography, econometrics, epidemiology, environmental science, forestry, etc.), where the influence of a vector of covariates on some response variable is to be studied in a context of spatial dependence. The literature on spatial models is relatively abundant, see for example, Guyon [15], Anselin and Florax [3], Cressie [7] or Ripley [29] for a list of references.
In our knowledge, only the papers of Koencker and Mizera [22], Hallin et al. [18], Abdi et al. ([1], [2]), Dabo-Niang and Thiam [12] have paid attention to the study of nonparametric quantile regression for finite dimensional random fields while Laksaci and Maref [24] have considered infinite dimensional fields. This last work deals with almost sure consistency of the conditional quantile regression. The work of Hallin et al. [16] deals with local linear spatial conditional quantile regression estimation. The method of Koencker and Mizera [22] is a spatial smoothing technique rather than a spatial (auto)regression one and do not take into account the spatial dependency structure of the data. The results of Abdi et al. ([1], [2]) concerned respectively, consistency in $p$-mean ($p > 1$) and asymptotic normality and of a kernel estimate of the conditional regression function for spatial processes. Dabo-Niang and Thiam [12] considered the $L_1$ consistency of the local linear and double kernel conditional quantile estimate.

As in the non-spatial case, conditional quantile estimation is useful for some non-parametric prediction models and is used as an alternative to classical spatial regression estimation models for non-functional data (see Biau and Cadre [4], Lu and Chen ([25], [26]), Hallin, Lu and Tran [16], Dabo-Niang and Yao [10]). Spatial conditional quantile is of wide interest in the modeling of spatial dependence and in the construction of confidence (predictive) intervals. The purpose of this paper is to estimate the conditional quantile regression for spatial functional data.

Recall that a recent and restrictive attention has been paid to nonparametric estimation of the conditional quantile of a scalar variable $Y$ given a functional variable $(X = X_t, t \in \mathbb{R})$ when observations are over an interval $T \in \mathbb{R}$. The first results concerning the nonparametric quantile estimation adapted to non-spatial functional data were obtained by Ferraty et al. [13]. Recently, Dabo-Niang and Laksaci [11] stated the convergence in $L^p$ norm under less restrictive conditions closely related to the concentration properties on small balls probability of the underlying explanatory variable.

The main purpose of this paper is to extend some of the results on quantile regression to the case of functional spatial processes. In our knowledge, this work is the first contribution on spatial quantile regression estimation for functional variables. Noting that, extending classical nonparametric conditional quantile estimation for dependent functional random variables to quantile regression for functional random fields, is far from being trivial. This is due to the absence of any canonical ordering in the space, and of obvious definition of tail sigma-fields.

The paper is organized as follows. In Section 2, we provide the notations and the kernel quantile estimates. Section 3 is devoted to assumptions. Section 4 is devoted to the $L_p$ convergence and the asymptotic normality results.
of the kernel quantile regression estimate, under mixing assumptions. Proofs and technical lemmas are given in Section 5.

2. THE MODEL

Consider \( Z_i = (X_i, Y_i) \), \( i \in \mathbb{N}^N \) be a \( \mathcal{F} \times \mathbb{R} \)-valued measurable strictly stationary spatial process, defined on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\), where \((\mathcal{F}, d)\) is a semi-metric space. Let \( d \) denotes the semi-metric and \( N \geq 1 \). A point \( i = (i_1, \ldots, i_N) \in \mathbb{N}^N \) will be referred to as a site. We assume that the process under study \((Z_i)\) is observed over a rectangular domain \( I_n = \{ i = (i_1, \ldots, i_N) \in \mathbb{Z}^N, 1 \leq i_k \leq n_k, k = 1, \ldots, N \} \), \( n = (n_1, \ldots, n_N) \in \mathbb{N}^N \). A point \( i \) will be referred to as a site. We will write \( n \to \infty \) if \( \min\{n_k\} \to \infty \) and \( \frac{|n_j n_k|}{n_k} < C \) for a constant \( C \) such that \( 0 < C < \infty \), for all \( j, k \) such that \( 1 \leq j, k \leq N \).

We assume that the \( Z_i \)'s have the same distribution as \((X, Y)\) and the regular version of the conditional probability of \( Y \) given \( X \) exists and admits a bounded probability density. For all \( x \in \mathcal{F} \), we denote respectively by \( F^x \) and \( f^x \) the conditional distribution function and density of \( Y \) given \( X = x \).

Let \( \alpha \in ]0, 1[ \), the \( \alpha \)th conditional quantile noted \( q_\alpha(x) \) is defined by

\[
F^x(q_\alpha(x)) = \alpha.
\]

To insure existence and unicity of \( q_\alpha(x) \), we assume that \( F^x \) is strictly increasing. This last is estimated by

\[
\hat{F}^x_n(y) = \begin{cases} 
\frac{\sum_{i \in I_n} K_1 \left( \frac{d(x, X_i)}{a_n} \right) K_2 \left( \frac{y_Y}{b_n} \right)}{\sum_{i \in I_n} K_1 \left( \frac{d(x, X_i)}{a_n} \right)} & \text{if } \sum_{i \in I_n} K_1 \left( \frac{d(x, X_i)}{a_n} \right) \neq 0, \\
0 & \text{else},
\end{cases}
\]

where \( K_1 \) is a kernel, \( K_2 \) is a distribution function, \( a_n \) (resp. \( b_n \)) is a sequence of real numbers which converges to 0 when \( n \to \infty \).

The kernel estimate \( \hat{q}_\alpha(x) \) of the conditional quantile \( q_\alpha(x) \) defined by

\[
\hat{F}^x_n(q_\alpha(x)) = \alpha.
\]

One can also use other methods to estimate \( q_\alpha \), such as the local linear method or the reproducing kernel Hilbert spaces method (see Preda, [28]).

In the following, we fix a point \( x \) in \( \mathcal{F} \) such that

\[
P(X \in B(x, r)) = \phi_x(r) > 0,
\]

where \( B(x, h) = \{ x' \in \mathcal{F} \mid d(x', x) < h \} \).
3. HYPOTHESES

Throughout the paper, when no confusion will be possible, we will denote by $C$ and $C'$ any generic positive constant, and we denote by $g^{(j)}$ the derivative of order $j$ of a function $g$. We will use the following hypotheses:

3.1. Nonparametric model conditions

$H_1$: $F^x$ is of class $C^1$ and $f^x(q_\alpha(x)) > 0$.

$H_2$: $\exists \delta_1 > 0$, $\forall (y_1, y_2) \in [q_\alpha(x) - \delta_1, q_\alpha(x) + \delta_1]^2$, $\forall (x_1, x_2) \in N_x \times N_x$,

$$|F^{x_1}(y_1) - F^{x_2}(y_2)| \leq C(d_1(x_1, x_2) + |y_1 - y_2|^{b_2}), \quad b_1 > 0, \quad b_2 > 0,$$

where $N_x$ is a small enough neighborhood of $x$.

$H_3$: There exist $C_1$ and $C_2$, $0 < C_1 < C_2 < \infty$ such that $C_1 I_{[0,1]}(t) < K_1(t) < C_2 I_{[0,1]}(t)$.

$H_4$: $K_2$ is of class $C^1$, of bounded derivative that verifies

$$\int_{\mathbb{R}} |t|^{b_2} K_2^{(1)}(t) dt < \infty.$$

3.2. Dependency conditions

In spatial dependent data analysis, the dependence of the observations has to be measured. Here, we will consider the following two dependence measures:

3.2.1. Local dependence condition

In order to establish the same convergence rate as in the i.i.d. case (see Dabo-Niang and Laksaci [10]), we need the following local dependency condition:

(2) \[
\begin{align*}
(i) & \text{ For all } i \neq j, \text{ the conditional density of } (Y_i, Y_j) \text{ given } (X_i, X_j) \\
& \text{ exists and is bounded.} \\
(ii) & \text{ For all } k \geq 2, \text{ we suppose that: there exists an increasing sequence } 0 < (v_k) < k: \\
& \max(\max_{i_1 \ldots i_k} \mathbb{P}(d(X_{i_j}, x) \leq r, 1 \leq j \leq k), \phi_{x}^{k}(r)) = O(\phi_{x}^{1+v_k}(r)).
\end{align*}
\]
3.2.2. Mixing condition

The spatial dependence of the process will be measured by means of $\alpha$-mixing. Then, we consider the $\alpha$-mixing coefficients of the field $(Z_i, i \in \mathbb{N}^N)$, defined by: there exists a function $\varphi(t) \downarrow 0$ as $t \to \infty$, such that whenever $E$, $E'$ subsets of $\mathbb{N}^N$ with finite cardinals,

$$\alpha(B(E), B(E')) = \sup_{B \in B(E), C \in B(E')} |P(B \cap C) - P(B)P(C)|$$

$$\leq \psi(\text{Card}(E), \text{Card}(E'))\varphi(\text{dist}(E, E')),$$

where $B(E)$ (resp. $B(E')$) denotes the Borel $\sigma$-field generated by $(X_i, i \in E)$ (resp. $(X_i, i \in E')$), Card(E) (resp. Card(E')) the cardinality of $E$ (resp. $E'$), dist($E, E'$) the Euclidean distance between $E$ and $E'$ and $\psi : \mathbb{N}^2 \to \mathbb{R}^+$ is a symmetric positive function nondecreasing in each variable.

Throughout the paper, it will be assumed that $\psi$ satisfies either

$$\psi(n, m) \leq C \min(n, m), \quad \forall n, m \in \mathbb{N}$$

or

$$\psi(n, m) \leq C(n + m + 1)^{\tilde{\beta}}, \quad \forall n, m \in \mathbb{N}$$

for some $\tilde{\beta} \geq 1$ and some $C > 0$. In the following, we will only consider Condition (4), one can extend easily the asymptotic results proved here in the case of (5).

We assume also that the process satisfies the following mixing condition: the process satisfies a polynomial mixing condition

$$\sum_{i=1}^\infty i^\delta \varphi(i) < \infty, \quad \delta > N(p + 2), \ p \geq 1.$$ 

If $N = 1$, then $X_1$ is called strongly mixing. Many stochastic processes, among them various useful time series models, satisfy strong mixing properties, which are relatively easy to check. Conditions (4)–(5) are used in Tran [30], Carbon et al. [5], and are satisfied by many spatial models (see Guyon [14] for some examples). In addition, we assume that

$$H_5: \exists 0 < \tau < (\delta - 5N)/2N, \ \eta_0, \ \eta_1 > 0, \text{ such that } \hat{n}^\tau b_n \to \infty \text{ and }$$

$$Cn^{(\delta + 2\tau)N-\delta + \eta_0} \leq \phi_x(a_n),$$

where $\delta$ is introduced in (6).

Remark 1. If (6) is satisfied, then $\varphi(i) \leq Ci^{-\delta}$.
4. MAIN RESULTS

4.1. Weak consistency

This section contains results on pointwise consistency in $p$-mean. Let $x$ be fixed, we give a rate of convergence of $\hat{q}_\alpha(x)$ to $q_\alpha(x)$ under some general conditions.

**Theorem 1.** Under the hypotheses $H_1$–$H_5$, (4), then, for all $p \geq 1$, we have

$$\|\hat{q}_\alpha(x) - q_\alpha(x)\|_p = (E|\hat{q}_\alpha(x) - q_\alpha(x)|^p)^{1/p} = 0 \left( (a_n)^{b_1} + (b_n)^{b_2} \right) + O \left( \frac{1}{\hat{n} \phi_x(a_n)} \right)^{\frac{1}{2}}.$$ 

Let $K_1 = K_1 \left( \frac{d(x, X_i)}{a_n} \right)$, $H_1(y) = K_2 \left( \frac{y - Y_i}{b_n} \right)$, $W_{ni} = W_{ni}(x) = \frac{K_1}{\sum_{i \in I_n} K_i}$,

$$\hat{F}_N^x(y) = \frac{1}{nEK_1} \sum_{i \in I_n} K_i H_1(y), \quad \hat{F}_D^x = \frac{1}{nEK_1} \sum_{i \in I_n} K_i.$$ 

By hypothesis $H_4$, $\hat{F}_N^x(y)$ is of class $C^1$; then, we can write the following Taylor development

$$\hat{F}_N^x(\hat{q}_\alpha(x)) = \hat{F}_N^x(q_\alpha(x)) + \hat{F}_N^{x(1)}(q_\alpha^*(x)) (\hat{q}_\alpha(x) - q_\alpha(x)),$$

where $q_\alpha^*(x)$ is in the interval of extremities $q_\alpha(x)$ and $\hat{q}_\alpha(x)$. Thus,

$$\hat{q}_\alpha(x) - q_\alpha(x) = \frac{1}{\hat{F}_N^{x(1)}(q_\alpha^*(x))} \left( \hat{F}_N^x(\hat{q}_\alpha(x)) - \hat{F}_N^x(q_\alpha(x)) \right) \left( \alpha \hat{F}_D^x - \hat{F}_N^x(q_\alpha(x)) \right).$$

It is shown in Laksaci and Maref (2009) that under (H1)–(H5), (2) and (6) that

$$\hat{q}_\alpha(x) - q_\alpha(x) \rightarrow 0, \text{ almost completely (a.co)}.$$

So, by combining this consistency and the result of Lemma 11.17 in Ferraty and Vieu ([13], p. 181), together with the fact that $q_\alpha^*(x)$ is lying between $\hat{q}_\alpha(x)$ and $q_\alpha(x)$, it follows that

$$\hat{F}_N^{x(1)}(q_\alpha^*(x)) - f^x(q_\alpha(x)) \rightarrow 0. \text{ a.co.}$$
Since \( f^x(q_\alpha(x)) > 0 \), we can write

\[
\exists C > 0 \text{ such that } \left| \frac{1}{\hat{F}_N^{x(1)}(q_\alpha^*(x))} \right| \leq C \text{ a.s.}
\]

It follows that

\[
\|\hat{q}_\alpha(x) - q_\alpha(x)\|_p \leq C\|\alpha\hat{F}_D^x - \hat{F}_N^x(q_\alpha(x))\|_p + (P(\hat{F}_D^x = 0))^{1/p}.
\]

So, the rest of the proof is deduce from the following three lemmas.

**Lemma 1.** Under \( H_2 - H_4 \), we have

\[
E \left[ \alpha\hat{F}_D^x - \hat{F}_N^x(q_\alpha(x)) \right] = O\left( a_n^{b_1} + b_n^{b_2} \right).
\]

**Lemma 2.** Under the hypotheses of Theorem 1, we have

\[
\|\alpha\hat{F}_D^x - \hat{F}_N^x(q_\alpha(x)) - E \left[ \alpha\hat{F}_D^x - \hat{F}_N^x(q_\alpha(x)) \right]\|_p = o\left( \left( \frac{1}{\hat{n} \phi_x(a_n)} \right)^{\lambda/2} \right).
\]

**Lemma 3.** Under the hypotheses of Lemma 2, we have

\[
(P(\hat{F}_D^x = 0))^{1/p} = o\left( \left( \frac{1}{\hat{n} \phi_x(a_n)} \right)^{\lambda/2} \right).
\]

### 4.2. Asymptotic normality

This section contains results on the asymptotic normality of the quantile estimator. For that we replace, respectively \( H_2 \) and \( H_4 \) by the following hypotheses.

**\( H'_2 \):** \( F^x \) satisfies \( H_2 \) and \( \forall z \in N_x, F^x \) is of class \( C^1 \) with respect to \( y \), the conditional density \( f^x \) is such that \( f^x(q_\alpha) > 0 \) and \( \forall(x_1, x_2) \in N_x \times N_x, \forall(y_1, y_2) \in \mathbb{R}^2 \)

\[
|f^{x_1}(y_1) - f^{x_2}(y_2)| \leq \left( \|x_1 - x_2\|^{d_1} + |y_1 - y_2|^{d_2} \right), \quad d_1, d_2 > 0.
\]

**\( H'_4 \):** \( K_2 \) satisfies \( H_4 \) and

\[
\int |t|^{d_2} K^{(1)}_2(t)dt < \infty.
\]

**Theorem 2.** Under the hypotheses of Theorem 1 and \( H'_2, H'_4 \), (4) then, for any \( x \in A \), we have

\[
\left( \frac{f^x(q_\alpha(x)) \hat{n}(\psi_{K_1}(a_n))^2}{\psi_{K_1}(a_n)(\alpha(1 - \alpha))} \right)^{(1/2)} \to_n \tilde{N}(0, 1).
\]
where
\[ C_n(x) = \]
\[ \frac{1}{f^x(q_0(x))\psi_k_1(a_n)} (\alpha \psi_k_1(a_n) - E[K_1((a_n)^{-1}d(x, X)) F^X(q_0(x))]) + O(b_n) \]
and
\[ A = \left\{ x \in F, \frac{\psi_k_2(a_n)}{(\psi_k_1(a_n))^2} \neq 0 \right\} \]
with \( \psi_y(h) = -\int_0^1 g'(t)\phi_x(ht)dt. \)

Firstly, observe that if \( H_5 \) is satisfied then, we have
\[ \exists 0 < \theta_1 < 1, \text{ such that } \hat{n}^{(1+\theta_1)(1+2N)} \leq \phi_x(a_n). \]

Let
\[ \Delta_1 = \frac{1}{\sqrt{EK_1^2}} [\alpha K_1 - K_1 H_1(q_0) - E(\alpha K_1 - K_1 H_1(q_0))]. \]

By hypothesis \( H_1' \), \( \hat{F}_N^x(y) \) is of class \( C^1 \), then, we can write the following Taylor development:
\[ \hat{F}_N^x(q_0) = \hat{F}_N^x(q_0) + \hat{F}_N^{x(1)}(q_0 \ast (\hat{q}_0 - q_0) \]
where \( g_\alpha \) is in the interval of extremities \( q_0 \) and \( \hat{q}_0 \). Thus,
\[ \hat{q}_0 - q_0 = \frac{1}{\hat{F}_N^{x(1)}(q_\alpha)} \left( \hat{F}_N^x(q_\alpha) - \hat{F}_N(q_\alpha) \right) = \frac{1}{\hat{F}_N^{x(1)}(q_\alpha)} \left( \alpha \hat{F}_B^x - \hat{F}_N(q_\alpha) \right), \]

\[ \left[ \frac{f^x(q_\alpha)^2 \hat{n} E^2 K_1}{\alpha(1 - \alpha) E K_1^2} \right]^{1/2} \left( [\hat{q}_0 - q_0 - C_n(x)] \right) = \]
\[ \left[ \frac{\hat{n} E^2 K_1}{\alpha(1 - \alpha) E K_1^2} \right]^{1/2} \left( \frac{f^x(q_\alpha)}{\hat{F}_N^{x(1)}(q_\alpha)} \right) \left( \alpha \hat{F}_B^x - \hat{F}_N(q_\alpha) \right) - E \left( \alpha \hat{F}_B^x - \hat{F}_N(q_\alpha) \right), \]

where
\[ C_n(x) = \frac{1}{f^x(q_\alpha)} E \left( \alpha \hat{F}_B^x - \hat{F}_N(q_\alpha) \right). \]

Consequently, the proof of the theorem is the consequence of the following lemmas and the convergence result (7).

**Lemma 4.** Under the hypotheses of Theorem 2, we have
i) \( Var(\Delta_1) \rightarrow \alpha(1 - \alpha); \)
ii) \( \sum_{j \in I_n} Cov(\Delta_1, \Delta_j) = o(\hat{n}) \) and
iii) \( \frac{1}{\hat{n}} var(\sum_{i \in I_n} \Delta_i) \rightarrow \alpha(1 - \alpha), \) when \( n \rightarrow \infty. \)
Lemma 5. Under the hypotheses of Theorem 2 we have
\[
\left[ \frac{\hat{\alpha} E_2 K_{i \alpha}}{\alpha(1 - \alpha) E K_{i \alpha}^2} \right]^{1/2} \left( \frac{\alpha \hat{F}_{\alpha}^x - \hat{F}_{\alpha}^x(q_\alpha)}{\alpha \hat{F}_{\alpha}^x - \hat{F}_{\alpha}^x(q_\alpha)} \right) \to N(0, 1).
\]

Lemma 6. Under the hypotheses \( H'_2 \) and \( H'_4 \), we have
\[
E \left[ \alpha \hat{F}_{\alpha}^x \right] = \alpha - \frac{1}{EK_1} E \left[ K \left( \frac{d, x - X}{a_n} \right) F^x(q_\alpha) \right] + O \left( b_n^2 \right)
\]
and
\[
E \left[ \alpha \hat{F}_{\alpha}^x \right] = O \left( a_n b_n + b_n^2 \right).
\]

It is easy to see that, if one imposes some additional assumptions on the function \( \phi_x(\cdot) \) and the bandwidth parameters \( (a_n, b_n) \) we can improved our asymptotic normality by explicit asymptotic expressions of dispersion terms or by removing the bias term \( C_n(x) \).

Corollary 1. Under the hypotheses of Theorem 2 and if the bandwidth parameters \( (a_n, b_n) \) and if the function \( \phi_x(a_n) \) satisfies
\[
\lim_{n \to \infty} (a_n b_1 + b_n b_2) \sqrt{n} \phi_x(a_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\phi_x(t a_n)}{\phi_x(a_n)} = \beta(t), \quad \forall t \in [0, 1],
\]
we have
\[
\left( \frac{1}{\delta_2(\alpha(1 - \alpha))} \right)^{1/2} \sqrt{n} \phi_x(a_n) \left( \hat{\alpha}_n(x) - t_\alpha(x) \right) \to_{n \to +\infty} N(0, 1),
\]
where \( \delta_j = -\int_0^1 (K^j)'(s)\beta(s)ds \), for \( j = 1, 2 \).

Remark 2. If we assume that (5) is satisfied instead of (4) then it is simple to have the results of Theorems 1 and 2, the only thing that changes is condition (H5) which will be replaced by some assumption that depend on \( \beta \).

5. APPENDIX

We first state the following lemmas which are due to Carbon et al. [6]. They are needed for the convergence of our estimates. Their proofs will then be omitted.

Lemma 7. Suppose \( E_1, \ldots, E_r \) be sets containing \( m \) sites each with \( \text{dist}(E_i, E_j) \geq \gamma \) for all \( i \neq j \), where \( 1 \leq i \leq r \) and \( 1 \leq j \leq r \). Suppose \( Z_1, \ldots, Z_r \) is a sequence of real-valued r.v.’s measurable with respect to \( \mathcal{B}(E_1), \ldots, \mathcal{B}(E_r) \) respectively, and \( Z_i \) takes values in \( [a, b] \). Then, there exists a sequence of independent r.v.’s \( Z_{i_1}^*, \ldots, Z_{i_r}^* \) independent of \( Z_1, \ldots, Z_r \) such
that $Z_i^*$ has the same distribution as $Z_i$ and satisfies

$$\sum_{i=1}^{r} E|Z_i - Z_i^*| \leq 2r(b - a)\psi((r - 1)m,H(\varphi(\gamma)).$$

**Lemma 8.** (i) Suppose that (3) holds. Denote by $L_r(F)$ the class of $F$-measurable r.v.’s $X$ satisfying $\|X\|_r = (E|X|^r)^{1/r} < \infty$. Suppose $X \in L_r(B(E))$ and $Y \in L_s(B(E'))$. Assume also that $1 \leq r, s, t < \infty$ and $r^{-1} + s^{-1} + t^{-1} = 1$. Then,

$$|EXY - EXEY| \leq C\|X\|_r\|Y\|_s\{\psi(Card(E),Card(E'))\varphi(dist(E,E'))\}^{1/t}. (9)$$

(ii) For r.v.’s bounded with probability 1, the right-hand side of (9) can be replaced by $C\psi(Card(E),Card(E'))\varphi(dist(E,E'))$.

*Proof of Lemma 1.* We have

$$E[\alpha \hat{F}_D^x - \hat{F}_N^x(q_0(x))] = \alpha - \frac{1}{EK_1}(E[K_1E[H_1(q_0(x)) | X_1]]).$$

We shall use the integration by parts and the usual change of variables $t = \frac{y - z}{b_n}$, to show that

$$E[\alpha \hat{F}_D^x - \hat{F}_N^x(q_0(x))] = \alpha - \frac{1}{EK_1}\left(K_1\int K_2^{(1)}(t)F^{-X_1}((q_0(x)) - b_nt)dt\right).$$

Hypotheses (H2) and (H4) allow to get

$$E[\alpha \hat{F}_D^x - \hat{F}_N^x(q_0(x))] \leq \frac{1}{EK_1}K_1\int K_2^{(1)}(t)\left|F^x((q_0(x)) - F^{-X_1}((q_0(x)) - b_nt)dt\right| \leq C\left(a_n^{b_1} + b_2^{b_2}\right).$$

*Proof of Lemma 2.* We have

$$\left\|\alpha \hat{F}_D^x - \hat{F}_N^x(q_0(x)) - E\left[\alpha \hat{F}_D^x - \hat{F}_N^x(q_0(x))\right]\right\|_p = \frac{1}{nEK_1}\left\|\sum_{i \in I_n} \theta_i\right\|_p,$$

where

$$\theta_i = K_1(\alpha - H_1(q_0(x))) - E[K_1(\alpha - H_1(q_0(x))].$$

We have $EK_1 = O(\phi_x(h_K))$, (because of H3), so it remains to show that

$$\left\|\sum_{i \in I_n} \theta_i\right\|_p = O(\sqrt{n}\phi_x(a_n)).$$

The evaluation of this quantity is based on ideas similar to that used by Gao et al. (2008), see also Abdi et al. (2010). More precisely, we prove the case
where \( p = 2m \) (for all \( m \in N^* \)) and we use the Hölder inequality for lower values of \( p \).

First of all, let us notice that the notations \( \theta_1 \) and \( \xi_1 \), deliberately introduced above, are the same as those used in Lemma 2.2 of Gao et al. (2008) or Abdi et al. (2010a). The proof of the lemma is completely modeled on that of Lemma 2.2 of Gao et al. To make easier the understanding of the effect of the boundedness of \( \theta_1 \) on the results, we opt to run along the lines of Gao et al.’s proof (keeping the same notations) and give the moment results in a simpler form. To start, note that

\[
E \left[ \left( \sum_{i \in I_n} \theta_1 \right)^{2r} \right] = \sum_{i \in I_n} E [\theta_1^{2m}] + \sum_{s=1}^{2m-1} \sum_{\nu_0 + \nu_1 + \cdots + \nu_s = 2m} V_s(\nu_0, \nu_1, \ldots, \nu_s),
\]

where \( \sum_{\nu_0 + \nu_1 + \cdots + \nu_s = 2m} \) is the summation over \((\nu_0, \nu_1, \ldots, \nu_s)\) with positive integer components satisfying \( \nu_0 + \nu_1 + \cdots + \nu_s = 2m \) and

\[
V_s(\nu_0, \nu_1, \ldots, \nu_s) = \sum_{k_0 \neq i_1 \neq \ldots \neq i_s} E \left[ \theta_{i_0}^{\nu_0} \theta_{i_1}^{\nu_1} \cdots \theta_{i_s}^{\nu_s} \right],
\]

where the summation \( \sum_{k_0 \neq i_1 \neq \ldots \neq i_s} \) is over indexes \((k_0, i_1, \ldots, i_s)\) with each index \( i_j \) taking value in \( I_n \) and satisfying \( i_j \neq i_l \) for any \( j \neq l, \ 0 \leq j, l \leq s \). By stationarity and the fact that \( K_2 \) is a distribution function, we have

\[
\sum_{i \in I_n} E (\theta_1)^{2m} \leq C \hat{n} E (|\theta_1|)^{2m} \leq \hat{n} E (K_1)^{2m} \leq C \hat{n} \phi_x(a_n).
\]

To control the term \( V_s(\nu_0, \nu_1, \ldots, \nu_s) \), we need to prove, for any positive integers \( \nu_0, \nu_1, \nu_2, \ldots, \nu_s \), the following results:

i) \( E [\theta_{i_1}^{\nu_1} \theta_{i_2}^{\nu_2} \cdots \theta_{i_s}^{\nu_s}] \leq C \phi_x(a_n)^{1+\nu_s} \);

ii) \( V_s(\nu_0, \nu_1, \ldots, \nu_s) = O((\hat{n} \phi_x(a_n))^{s+1}) \), for \( s = 1, 2, \ldots, m-1 \) and \( m > 1 \);

iii) \( V_s(\nu_0, \nu_1, \ldots, \nu_s) = O((\hat{n} \phi_x(a_n))^m) \), for \( m \leq s \leq 2m-1 \).

To show the result i), remark that the boundness of \( K_2 \) and (2) yield

\[
E |\theta_{i_1}^{\nu_1} \theta_{i_2}^{\nu_2} \cdots \theta_{i_s}^{\nu_s}| \leq C \phi_x^{1+\nu_s}(a_n).
\]

Proof of ii). Note that we can write

\[
V_s(\nu_0, \nu_1, \ldots, \nu_s) = \sum_{k_0 \neq i_1 \neq \ldots \neq i_s} \left[ E \left( \prod_{j=0}^{s} \theta_{i_j}^{\nu_j} \right) - \prod_{j=0}^{s} E \theta_{i_j}^{\nu_j} \right] + \sum_{k_0 \neq i_1 \neq \ldots \neq i_s} \prod_{j=0}^{s} E \theta_{i_j}^{\nu_j} =: V_{s1} + V_{s2}.
\]

Clearly, we have \( |V_{s2}| \leq C \sum_{k_0 \neq i_1 \neq \ldots \neq i_s} (\phi_x(a_n))^{s+1} \leq C (\hat{n} \phi_x(a_n))^{s+1}. \)
For the term $V_{s,1}$, notice that

$$E\left( \prod_{j=0}^{s} \theta_{i_j}^{\nu_i} \right) - \prod_{j=0}^{s} E\theta_{i_j}^{\nu_i} =$$

$$= \sum_{l=0}^{s-1} \left( \prod_{j=0}^{l-1} E\theta_{i_j}^{\nu_i} \right) \left( E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] - E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] \right).$$

where we define $\prod_{j=0}^{s} \theta_{i_j}^{\nu_i} = 1$ if $l > s$. Then, we obtain

$$|V_{s,1}| \leq \sum_{l=0}^{s-1} \left( \tilde{\phi}_x(a_n) \right)^l \sum_{l \neq \ldots \neq s} \left| E\left[ \prod_{j=l}^{s} \xi_{i_j}^{\nu_i} \right] - E\left[ \prod_{j=l}^{s} \xi_{i_j}^{\nu_i} \right] \right| =$$

$$= \sum_{l=0}^{s-1} \left( \tilde{\phi}_x(a_n) \right)^l V_{ls1}.$$

Let $P$ be some positive real, we have

$$V_{ls1} = \sum_{0 < \text{dist}(\{i_l, \ldots, i_s\}) \leq P} \left| E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] - E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] \right| +$$

$$+ \sum_{0 < \text{dist}(\{i_l, \ldots, i_s\}) > P} \left| E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] - E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] \right| =: V_{ls11} + V_{ls12}.$$

Using the result i) above leads to

$$\left| E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] - E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] \right| \leq \left| E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] \right| + \left| E\left[ \prod_{j=l}^{s} \theta_{i_j}^{\nu_i} \right] \right| \leq$$

$$\leq C\phi_x(a_n)^{1+v_{s+1-l}}.$$

Thus, we have

$$V_{ls11} \leq \sum_{0 < \text{dist}(\{i_l, \ldots, i_s\}) \leq P} C\phi_x(a_n)^{1+v_{s+1-l}} \leq$$

$$\leq C\phi_x(a_n)^{1+v_{s+1-l}} \sum_{k=1}^{P} \sum_{1 \leq \text{dist}(\{i_k, \ldots, i_s\}) \leq t < k+1} 1.$$
Since \( \text{dist}(\{i_j\}, \{i_t, \ldots, i_s\}) = t \), it follows that there exists some \( i_j \in \{i_t, \ldots, i_s\} \), say \( i_{t+1} \), such that \( \text{dist}(\{i_j\}, \{i_{t+1}\}) = t \), and therefore,

\[
\sum_{k=1}^{P} \sum_{1 \leq \sum_{l=1}^{P} k \leq \text{dist}(\{i_j\}, \{i_{t+1}\}) = t < k+1} 1 \leq \hat{n}^{(s-1)} \sum_{k=1}^{P} \sum_{k \leq \text{dist}(\{i_j\}, \{i_{t+1}\}) = t < k+1} 1.
\]

Thus,

\[
V_{ls11} \leq C \phi_x(a_n)^{1+v_{s+1-t}} \hat{n}^{(s-l)} \sum_{k=1}^{P} \sum_{k \leq \|l\| = t < k+1} 1 \leq C \phi_x(a_n)^{1+v_{s+1-t}} \hat{n}^{(s-l)} p^N.
\]

For the term \( V_{ls12} \), notice that since the variables \( \theta_i \) are bounded, we have (see Lemma 8)

\[
|E \left[ \prod_{j=1}^{s} \theta_{i_{j+1}}^{s-j} \right] - E \left[ \prod_{j=1}^{s} \theta_{i_{j+1}}^{s-j} \right] | \leq C \psi(1, s-l) \phi(t),
\]

where \( t = \text{dist}(\{i_t\}, \{i_{t+1}, \ldots, i_s\}) \). Then, under (4) or (5), we have

\[
V_{ls12} \leq C \sum_{k=P+1}^{\infty} \sum_{k \leq \text{dist}(\{i_t\}, \{i_{t+1}\}) = t < k+1} \phi(t) \leq C \sum_{k=P+1}^{\infty} \hat{n}^{(s-l)} \sum_{k \leq \|l\| = t < k+1} \phi(t) \leq C \hat{n}^{(s-l)} \sum_{t=P+1}^{\infty} t^{N-1} \phi(t).
\]

Combining the upper bounds of \( V_{ls11} \) and \( V_{ls12} \), we have

\[
|V_{s1}| \leq C \left( \sum_{l=0}^{s-1} (\hat{n} \phi_x(a_n))^{l} \phi_x(a_n)^{1+v_{s+1-t}} \hat{n}^{(s-l)} p^N + \hat{n}^{(s-l)} \sum_{t=P+1}^{\infty} t^{N-1} \phi(t) \right) \leq C (\hat{n} \phi_x(a_n))^{(s+1)} \sum_{l=0}^{s-1} (\hat{n} \phi_x(a_n))^{l-s-1} \phi_x(a_n)^{1+v_{s+1-t}} \hat{n}^{(s-l)} p^N + \hat{n}^{(s-l)} \sum_{t=P+1}^{\infty} t^{N-1} \phi(t) = \]

\[ C(\hat{n}\phi_x(a_n))^{(s+1)} \sum_{l=0}^{s-1} \left[ \hat{n}^{-1} P_N \phi_x(a_n)^{1+s_{s-1-l}} \phi_x(a_n)^{l-s-1} \right. \\
+ \left. \hat{n}^{-1} \phi_x(a_n)^{l-s-1} \sum_{t=P+1}^{t=\infty} t^{N-1} \varphi(t) \right] \leq \]
\[ \leq C(\hat{n}\phi_x(a_n))^{(s+1)} \sum_{l=0}^{s-1} \left[ \hat{n}^{-1} P_N \phi_x(a_n)^{1+s_{s-1-l}} \phi_x(a_n)^{l-s-1} \right. \\
+ \left. \hat{n}^{-1} \phi_x(a_n)^{l-s-1} P^{sN} \sum_{t=P+1}^{t=\infty} t^{sN+N-1} \varphi(t) \right]. \]

Taking \( P = \phi_x(a_n)^{-1/N} \), we obtain
\[ |V_{s1}| \leq C(\hat{n}\phi_x(a_n))^{(s+1)} \sum_{l=0}^{s-1} \left[ \frac{1}{\hat{n}\phi_x(a_n)} \phi_x(a_n)^{1+s_{s-1-l}} \phi_x(a_n)^{l-s-1} \right. \\
+ \left. \frac{1}{\phi_x(a_n)} \phi_x(a_n)^{l-s-1} \sum_{t=P+1}^{t=\infty} t^{sN+N-1-\delta} \right] \leq C(\hat{n}\phi_x(a_n))^{(s+1)} \]

since \( \delta > N(p+2) \).

**Proof of iii.** As indicated in Gao et al. (2008), since the arguments are similar for any \( m \leq s \leq 2m - 1 \), the proof is given only for \( s = 2m - 1 \) and \( N = 2 \) for simplicity. In this case, \( V_{2m-1}(n_0, n_1, \ldots, n_{2m-1}) \) is denoted \( W \). To simplify the notations, we write \( i = (i, j) \in \mathbb{Z}^2 \) and \( i_k = (i_k, j_k) \in \mathbb{Z}^2 \). The main difficulty is to cope with the summation
\[ \sum_{i_0 \neq i_1 \neq \ldots \neq i_{2m-1}} E[\theta_{i_0} \theta_{i_1} \ldots \theta_{i_{2m-1}}] = \]
\[ = \sum_{(i_0, j_0) \neq (i_1, j_1) \neq \ldots \neq (i_{2m-1}, j_{2m-1})} E[\theta_{i_0} \theta_{i_1} \theta_{i_2} \ldots \theta_{i_{2m-1}}] \cdot \]

To this end, a novel ordering in \( \mathbb{Z}^2 \) (see Gao et al. (2008)), makes possible to separate the indexes into two (or more) sets whose distance is greater or smaller than \( P \) (usually larger than 1) is considered. Arrange each of the index sets \( \{i_0, i_1, \ldots, i_{2m-1}\} \) and \( \{j_0, j_1, \ldots, j_{2m-1}\} \) in ascending orders as (retaining the same notation for the first ordered index set for simplicity) \( i_0 \leq i_1 \leq \ldots \leq i_{2m-1} \) and \( j_0 \leq j_1 \leq \ldots \leq j_{2m-1} \), where \( l_k \) is to indicate that \( l_k \) may not be equal to \( k \). The number of such arrangements is at most \((2m)!\). Let \( \Delta_k = i_k - i_{k-1} \) and \( \Delta_j = j_k - j_{k-1} \) and arrange \( \{\Delta_i_1, \ldots, \Delta_i_{2m-1}\} \) and \( \{\Delta_j_1, \ldots, \Delta_j_{2m-1}\} \) in decreasing orders, respectively as \( \Delta_i_1 \geq \cdots \geq \Delta_i_{2m-1} \) and \( \Delta_j_1 \geq \cdots \geq \Delta_j_{2m-1} \). Let \( t_1 = \Delta_i_{a_1}, t_2 = \Delta_j_{b_1} \) and \( t = \max\{t_1, t_2\} \). If
\( t_1 \geq t_2 \) then \( t = t_1 \), and \( 0 \leq i_{a_k} - i_{a_{k-1}} \leq t_1 \leq n_1 \) for \( k = m + 1, \ldots, 2m - 1 \), \( 0 \leq j_{b_k} - j_{b_{k-1}} \leq t \leq n_2 \) for \( k = m, m + 1, \ldots, 2m - 1 \). Therefore,
\[
(10) \quad i_{a_{k-1}} \leq i_{a_k} \leq i_{a_{k-1}} + t, \quad j_{b_{k-1}} \leq j_{b_k} \leq j_{b_{k-1}} + t \quad \text{for} \quad k = m + 1, \ldots, 2m - 1.
\]

We arrange \( i_0 \neq i_1 \neq \cdots \neq i_{2m-1} \) according to the order of \( i_0 \leq i_1 \leq \cdots \leq i_{2m-1} \). If \( t_1 < t_2 \), arrange according to the order of \( j_0 \leq j_1 \leq \cdots \leq j_{2m-1} \) and the proof is similar.

Let \( \mathcal{I} = \{i_1, \ldots, i_{2m-1}\} \), \( \mathcal{I}_a = \{i_{a_1}, \ldots, i_{a_m}\} \), \( \mathcal{I}_c = \mathcal{I} - \mathcal{I}_a = \{i_{a_{m+1}}, \ldots, i_{a_{2m-1}}\} \), \( \mathcal{J} = \{j_1, \ldots, j_{2m-1}\} \), \( \mathcal{J}_b = \{j_{b_1}, \ldots, j_{b_m}\} \) and \( \mathcal{J}_c = \mathcal{J} - \mathcal{J}_b = \{j_{b_{m+1}}, \ldots, j_{b_{2m-1}}\} \). Remark that \((i_{a_1}, \ldots, i_{a_{2m-1}})\) and \((j_{b_1}, \ldots, j_{b_{2m-1}})\) are permutations of respectively \( \mathcal{I} \) and \( \mathcal{J} \). Then, from (10), \( t = i_{a_m} - i_{a_{m-1}} \) and \( t \geq j_{b_m} - j_{b_{m-1}} \), we deduce that
\[
W = \sum_{i_0 \neq i_1 \neq \cdots \neq i_{2m-1}} E \left[ \theta_{i_0} \theta_{i_1} \cdots \theta_{i_{2m-1}} \right]
\leq C \sum_{1 \leq i_0 \leq i_1 \leq \cdots \leq i_{2m-1} \leq n_1} \sum_{1 \leq j_0 \leq j_1 \leq \cdots \leq j_{2m-1} \leq n_2} \left| E \left[ \theta_{i_0} \theta_{i_1} \cdots \theta_{i_{2m-1}} \right] \right|
\leq C \max(n_1, n_2) \sum_{t=1}^{n_1} \sum_{i_0=1}^{n_1} \sum_{i_{a_k}=i_{a_{k-1}}}^{n_1} \sum_{j_0=1}^{n_2} \sum_{j_{b_k}=j_{b_{k-1}}}^{n_2} \sum_{j_{b_{k-1}}=j_{b_{k-1}}}^{n_2} \sum_{j_{b_{k-1}}=j_{b_{k-1}}}^{n_2} \sum_{j_{b_{k-1}}=j_{b_{k-1}}}^{n_2} \sum_{j_{b_{k-1}}=j_{b_{k-1}}}^{n_2}
\cdot \left| E \left[ \theta_{i_0} \theta_{i_1} \cdots \theta_{i_{2m-1}} \right] \right|.
\]

Take a positive constant \( P \) such that \( 1 \leq P \leq \max(n_1, n_2) \) and divide the right hand side of the previous inequality into two parts denoted by \( W_1 \) and \( W_2 \) according to \( 1 \leq t \leq P \) and \( t > P \). Then, \( W \leq W_1 + W_2 \). In one hand, use the result i) with \( s = 2m \), and get
\[
W_1 = C \sum_{t=1}^{P} \sum_{i_0=1}^{n_1} \sum_{i_{a_k}=i_{a_{k-1}}}^{n_1} \sum_{j_0=1}^{n_2} \sum_{j_{b_k}=j_{b_{k-1}}}^{n_2} \sum_{j_{b_{k-1}}=j_{b_{k-1}}}^{n_2} \sum_{j_{b_{k-1}}=j_{b_{k-1}}}^{n_2} \sum_{j_{b_{k-1}}=j_{b_{k-1}}}^{n_2} \phi_x(a_n)^{1+v_2m}
\leq C(n_1 n_2)^m \sum_{t=1}^{P} t^{2m-1} \phi_x(a_n)^{1+v_2m} \leq C(n_1 n_2)^m P^{2m} \phi_x(a_n)^{1+v_2m}.
\]
In another hand, assume that neither $i_1$ nor $i_{2m-1}$ belongs to $\mathcal{I}_a$ (if $i_1$ or $i_{2m-1}$ is in $\mathcal{I}_a$ the proof is similar). In this case, $\mathcal{I}_a$ is a subset of size $m$ chosen from the $2m - 3$ remaining indexes (besides $i_0$, $i_1$ and $i_{2m-1}$). As raised by Gao et al., this is due to the fact that there must be two successive indexes because there are not enough elements in the set of remaining indices to allow a gap between every two elements of $\mathcal{I}_a$. The first components of $i_j$’s are ordered as $i_0 \leq i_1 \leq \cdots \leq i_{k^*} \leq i_{k^*+1} \leq \cdots \leq i_{2m-1}$ for some $k^* \geq 1$ and $\Delta i_j = i_j - i_{j-1}$. Then, we have, either $\text{dist}(\{i_0, \ldots, i_{k^*-1}, i_{k^*}\}) \geq \Delta i_{k^*} \geq t$, $\text{dist}(\{i_{k^*}, \ldots, i_{2m-1}\}) \geq \Delta i_{k^*+1} \geq t$ and $\text{dist}(\{i_0, \ldots, i_{k^*-1}, i_{k^*}, \ldots, i_{2m-1}\}) \geq \Delta i_{k^*+1} \geq t$, or $\text{dist}(\{i_0, \ldots, i_{k^*-1}\}) \geq \Delta i_1 \geq t$ or $\text{dist}(\{i_0, \ldots, i_{2m-2}\}) \geq \Delta i_{2m-1} \geq t$. Let $A_{i_{k^*}} = \theta_{k^*} \theta_{1} \ldots \theta_{i_{k^*}-1}$ and $B_{i_{k^*}+1} = \theta_{i_{k^*}+1} \ldots \theta_{i_{2m-1}}$, then, for the case of $i_{k^*}$ and $i_{k^*+1}$ in $\mathcal{I}_a$, we have

$$|E[\theta_{i_1} \ldots \theta_{i_{2m-1}}]| = |E[A_{i_{k^*}+1} \theta_{k^*} B_{i_{k^*}+1}]|$$

$$\leq |E[(A_{i_{k^*}+1} - EA_{i_{k^*}+1})(\theta_{k^*} B_{i_{k^*}+1} - E\theta_{k^*} B_{i_{k^*}+1})]|$$

$$+ |E[A_{i_{k^*}+1} E[(\theta_{k^*} B_{i_{k^*}+1})]|$$

$$= |\text{Cov}(A_{i_{k^*}+1}, \theta_{k^*} B_{i_{k^*}+1})| + |E[A_{i_{k^*}+1}]| |\text{Cov}(\theta_{k^*}, B_{i_{k^*}+1})|$$

$$\leq C \varphi(t) + C \phi(x(a_n)^{1+v_n} \varphi(t) \leq C \varphi(t).$$

Thus,

$$W_2 = C \sum_{t = P+1}^{\max(n_1, n_2)} \sum_{i=1}^{n_1} \sum_{i_k = i_{k-1}^i + t}^{i_{k-1}^i + t} \sum_{j=1}^{n_2} \sum_{j_{k-1}^j = j_{k-1}^j}^{j_{k-1}^j + m} \sum_{i_k = i_{k-1}^i}^{i_{k-1}^i + t} |E[\theta_{i_1} \ldots \theta_{i_{2m-1}}]| \leq C(n_1 n_2)^m \sum_{t=1}^{P+1} t^{2m-1} \varphi(t).$$

It follows that

$$W \leq W_1 + W_2 \leq C(n_1 n_2)^m P^{2m} \phi_\nu(x(a_n)^{1+v_2}) + C(n_1 n_2)^m \sum_{t = P+1}^{\infty} t^{2m-1} \varphi(t)$$

$$\leq (n_1 n_2)^m \left( P^{2m} \phi_\nu(x(a_n)^{1+v_2}) + P^{2m-1-\delta} \right).$$

For general $N$, we obtain by similar arguments

$$W \leq C(n)^m \left( P^{Nm} \phi_\nu(x(a_n)^{1+v_{Nm}}) + P^{Nm-1-\delta} \right).$$
Taking $P = \phi_x(a_n)^{-(1+\nu_N)m}/(1+\delta)$, we get

$$W \leq C(\hat{n}\phi_x(a_n))^m \left( \hat{n}^{-1}\phi_x(a_n)^{-N_m-1+(1+\nu_N)} + (\hat{n}\phi_x(a_n))^{-1} \sum_{t=P+1}^{\infty} t^{N_m-1-\delta} \right)$$

$$\leq C(\hat{n}\phi_x(a_n))^m$$

because $\delta > N(p+2)$. This ends the proof of the lemma.

**Proof of Lemma 3.** We have for all $\epsilon < 1$,

$$P \left( \hat{F}_D = 0 \right) \leq P \left( \hat{F}_D \leq 1 - \epsilon \right) \leq P \left( |\hat{F}_D - E[\hat{F}_D]| \geq \epsilon \right).$$

Markov's inequality allows to get, for any $p > 0$,

$$P \left( |\hat{F}_D - E[\hat{F}_D]| \geq \epsilon \right) \leq \frac{E \left[ |\hat{F}_D - E[\hat{F}_D]|^p \right]}{\epsilon^p}.$$  

So,

$$\left( P(\hat{F}_D = 0) \right)^{1/p} = O \left( \|\hat{F}_D - E[\hat{F}_D]\|_p \right).$$

The computation of $\|\hat{F}_D - E[\hat{F}_D]\|_p$ can be done by following the same arguments as those used to prove Lemma 2. This yields the proof.

**Proof of Lemma 4.** Let us calculate the variance $Var(\Delta_i)$. We have

$$Var(\Delta_i) = \frac{1}{EK_i^2} \left[ E K_i^2 (\alpha - H_1(q_0))^2 - (EK_i (\alpha - H_1(q_0)))^2 \right]$$

$$= \frac{1}{EK_i^2} E K_i^2 (\alpha - H_1(q_0))^2 - (EK_i)^2 \left[ E \frac{K_i (H_1(q_0) - \alpha)}{EK_i} \right]^2 = A_1 - A_2.$$

Let us first consider $A_2$. We deduce from the hypothesis $H_3$ that there exist two positive constants $C$ and $C'$ such that $C\phi_x(a_n) \leq E K_i^2 \leq C'\phi_x(a_n)$, $r > 1$, thus, $(EK_i)^2 = o(1)$. If we take the conditional expectation with respect to $X$, we get

$$\left| E \left[ \frac{K_i (H_1(q_0) - \alpha)}{EK_i} \right] \right| = \left| E \frac{K_i}{EK_i} \left[ E (H_1(q_0)|X) - \alpha \right] \right| \leq E \frac{K_i}{EK_i} \left| E (H_1(q_0)|X) - \alpha \right|.$$

It is easy to see that by hypothesis $H'_2$ (ii)

$$|E (H_1(q_0)|X) - \alpha| = |E (H_1(q_0)|X) - F^x(q_0)|$$

$$\leq C \left( a_n^{b_1} + b_n \int |t^{b_2}| K_2^{(1)}(t) dt \right),$$

$$\left| E \frac{K_i (H_1(q_0) - \alpha)}{EK_i} \right| = O \left( a_n^{b_1} + b_n \right).$$
Then, we deduce that $A_2$ tends to 0. Concerning $A_1$, we have

$$(\alpha - H_1(q_\alpha))^2 = (H_1^2(q_\alpha) - \alpha) - 2\alpha (H_1(q_\alpha) - \alpha) + \alpha - \alpha^2.$$ 

Then, we can write

$$A_1 = \frac{1}{EK_i} \left[ E K_i \left( H_1^2(q_\alpha) - \alpha \right) - 2\alpha E K_i \left( H_1(q_\alpha) - \alpha \right) \right] + \alpha (1 - \alpha).$$

The conditional expectation with respect to $X_i$, permits to obtain

$$A_1 = E K_i \left( E \left( H_1^2(q_\alpha) | X_1 \right) - \alpha \right) - 2\alpha E K_i \left( E \left( H_1(q_\alpha) | X_1 \right) - \alpha \right) + \alpha (1 - \alpha).$$

The same argument as above, gives

$$\left| E K_i \left( E \left( H_1^2(q_\alpha) | X_1 \right) - \alpha \right) \right| = O \left( a_n^{b_1} + b_n^{b_2} \right).$$

It remains to show that

$$\left| E \left( H_1^2(q_\alpha) | X_1 \right) - \alpha \right| = O \left( a_n^{b_1} + b_n^{b_2} \right).$$

By an integration by part and hypotheses $H_2$ and $H_4$, we have

$$\left| E \left( H_1^2(q_\alpha) | X_1 \right) - \alpha \right| = \int K_i^2 \left( \frac{q_\alpha - z}{b_n} \right) f(X_1(z)) dz - F_X(q_\alpha)$$

$$= \int K_i^2(t) K_i^{(1)}(t) \left( F_X(q_\alpha - b_n t) - F_X(q_\alpha) \right) dt$$

$$\leq a_n^{b_1} \int K_i^2(t) K_i^{(1)}(t) dt + b_n^{b_2} \int K_i^2(t) |t|^{b_2} K_i^{(1)}(t) dt$$

$$\leq C a_n^{b_1} + b_n^{b_2} \int |t|^{b_2} K_i^{(1)}(t) dt = O \left( a_n^{b_1} + b_n^{b_2} \right).$$

We deduce from above that $A_1$ converges to $\alpha (1 - \alpha)$; then,

$$\text{Var} (\Delta_1) \to \alpha (1 - \alpha).$$

Let us focus now on the covariance term. We consider

$$E_1 = \{i, j \in \mathcal{I}_n : 0 < |i - j| \leq c_n\},$$

$$E_2 = \{i, j \in \mathcal{I}_n : |i - j| > c_n\}.$$
We have

\[
\text{Cov}(\Delta_i, \Delta_j) = E\Delta_i \Delta_j =
\]

\[
= \frac{1}{EK_1^2} \left[ EK_1K_j (\alpha - H_i(q_\alpha)) (\alpha - H_j(q_\alpha)) - (EK_1 (\alpha - H_i(q_\alpha)))^2 \right] \leq
\]

\[
\leq \frac{1}{EK_1^2} |EK_1K_j (\alpha - H_i(q_\alpha)) (\alpha - H_j(q_\alpha))| + \frac{1}{EK_1^2} |EK_1 (\alpha - H_i(q_\alpha)))^2 .
\]

The conditional expectation with respect to \(X_i\), gives

\[
\frac{1}{EK_1^2} |EK_1 (\alpha - H_i(q_\alpha))|^2 =
\]

\[
= \frac{1}{EK_1^2} |EK_1 (\alpha - E(H_i(q_\alpha)|X_i))|^2 \leq \frac{1}{EK_1^2} |EK_1 |E (H_i(q_\alpha)|X_i) - \alpha||^2 .
\]

Recall that

\[
|E (H_i(q_\alpha)|X) - \alpha| = O \left(b_{n1}^n + b_{n2}^2\right) ; \text{ then, } |E (H_i(q_\alpha)|X) - \alpha| \leq C .
\]

So,

\[
\frac{1}{EK_1^2} |EK_1 (\alpha - H_i(q_\alpha))|^2 \leq C \phi_x(a_n) .
\]

Since \(K_1\) is bounded, we get

\[
\frac{1}{EK_1^2} |EK_1K_j (\alpha - H_i(q_\alpha)) (\alpha - H_j(q_\alpha))| \leq C \frac{1}{EK_1^2} |EK_1K_j| \leq
\]

\[
\leq C \frac{1}{EK_1^2} P [(X_i, X_j) \in B(x, a_n) \times B(x, a_n)] .
\]

Then, we deduce from (2)

\[
\frac{1}{EK_1^2} |EK_1K_j (\alpha - H_i(q_\alpha)) (\alpha - H_j(q_\alpha))| \leq
\]

\[
\leq C \frac{1}{EK_1^2} (\phi_x(a_n))^{1+v_2} \leq (\phi_x(a_n))^{v_2} .
\]

Then, we have since \(v_2 > 1\): \(Cov(\Delta_i, \Delta_j) \leq C(\phi_x(a_n) + (\phi_x(a_n)))^{v_2} \leq C(\phi_x(a_n))\) and \(\sum_{E_i} Cov(\Delta_i, \Delta_j) \leq C\hat{\mathcal{N}}_n^{\phi_x(a_n)}\).

Lemma 8 and \(|\Delta_i| \leq C\phi_x(a_n)^{-1/2}\), permit to write that

\[
|Cov(\Delta_i, \Delta_j)| \leq C\phi_x(a_n)^{-1} \varphi (||i - j||)
\]
and 
\[
\sum_{E_2} \text{Cov}(\Delta_i, \Delta_j) \leq C\phi_x(a_n)^{-1} \sum_{(i,j) \in E_2} \varphi(||i-j||) \leq C\hat{n}\phi_x(a_n)^{-1} \sum_{i : ||i|| > c_n} \varphi(||i||) \\
\leq C\hat{n}\phi_x(a_n)^{-1}c_n^{-\delta} \sum_{i : ||i|| > c_n} ||i||^{\delta} \varphi(||i||).
\]

Finally, for \(\delta > 0\) we have 
\[
\sum_{E_2} \text{Cov}(\Delta_i, \Delta_j) \leq \left( C\hat{n}c_n^{N} + C\hat{n}\phi_x(a_n)^{-1} \sum_{i : ||i|| > c_n} ||i||^{\delta} \varphi(||i||) \right).
\]

Let \(c_n = \phi_x(a_n)^{-1/N}\), then, we have 
\[
\sum_{E_2} \text{Cov}(\Delta_i, \Delta_j) \leq \left( C\hat{n} + C\hat{n}\phi_x(a_n)^{\delta/N-1} \sum_{i : ||i|| > c_n} ||i||^{\delta} \varphi(||i||) \right).
\]

Hence, we obtain that 
\[
\sum_{E_2} \text{Cov}(\Delta_i, \Delta_j) = o(\hat{n}).
\]

In conclusion, we have 
\[
\frac{1}{n} \text{var} \left( \sum_{i \in \mathcal{I}_n} \Delta_i \right) = \left( \text{var}(\Delta_i) + \frac{1}{n} \sum_{i,j \in \mathcal{I}_n} \text{Cov}(\Delta_i, \Delta_j) \right) \to \alpha(1-\alpha) \text{ when } n \to \infty.
\]

This yields the proof.

**Proof of Lemma 5.** Let 
\[
S_n = \sum_{k=1}^{n_k} \Delta_j
\]

with 
\[
\Delta_j = \frac{1}{\sqrt{E K_1}} [\alpha K_1 - K_1 H_1(q_\alpha) - E (\alpha K_1 - K_1 H_1(q_\alpha))].
\]

Then, we can write 
\[
\left[ \frac{\hat{n}E^2K_1}{\alpha(1-\alpha)EK_1^2} \right]^{1/2} \left[ \alpha \hat{F}_\alpha^x - \hat{F}_\alpha^x(q_\alpha) - E \left[ \alpha \hat{F}_\alpha^x - \hat{F}_\alpha^x(q_\alpha) \right] \right] = (\hat{n}\alpha(1-\alpha))^{-1/2} S_n.
\]

Consider the same spatial block decomposition (due to Tran (1990)) as Lemma 2, with \(q_n = o\left(\hat{n}\phi_x(a_n)^{(1+2N)}\right)^{1/(2N)}\), \(m_n = \left(\hat{n}\phi_x(a_n)^{(1/(2N)}}/s_n\right)\).
where \( s_n = o \left( \left[ \hat{n} \phi_x (a_n)^{1+2N} q_n^{-1} \right]^{1/(2N)} \right) \). Then, we have

\[ S_n = \sum_{i=1}^{2N} T(n, x, i), \]

where

\[ T(n, x, i) = \sum_{j \in J} U(i, n, x, j). \]

Hence,

\[ S_n / (\hat{n} \alpha (1 - \alpha))^{1/2} = T(n, x, 1) / (\hat{n} \alpha (1 - \alpha))^{1/2} + \sum_{i=2}^{2N} T(n, x, i) / (\hat{n} \alpha (1 - \alpha))^{1/2}. \]

Thus, the proof of the asymptotic normality of \((\hat{n} \alpha (1 - \alpha))^{-1/2} S_n\) is reduced to the proofs of the following results

(11) \[ Q_1 \equiv \left| E \exp \left[ iu T(n, x, 1) \right] - \prod_{k=1}^{r_k-1} E \exp \left[ iu U(1, n, x, j) \right] \right| \to 0 \]

(12) \[ Q_2 \equiv \hat{n}^{-1} E \left( \sum_{i=2}^{2N} T(n, x, i) \right)^2 \to 0 \]

(13) \[ Q_3 \equiv \hat{n}^{-1} \sum_{j \in J} E \left[ U(1, n, x, j) \right]^2 \to \alpha (1 - \alpha) \]

(14) \[ Q_4 \equiv \hat{n}^{-1} \sum_{j \in J} E \left[ (U(1, n, x, j))^2 1_{\{U(1, n, x, j) > \epsilon (\alpha (1 - \alpha) \hat{n})^{1/2}\}} \right] \to 0, \text{ for all } \epsilon > 0. \]

**Proof of (11).** Let us numerate the \( M = \prod_{k=1}^{N} r_k = \hat{n} (m_n + q_n)^{-N} \leq \hat{n} m_n^{-N} \) random variables \( U(1, n, x, j); j \in J \) in the arbitrary way \( \tilde{U}_1, \ldots, \tilde{U}_M \). For \( j \in J \), let

\[ I(1, n, x, j) = \{ i : j_k (m_n + q_n) + 1 \leq i_k \leq j_k (m_n + q_n) + m_n ; k = 1, \ldots, N \} \]

then, we have \( U(1, n, x, j) = \sum_{i \in I(1, n, x, j)} \Delta_i \). Note that each of the sets of site \( I(1, n, x, j) \) contains \( m_n^N \), these sets are distant of \( m_n \) at least.
Let us apply the lemma of Volkonski and Rozanov (1959) to the variable $(\exp(iu\tilde{U}_1), \ldots, \exp(iu\tilde{U}_M))$. The fact that $\left| \prod_{s=j+1}^{M} \exp[iu\tilde{U}_s] \right| \leq 1$, implies

$$Q_1 = \left| E \exp \left[ iuT(n, x, 1) \right] - \prod_{j=0}^{r_k-1} E \exp \left[ iuU(1, n, x, j) \right] \right|$$

$$\leq \sum_{k=1}^{M-1} \sum_{j=k+1}^{M} \left| E(\exp[iu\tilde{U}_k] - 1)(\exp[iu\tilde{U}_j] - 1) \prod_{s=j+1}^{M} \exp[iu\tilde{U}_s] \right|$$

$$\leq \sum_{k=1}^{M-1} \sum_{j=k+1}^{M} \left| E(\exp[iu\tilde{U}_k] - 1)(\exp[iu\tilde{U}_j] - 1) \right| \prod_{s=j+1}^{M} \exp[iu\tilde{U}_s]$$

Let $\tilde{I}_j$ be the set of sites among the $I(1, n, x, j)$ such that $\tilde{U}_j = \sum_{i \in \tilde{I}_j} \Delta_i$. The lemma of Carbon et al. (1997) and assumption (3), give

$$\left| E(\exp[iu\tilde{U}_k] - 1)(\exp[iu\tilde{U}_j] - 1) - E(\exp[iu\tilde{U}_k] - 1)E(\exp[iu\tilde{U}_j] - 1) \right| \leq C\varphi(d(\tilde{I}_j, \tilde{I}_k))m_n^N.$$
Then,
\[
Q_1 \leq Cm_n^N \sum_{k=1}^{M-1} \sum_{j=k+1}^{M} \varphi \left( d(\tilde{I}_j, \tilde{I}_k) \right) \leq Cm_n^N M \sum_{k=2}^{M} \varphi \left( d(\tilde{I}_1, \tilde{I}_k) \right)
\]
\[
\leq Cm_n^N M \sum_{i=1}^{\infty} \sum_{k: iq_n \leq d(\tilde{I}_1, \tilde{I}_k) < (i+1)q_n} \varphi \left( d(\tilde{I}_1, \tilde{I}_k) \right)
\]
\[
\leq Cm_n^N M \sum_{i=1}^{\infty} i^{N-1} \varphi(iq_n) \leq C \hat{n}^{-N\delta} \sum_{i=1}^{\infty} i^{N-1-N\delta},
\]
by (6). This last tends to zero by the fact that $\hat{n}^{-N\delta} \to 0$ (see (H5)).

Proof of (12). We have
\[
Q_2 = \hat{n}^{-1} E \left( \sum_{i=2}^{2^N} T(n, x, i) \right)^2 =
\]
\[
= \hat{n}^{-1} \left( \sum_{i=2}^{2^N} E[T(n, x, i)]^2 + \sum_{i,j=2,...,2^N, i \neq j} E[T(n, x, i)] E[T(n, x, j)] \right).
\]

By Cauchy-Schwartz inequality, we get $\forall 2 \leq i \leq 2^N$:
\[
\hat{n}^{-1} E[T(n, x, i)] E[T(n, x, j)] \leq \left( \hat{n}^{-1} E[T(n, x, i)]^2 \right)^{1/2} \left( \hat{n}^{-1} E[T(n, x, j)]^2 \right)^{1/2}.
\]

Then, it suffices to prove that
\[
\hat{n}^{-1} E[T(n, x, i)]^2 \to 0, \quad \forall 2 \leq i \leq 2^N.
\]
We will prove this for $i = 2$, the case where $i \neq 2$ is similar. We have $T(n, x, 2) = \sum_{j \in J} U(2, n, x, j) = \sum_{j=1}^M \hat{U}_j$, where we enumerate the $U(2, n, x, j)$ in the arbitrary way $\hat{U}_1, \ldots, \hat{U}_M$. Then,
\[
E[T(n, x, 2)]^2 = \sum_{i=1}^M \text{Var} (\hat{U}_i) + \sum_{i=1}^M \sum_{j=1}^M \text{Cov}(\hat{U}_i, \hat{U}_j) = A_1 + A_2.
\]
The stationarity of the process \((X_i, Y_i)_{i \in \mathbb{Z}^N}\), implies that

\[
\text{Var}(\hat{U}_i) = \text{Var}\left( \sum_{i_k=1}^{m_n} \sum_{i_N=1}^{q_n} \Delta_i \right)^2
= m_n^{N-1} q_n \text{Var}(\Delta_i) + \sum_{k=1}^{m_n} \sum_{i_N=1}^{q_n} \sum_{j=1}^{m_n} \sum_{k \neq j}^{j_N=1} E \Delta_i \Delta_j.
\]

We proved above that \(\text{Var}(\Delta_i) < C\). By Lemma 8, we have

\[
|E \Delta_i(x) \Delta_j(x)| \leq C \phi_x(a_n)^{-1} \varphi(||i-j||).
\]

Then, we deduce that

\[
\text{Var}(\hat{U}_i) \leq C m_n^{N-1} q_n \left( 1 + \phi_x(a_n)^{-1} \sum_{i_k=1}^{m_n} \sum_{i_N=1}^{q_n} (\varphi(||i||)) \right)
\leq C m_n^{N-1} q_n \phi_x(a_n)^{-1} \sum_{i_k=1}^{m_n} \sum_{i_N=1}^{q_n} (\varphi(||i||)).
\]

Consequently, we have

\[
A_1 \leq C M m_n^{N-1} q_n \phi_x(a_n)^{-1} \sum_{i=1}^{\infty} i^{N-1} (\varphi(i)).
\]

Let

\[
I(2, n, x, j) = \{ i : j_k(m_n+q_n) + 1 \leq i_k \leq j_k(m_n+q_n) + m_n, 1 \leq k \leq N-1; \\
+ j_N(m_n+q_n) + m_n + 1 \leq i_N \leq (J_N+1)(m_n+q_n) \}.
\]

The variable \(U(2, n, x, j)\) is the sum of the \(\Delta_i\) such that \(i\) is in \(I(2, n, x, j)\). Since \(m_n > q_n\), if \(i\) and \(i'\) are respectively in the two different sets \(I(2, n, x, j)\) and \(I(2, n, x, j')\); then, \(i_k \neq i'_k\), for a certain \(k\) such that \(1 \leq k \leq N\) and \(||i-i'|| > q_n\).

By using the definition of \(A_2\), the stationarity of the process and (15), we have

\[
A_2 \leq \sum_{k=1}^{n_k} \sum_{i_k=1}^{n_k} \sum_{k=1}^{m_n} \sum_{i_N=1}^{q_n} E \Delta_i \Delta_j \leq C \phi_x(a_n)^{-1} \hat{n} \sum_{k=1}^{n_k} (\varphi(||i||))
\]
Lemma 4(iii) and (12) imply, respectively, that $\hat{q} \leq q$ by the hypothesis on $\alpha$. We deduce that

$$A_2 \leq C\phi_x(a_n)^{-1}\hat{m} \sum_{i=q_n}^\infty i^{N-1} (\varphi(i)).$$

We deduce that

$$\hat{n}^{-1} E[T(n, x, 2)]^2 \leq CM m_n^{-1} q_n \hat{n}^{-1} \phi_x(a_n)^{-1} \sum_{i=1}^{\infty} i^{N-1-\delta} + C\phi_x(a_n)^{-1} \sum_{i=q_n}^{\infty} i^{N-1-\delta}.$$

From $(m_n + q_n)^{-N} m_n^{-1} q_n = (m_n + q_n)^{-N} m_n^N \left( \frac{q_n}{m_n} \right) \leq \frac{q_n}{m_n}$, we get

$$CM m_n^{-1} q_n \hat{n}^{-1} \phi_x(a_n)^{-1} = \hat{n}(m_n + q_n)^{-N} m_n^{-1} q_n \hat{n}^{-1} \phi_x(a_n)^{-1} \leq \left( \frac{q_n}{m_n} \right) \phi_x(a_n)^{-1} = q_n s_n (\hat{n} \phi_x(a_n)) \frac{1}{\sqrt{N}} \phi_x(a_n)^{-1} = q_n s_n \left( \hat{n} \phi_x(a_n)^{(1+2N)} \right)^{\frac{1}{2}}.$$

By the hypothesis on $q_n s_n$, this last term converges to 0. Finally, we have

$$C\phi_x(a_n)^{-1} \sum_{i=q_n}^{\infty} i^{N-1-\delta} \leq C\phi_x(a_n)^{-1} \int_{q_n}^{\infty} i^{N-1-\delta} dt = C\phi_x(a_n)^{-1} q_n^{-\delta}.$$

This last term converges to zero by (8) and ends the proof of (12).

**Proof of (13).** Let us use the following decomposition of small and big blocks

$$S'_n = T(n, x, 1), \quad S''_n = \sum_{i=2}^{2N} T(n, x, i).$$

Then, we can write

$$\hat{n}^{-1} E(S'_n)^2 = \hat{n}^{-1} ES'_n^2 + \hat{n}^{-1} E(S''_n)^2 - 2\hat{n}^{-1} ES_n S''_n.$$

Lemma 4(iii) and (12) imply, respectively, that $\hat{n}^{-1} E(S_n)^2 = \hat{n}^{-1} \text{var}(S_n) \to \alpha(1 - \alpha)$ and $\hat{n}^{-1} E(S''_n)^2 \to 0$. Then, to show that $\hat{n}^{-1} E(S'_n)^2 \to \alpha(1 - \alpha)$, it suffices to remark that $\hat{n}^{-1} ES_n S''_n \to 0$ because, by Cauchy-Schwartz’s inequality, we can write

$$|\hat{n}^{-1} ES_n S''_n| \leq \hat{n}^{-1} E|S_n S''_n| \leq (\hat{n}^{-1} ES_n^2)^{1/2} (\hat{n}^{-1} ES_n^2)^{1/2}. $$
Recall that $T(n, x, 1) = \sum_{j \in J} U(1, n, x, j)$, so

$$\hat{n}^{-1} E(S'_n)^2 = \hat{n}^{-1} \sum_{j_k=0}^{r_k-1} E[U(1, n, x, j)]^2 + \hat{n}^{-1} \sum_{j_k=0}^{r_k-1} \sum_{k=1, \ldots, N, i_k=0, i_k \neq j_k \text{ for some } k}^{r_k-1} \text{cov}[U(1, n, x, j), U(1, n, x, i)].$$

By similar arguments used above for $A_2$, this last term is not greater than

$$C\phi_x(a_n)^{-1} \sum_{i_k=1}^{r_k-1} (\varphi(||i||)) \leq C\phi_x(a_n)^{-1} \sum_{i=\alpha n}^{\infty} i^{N-1}(\varphi(i)) \leq C\phi_x(a_n)^{-1} q_n^{N-\delta} \to 0.$$

So, $Q_3 \to \alpha(1 - \alpha)$. This ends the proof.

Proof of (14). Since $|\Delta_i| \leq C\phi_x(a_n)^{-1/2}$, we have $|U(1, n, x, j)| \leq Cm_n^{1/2} \phi_x(a_n)^{-1/2}$. Then, we deduce that

$$Q_4 \leq Cm_n^{2N} \phi_x(a_n)^{-1} \hat{n}^{-1} \sum_{k=1, \ldots, N}^{r_k-1} P\left[U(1, n, x, j) > \epsilon \alpha(1 - \alpha)\hat{n}^{1/2}\right].$$

We have $|U(1, n, x, j)|/(\alpha(1 - \alpha)\hat{n})^{1/2} \leq Cm_n^{1/2} \hat{n}^{1/2} = C(s_n)^{-N} \to 0$, because $m_n = (\hat{n}\phi_x(a_n))^{1/(2N)} / s_n$ and $s_n \to \infty$. So, for all $\epsilon$ and $j \in J$; if $\hat{n}$ is great enough, then $P[U(1, n, x, j) > \epsilon \alpha(1 - \alpha)\hat{n}^{1/2}] = 0$. Then, $Q_4 = 0$ for $\hat{n}$ great enough. This yields the proof.

Proof of Lemma 6. By change of variables, using the stationarity of the process, we have

$$E\left[\alpha \hat{F}^x_D - \hat{F}^x_N(q_\alpha)\right] = \alpha - \frac{1}{EK_1} E[K_1 H_1(q_\alpha)] = \alpha - \frac{1}{EK_1} E[K_1 H_1(q_\alpha) | X_i]$$

$$= \alpha - \frac{1}{EK_1} E\left[K_1 \int_{\mathbb{R}} K_2\left(\frac{q_\alpha - y}{b_n}\right) F_X(y)dy\right]$$

$$= \alpha - \frac{1}{EK_1} E\left[K_1 \int_{\mathbb{R}} b_n^{(-1)} K_2^{(1)}\left(\frac{q_\alpha - y}{b_n}\right) F_X(y)dy\right]$$

$$= \alpha - \frac{1}{EK_1} E\left[K_1 \int_{\mathbb{R}} K_2^{(1)}(t) F_X(q_\alpha - b_n t) dt\right]$$

$$= \alpha + \beta_1 + \beta_2,$$
where

$$\beta_1 = -\frac{1}{EK_1} E \left( K_1 \int_{\mathbb{R}} K_2^{(1)}(t) F^{X_i}(q_\alpha) dt \right) = -\frac{1}{EK_1} E \left( K_1 \left( \frac{d(x, X)}{a_n} \right) F^X(q_\alpha) \right)$$

and

$$\beta_2 = \frac{1}{EK_1} E \left( K_1 \int_{\mathbb{R}} K_2^{(1)}(t) \left[ F^{X_i}(q_\alpha) - F^{X_i}(q_\alpha - b_n t) \right] dt \right)$$

$$\leq \frac{1}{EK_1} E \left( K_1 \int_{\mathbb{R}} K_2^{(1)}(t) \left| F^{X_i}(q_\alpha) - F^{X_i}(q_\alpha - b_n t) \right| dt \right)$$

$$\leq C \frac{1}{EK_1} E \left( K_1 (b_n)^{b_2} \int_{\mathbb{R}} \left| t \right|^{b_2} K_2^{(1)}(t) \right) dt \leq C (b_n)^{b_2}.$$

This yields the proof of the first result of the lemma. The following result ends the proof of the second result

$$\beta_1 = \frac{1}{EK_1} E \left( K_1 \left( \frac{d(x, X)}{a_n} \right) \left[ F^x(q_\alpha) - F^X(q_\alpha) \right] \right) -$$

$$- \frac{1}{EK_1} E \left( K_1 \left( \frac{d(x, X)}{a_n} \right) F^x(q_\alpha) \right)$$

$$\leq \frac{1}{EK_1} E \left( K_1 \left( \frac{d(x, X)}{a_n} \right) \left| F^x(q_\alpha) - F^X(q_\alpha) \right| \right) - \alpha$$

$$\leq C (a_n)^{b_1} \frac{1}{EK_1} E \left( K_1 \left( \frac{d(x, X)}{a_n} \right) \right) - \alpha \leq C (a_n)^{b_1} - \alpha.$$

REFERENCES


Received 14 April 2011

Université Lille 3
Labo. EQUIPPE, Maison de la Recherche
BP 60149
59653 Villeneuve d’Ascq Cedex, France
sophie.dabo@univ-lille3.fr
zoulikha.kaid@etu.univ-lille3.fr

Université Djillali Liabès
Laboratoire de Statistique et Processus Stochastiques
BP 89, 22000 Sidi Bel Abbès, Algeria
alilak@yahoo.fr