SPATIAL CONDITIONAL QUANTILE REGRESSION: WEAK CONSISTENCY OF A KERNEL ESTIMATE

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We consider a conditional quantile regression model for spatial data. More precisely, given a strictly stationary random field $Z_{\mathbf{i}} = (X_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^N}$, we investigate a kernel estimate of the conditional quantile regression function of the univariate response variable $Y_{\mathbf{i}}$ given the functional variable $X_{\mathbf{i}}$. The main purpose of the paper is to prove the convergence (with rate) in L^p norm and the asymptotic normality of the estimator.

AMS 2010 Subject Classification: 62M30, 62M20, 62G08, 62G05.

Key words: Kernel conditional quantile estimation, Kernel regression estimation, spatial process.

1. INTRODUCTION

Conditional quantile estimation is an important field in statistics which dates back to Stone (1977) and has been widely studied in the non-spatial case. It is useful in all domain of statistics, such as time series, survival analysis and growth charts, among others, see Koenker ([20], [21]) for a review. There exist an extensive literature and various nonparametric approaches in conditional quantile estimation in the non spatial case for independent samples and dependent non-functional or functional observations. Among the many papers dealing with conditional quantile estimation in finite dimension, one can refer, for example, to key works of Portnoy [27], Koul and Mukherjee [23], Honda [19].

Potential applications of quantile regression to spatial data are number less. Indeed, there is an increasing number of situations coming from different fields of applied sciences (soil science, geology, oceanography, econometrics, epidemiology, environmental science, forestry, etc.), where the influence of a vector of covariates on some response variable is to be studied in a context of spatial dependence. The literature on spatial models is relatively abundant, see for example, Guyon [15], Anselin and Florax [3], Cressie [7] or Ripley [29] for a list of references.

REV. ROUMAINE MATH. PURES APPL., **57** (2012), *4*, 311-339

In our knowledge, only the papers of Koencker and Mizera [22], Hallin *et al.* [18], Abdi *et al.* ([1], [2]), Dabo-Niang and Thiam [12] have paid attention to the study of nonparametric quantile regression for finite dimensional random fields while Laksaci and Maref [24] have considered infinite dimensional fields. This last work deals with almost sure consistency of the conditional consistency of a kernel conditional quantile estimate. The work of Hallin *et al.* [16] deals with local linear spatial conditional quantile regression estimation. The method of Koencker and Mizera [22] is a spatial smoothing technique rather than a spatial (auto)regression one and do not take into account the spatial dependency structure of the data. The results of Abdi *et al.* ([1], [2]) concerned respectively, consistency in p-mean ($p > 1$) and asymptotic normality and of a kernel estimate of the conditional regression function for spatial processes. Dabo-Niang and Thiam [12] considered the L_1 consistency of the local linear and double kernel conditional quantile estimate.

As in the non-spatial case, conditional quantile estimation is useful for some non-parametric prediction models and is used as an alternative to classical spatial regression estimation models for non-functional data (see Biau and Cadre [4], Lu and Chen ([25], [26]), Hallin, Lu and Tran [16], Dabo-Niang and Yao [10]). Spatial conditional quantile is of wide interest in the modeling of spatial dependence and in the construction of confidence (predictive) intervals. The purpose of this paper is to estimate the conditional quantile regression for spatial functional data.

Recall that a recent and restrictive attention has been paid to nonparametric estimation of the conditional quantile of a scalar variable Y given a functional variable ($X = X_t$, $t \in \mathbb{R}$) when observations are over an interval $T \in \mathbb{R}$. The first results concerning the nonparametric quantile estimation adapted to non-spatial functional data were obtained by Ferraty *et al.* [13]. Recently, Dabo-Niang and Laksaci [11] stated the convergence in L^p norm under less restrictive conditions closely related to the concentration properties on small balls probability of the underlying explanatory variable.

The main purpose of this paper is to extend some of the results on quantile regression to the case of functional spatial processes. In our knowledge, this work is the first contribution on spatial quantile regression estimation for functional variables. Noting that, extending classical nonparametric conditional quantile estimation for dependent functional random variables to quantile regression for functional random fields, is far from being trivial. This is due to the absence of any canonical ordering in the space, and of obvious definition of tail sigma-fields.

The paper is organized as follows. In Section 2, we provide the notations and the kernel quantile estimates. Section 3 is devoted to assumptions. Section 4 is devoted to the L_p convergence and the asymptotic normality results of the kernel quantile regression estimate, under mixing assumptions. Proofs and technical lemmas are given in Section 5.

2. THE MODEL

Consider $Z_{\mathbf{i}} = (X_{\mathbf{i}}, Y_{\mathbf{i}})$, $\mathbf{i} \in \mathbb{N}^N$ be a $\mathcal{F} \times \mathbb{R}$-valued measurable strictly stationary spatial process, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, where $(\mathcal{F}, d)$ is a semi-metric space. Let d denotes the semi-metric and $N \geq 1$. A point $\mathbf{i} = (i_1, \dots, i_N) \in \mathbb{N}^N$ will be referred to as a site. We assume that the process under study $(Z_{\mathbf{i}})$ is observed over a rectangular domain $I_{\mathbf{n}} = \{\mathbf{i} = (i_1, \dots, i_N) \in \mathbb{Z}^N,\ 1 \leq i_k \leq n_k,\ k = 1, \dots, N\}$, $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{N}^N$. A point $\mathbf{i}$ will be referred to as a *site*. We will write $\mathbf{n} \to \infty$ if $\min\{n_k\} \to \infty$ and $|\frac{n_j}{n_k}| < C$ for a constant C such that $0 < C < \infty$, for all j, k such that $1 \leq j, k \leq N$. For $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{N}^N$, we set $\widehat{\mathbf{n}} = n_1 \times \cdots \times n_N$.

We assume that the $Z_{\mathbf{i}}$'s have the same distribution as (X, Y) and the regular version of the conditional probability of Y given X exists and admits a bounded probability density. For all $x \in \mathcal{F}$, we denote respectively by F^x and f^x the conditional distribution function and density of Y given $X = x$.

Let $\alpha \in]0, 1[$, the α^{th} conditional quantile noted $q_\alpha(x)$ is defined by

$$F^x(q_\alpha(x)) = \alpha.$$

To insure existence and unicity of $q_\alpha(x)$, we assume that F^x is strictly increasing. This last is estimated by

$$(1) \qquad \widehat{F}^x_{\mathbf{n}}(y) = \begin{cases} \dfrac{\sum\limits_{\mathbf{i} \in \mathcal{I}_{\mathbf{n}}} K_1\left(\frac{d(x, X_{\mathbf{i}})}{a_{\mathbf{n}}}\right) K_2\left(\frac{y - Y_{\mathbf{i}}}{b_{\mathbf{n}}}\right)}{\sum\limits_{\mathbf{i} \in \mathcal{I}_{\mathbf{n}}} K_1\left(\frac{d(x, X_{\mathbf{i}})}{a_{\mathbf{n}}}\right)} & \text{if } \sum\limits_{\mathbf{i} \in \mathcal{I}_{\mathbf{n}}} K_1\left(\frac{d(x, X_{\mathbf{i}})}{a_{\mathbf{n}}}\right) \neq 0, \\ 0 & \text{else,} \end{cases}$$

where K_1 is a kernel, K_2 is a distribution function, $a_{\mathbf{n}}$ (resp. $b_{\mathbf{n}}$) is a sequence of real numbers which converges to 0 when $\mathbf{n} \to \infty$.

The kernel estimate $\widehat{q}_\alpha(x)$ of the conditional quantile $q_\alpha(x)$ defined by

$$\widehat{F}^x(\widehat{q}_\alpha(x)) = \alpha.$$

One can also use other methods to estimate q_α, such as the local linear method or the reproducing kernel Hilbert spaces method (see Preda, [28]).

In the following, we fix a point x in $\mathcal{F}$ such that

$$P(X \in B(x, r)) = \phi_x(r) > 0,$$

where $B(x, h) = \{x' \in \mathcal{F} \mid d(x', x) < h\}$.

3. HYPOTHESES

Throughout the paper, when no confusion will be possible, we will denote by C and C' any generic positive constant, and we denote by $g^{(j)}$ the derivative of order j of a function g. We will use the following hypotheses:

3.1. Nonparametric model conditions

H_1: F^x is of class $\mathcal{C}^1$ and $f^x(q_\alpha(x)) > 0$.

H_2: $\exists \delta_1 > 0$, $\forall\,(y_1, y_2) \in [q_\alpha(x) - \delta_1, q_\alpha(x) + \delta_1]^2$, $\forall\,(x_1, x_2) \in N_x \times N_x$,

$$|F^{x_1}(y_1) - F^{x_2}(y_2)| \leq C\big(d^{b_1}(x_1, x_2) + |y_1 - y_2|^{b_2}\big), \quad b_1 > 0,\, b_2 > 0,$$

where N_x is a small enough neighborhood of x.

H_3: There exist C_1 and C_2, $0 < C_1 < C_2 < \infty$ such that $C_1 \mathbb{I}_{[0,1]}(t) < K_1(t) < C_2 \mathbb{I}_{[0,1]}(t)$.

H_4: K_2 is of class $\mathcal{C}^1$, of bounded derivative that verifies

$$\int_{\mathbb{R}} |t|^{b_2}\, K_2^{(1)}(t)\mathrm{d}t < \infty.$$

3.2. Dependency conditions

In spatial dependent data analysis, the dependence of the observations has to be measured. Here, we will consider the following two dependence measures:

3.2.1. *Local dependence condition*

In order to establish the same convergence rate as in the i.i.d. case (see Dabo-Niang and Laksaci [10]), we need the following local dependency condition:

$$(2)\quad \begin{cases} \text{(i) For all } \mathbf{i} \neq \mathbf{j}, \text{ the conditional density of } (Y_{\mathbf{i}}, Y_{\mathbf{j}}) \text{ given } (X_{\mathbf{i}}, X_{\mathbf{j}}) \\ \text{ exists and is bounded.} \\ \text{(ii) For all } k \geq 2, \text{ we suppose that: there exists an increasing} \\ \text{sequence } 0 < (v_k) < k : \\ \max(\max_{\mathbf{i}_1 \ldots \mathbf{i}_k \in \mathcal{I}_{\mathbf{n}}} P(d(X_{\mathbf{i}_j}, x) \leq r,\ 1 \leq j \leq k),\ \phi_x^k(r)) = O(\phi_x^{1+v_k}(r)). \end{cases}$$

3.2.2. *Mixing condition*

The spatial dependence of the process will be measured by means of α-mixing. Then, we consider the α-mixing coefficients of the field $(Z_{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^N)$, defined by: there exists a function $\varphi(t) \downarrow 0$ as $t \to \infty$, such that whenever E, E' subsets of $\mathbb{N}^N$ with finite cardinals,

$$\alpha(\mathcal{B}(E), \mathcal{B}(E')) = \sup_{B\in\mathcal{B}(E),\, C\in\mathcal{B}(E')} |\mathbf{P}(B \cap C) - \mathbf{P}(B)\mathbf{P}(C)| \le \psi(\mathrm{Card}(E), \mathrm{Card}(E'))\varphi(\mathrm{dist}(E, E')), \tag{3}$$

where $\mathcal{B}(E)$ (resp. $\mathcal{B}(E')$) denotes the Borel σ-field generated by $(X_{\mathbf{i}},\ \mathbf{i} \in E)$ (resp. $(X_{\mathbf{i}},\ \mathbf{i} \in E')$), $\mathrm{Card}(E)$ (resp. $\mathrm{Card}(E')$) the cardinality of E (resp. E'), $\mathrm{dist}(E, E')$ the Euclidean distance between E and E' and $\psi : \mathbb{N}^2 \to \mathbb{R}^+$ is a symmetric positive function nondecreasing in each variable.

Throughout the paper, it will be assumed that ψ satisfies either

$$\psi(n, m) \le C \min(n, m), \quad \forall n, m \in \mathbb{N} \tag{4}$$

or

$$\psi(n, m) \le C(n + m + 1)^{\widetilde{\beta}}, \quad \forall n, m \in \mathbb{N} \tag{5}$$

for some $\widetilde{\beta} \ge 1$ and some $C > 0$. In the following, we will only consider Condition (4), one can extend easily the asymptotic results proved here in the case of (5).

We assume also that the process satisfies the following mixing condition: the process satisfies a polynomial mixing condition

$$\sum_{i=1}^{\infty} i^{\delta}\varphi(i) < \infty, \quad \delta > N(p + 2),\ p \ge 1. \tag{6}$$

If $N = 1$, then $X_{\mathbf{i}}$ is called strongly mixing. Many stochastic processes, among them various useful time series models, satisfy strong mixing properties, which are relatively easy to check. Conditions (4)–(5) are used in Tran [30], Carbon *et al.* [5], and are satisfied by many spatial models (see Guyon [14] for some examples). In addition, we assume that

H_5: $\exists 0 < \tau < (\delta - 5N)/2N$, η_0, $\eta_1 > 0$, such that $\widehat{\mathbf{n}}^{\tau} b_{\mathbf{n}} \to \infty$ and

$$C\widehat{\mathbf{n}}^{\frac{(5+2\tau)N-\delta}{\delta}+\eta_0} \le \phi_x(a_{\mathbf{n}}),$$

where δ is introduced in (6).

Remark 1. If (6) is satisfied, then $\varphi(i) \le Ci^{-\delta}$.

4. MAIN RESULTS

4.1. Weak consistency

This section contains results on pointwise consistency in p-mean. Let x be fixed, we give a rate of convergence of $\widehat{q}_\alpha(x)$ to $q_\alpha(x)$ under some general conditions.

THEOREM 1. *Under the hypotheses H_1–H_5, (4), then, for all $p \geq 1$, we have*

$$\|\widehat{q}_\alpha(x) - q_\alpha(x)\|_p = (E|\widehat{q}_\alpha(x) - q_\alpha(x)|^p)^{1/p} =$$
$$= O\left((a_{\mathbf{n}})^{b_1} + (b_{\mathbf{n}})^{b_2}\right) + O\left(\left(\frac{1}{\widehat{\mathbf{n}}\,\phi_x(a_{\mathbf{n}})}\right)^{\frac{1}{2}}\right).$$

Let

$$K_{\mathbf{i}} = K_1\left(\frac{d(x, X_{\mathbf{i}})}{a_{\mathbf{n}}}\right), \quad H_{\mathbf{i}}(y) = K_2\left(\frac{y - Y_{\mathbf{i}}}{b_{\mathbf{n}}}\right), \quad W_{\mathbf{ni}} = W_{\mathbf{ni}}(x) = \frac{K_{\mathbf{i}}}{\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} K_{\mathbf{i}}},$$

$$\widehat{F}_N^x(y) = \frac{1}{\widehat{\mathbf{n}} E K_{\mathbf{1}}} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} K_{\mathbf{i}} H_{\mathbf{i}}(y), \quad \widehat{F}_D^x = \frac{1}{\widehat{\mathbf{n}} E K_{\mathbf{1}}} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} K_{\mathbf{i}}.$$

By hypothesis H_4, $\widehat{F}_N^x(y)$ is of class $\mathcal{C}^1$; then, we can write the following Taylor development

$$\widehat{F}_N^x(\widehat{q}_\alpha(x)) = \widehat{F}_N^x(q_\alpha(x)) + \widehat{F}_N^{x(1)}(q_\alpha^*(x))\,(\widehat{q}_\alpha(x) - q_\alpha(x)),$$

where $q_\alpha^*(x)$ is in the interval of extremities $q_\alpha(x)$ and $\widehat{q}_\alpha(x)$. Thus,

$$\widehat{q}_\alpha(x) - q_\alpha(x) = \frac{1}{\widehat{F}_N^{x(1)}\big(q_\alpha^*(x)\big)}\left(\widehat{F}_N^x\big(\widehat{q}_\alpha(x)\big) - \widehat{F}_N^x\big(q_\alpha(x)\big)\right)$$
$$= \frac{1}{\widehat{F}_N^{x(1)}\big(q_\alpha^*(x)\big)}\left(\alpha\widehat{F}_D^x - \widehat{F}_N^x\big(q_\alpha(x)\big)\right).$$

It is shown in Laksaci and Maref (2009) that under (H1)–(H5), (2) and (6) that

$$\widehat{q}_\alpha(x) - q_\alpha(x) \to 0, \text{ almost completely (a.co).}$$

So, by combining this consistency and the result of Lemma 11.17 in Ferraty and Vieu ([13], p. 181), together with the fact that $q_\alpha^*(x)$ is lying between $\widehat{q}_\alpha(x)$ and $q_\alpha(x)$, it follows that

$$\widehat{F}_N^{x(1)}(q_\alpha^*(x)) - f^x(q_\alpha(x)) \to 0. \quad \text{a.co.} \tag{7}$$

Since $f^x(q_\alpha(x)) > 0$, we can write

$$\exists C > 0 \text{ such that } \left| \frac{1}{\widehat{F}_N^{x(1)}(q_\alpha^*(x))} \right| \leq C \quad \text{a.s.}$$

It follows that

$$\|\widehat{q}_\alpha(x) - q_\alpha(x)\|_p \leq C \left\| \alpha \widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x)) \right\|_p + \left(P(\widehat{F}_D^x = 0)\right)^{1/p}.$$

So, the rest of the proof is deduce from the following three lemmas.

Lemma 1. *Under $H_2 - H_4$, we have*

$$E\left[\alpha \widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x))\right] = O\left(a_{\mathbf{n}}^{b_1} + b_{\mathbf{n}}^{b_2}\right).$$

Lemma 2. *Under the hypotheses of Theorem 1, we have*

$$\left\| \alpha \widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x)) - E\left[\alpha \widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x))\right] \right\|_p = o\left(\left(\frac{1}{\widehat{\mathbf{n}}\, \phi_x(a_{\mathbf{n}})} \right)^{\frac{1}{2}} \right).$$

Lemma 3. *Under the hypotheses of Lemma 2, we have*

$$\left(P\left(\widehat{F}_D^x = 0\right)\right)^{1/p} = o\left(\left(\frac{1}{\widehat{\mathbf{n}}\, \phi_x(a_{\mathbf{n}})} \right)^{\frac{1}{2}} \right).$$

4.2. **Asymptotic normality**

This section contains results on the asymptotic normality of the quantile estimator. For that we replace, respectively H_2 and H_4 by the following hypotheses.

H_2': F^x satisfies H_2 and $\forall z \in N_x$, F^z is of class $\mathcal{C}^1$ with respect to y, the conditional density f^x is such that $f^x(q_\alpha) > 0$ and $\forall (x_1, x_2) \in N_x \times N_x$, $\forall (y_1, y_2) \in \mathbb{R}^2$

$$|f^{x_1}(y_1) - f^{x_2}(y_2)| \leq \left(\|x_1 - x_2\|^{d_1} + |y_1 - y_2|^{d_2}\right), \quad d_1, d_2 > 0.$$

H_4': K_2 satisfies H_4 and

$$\int |t|^{d_2} K_2^{(1)}(t)\mathrm{d}t < \infty.$$

Theorem 2. *Under the hypotheses of Theorem 1 and H_2', H_4', (4) then, for any $x \in \mathcal{A}$, we have*

$$\left(\frac{(f^x(q_\alpha(x)))^2 \widehat{\mathbf{n}}(\psi_{K_1}(a_{\mathbf{n}}))^2}{\psi_{K_1^2}(a_{\mathbf{n}})(\alpha(1-\alpha))} \right)^{(1/2)} (\widehat{q_\alpha}(x) - q_\alpha(x) - C_{\mathbf{n}}(x)) \to_{n \to +\infty} N(0,1),$$

where

$$C_{\mathbf{n}}(x) =$$

$$= \frac{1}{f^x(q_\alpha(x))\psi_{K_1}(a_{\mathbf{n}})}\left(\alpha\psi_{K_1}(a_{\mathbf{n}}) - E\big[K_1\big((a_{\mathbf{n}})^{-1}d(x,X)\big)\,F^X(q_\alpha(x))\big]\right) + O(b_{\mathbf{n}})$$

and

$$\mathcal{A} = \left\{x \in \mathcal{F},\ \frac{\psi_{K_1^2}(a_{\mathbf{n}})}{(\psi_{K_1}(a_{\mathbf{n}}))^2} \neq 0\right\} \quad \textit{with} \quad \psi_g(h) = -\int_0^1 g'(t)\phi_x(ht)\mathrm{d}t.$$

Firstly, observe that if H_5 is satisfied then, we have

$$\exists\, 0 < \theta_1 < 1, \text{ such that } \widehat{\mathbf{n}}^{(-1+\theta_1)/(1+2N)} \leq \phi_x(a_{\mathbf{n}}). \tag{8}$$

Let

$$\Delta_{\mathbf{i}} = \frac{1}{\sqrt{EK_{\mathbf{1}}^2}}\left[\alpha K_{\mathbf{i}} - K_{\mathbf{i}}H_{\mathbf{i}}(q_\alpha) - E\left(\alpha K_{\mathbf{i}} - K_{\mathbf{i}}H_{\mathbf{i}}(q_\alpha)\right)\right].$$

By hypothesis H_4', $\widehat{F}_N^x(y)$ is of class $\mathcal{C}^1$, then, we can write the following Taylor development:

$$\widehat{F}_N^x(\widehat{q}_\alpha) = \widehat{F}_N^x(q_\alpha) + \widehat{F}_N^{x(1)}(q_\alpha^*)\,(\widehat{q}_\alpha - q_\alpha)$$

where q_α^* is in the interval of extremities q_α and $\widehat{q}_\alpha$. Thus,

$$\widehat{q}_\alpha - q_\alpha = \frac{1}{\widehat{F}_N^{x(1)}(q_\alpha^*)}\left(\widehat{F}_N^x(\widehat{q}_\alpha) - \widehat{F}_N^x(q_\alpha)\right) = \frac{1}{\widehat{F}_N^{x(1)}(q_\alpha^*)}\left(\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha)\right),$$

$$\left[\frac{f^x(q_\alpha)^2\widehat{\mathbf{n}}E^2K_{\mathbf{i}}}{\alpha(1-\alpha)EK_{\mathbf{i}}^2}\right]^{1/2}\left(\left[\widehat{q}_\alpha - q_\alpha - C_{\mathbf{n}}(x)\right]\right) =$$

$$= \left[\frac{\widehat{\mathbf{n}}E^2K_{\mathbf{i}}}{\alpha(1-\alpha)EK_{\mathbf{i}}^2}\right]^{1/2}\left(\frac{f^x(q_\alpha)}{\widehat{F}_N^{x(1)}(q_\alpha^*)}\left(\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha)\right) - E\left(\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha)\right)\right),$$

where

$$C_{\mathbf{n}}(x) = \frac{1}{f^x(q_\alpha)}E\left(\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha)\right).$$

Consequently, the proof of the theorem is the consequence of the following lemmas and the convergence result (7).

LEMMA 4. *Under the hypotheses of Theorem* 2, *we have*
i) $Var\left(\Delta_{\mathbf{i}}\right) \to \alpha(1-\alpha)$;
ii) $\sum_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{\mathbf{n}}} Cov\left(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}\right) = o\left(\widehat{\mathbf{n}}\right)$ *and*
iii) $\frac{1}{\widehat{\mathbf{n}}}var\left(\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\Delta_{\mathbf{i}}\right) \to \alpha(1-\alpha)$, *when* $\mathbf{n} \to \infty$.

LEMMA 5. *Under the hypotheses of Theorem* 2 *we have*

$$\left[\frac{\widehat{\mathbf{n}}E^2K_{\mathbf{i}}}{\alpha(1-\alpha)EK_{\mathbf{i}}^2}\right]^{1/2}\left(\left[\alpha\widehat{F}_D^x-\widehat{F}_N^x(q_\alpha)\right]-E\left[\alpha\widehat{F}_D^x-\widehat{F}_N^x(q_\alpha)\right]\right)\to N(0,1).$$

LEMMA 6. *Under the hypotheses* H_2' *and* H_4', *we have*

$$E\left[\alpha\widehat{F}_D^x-\widehat{F}_N^x(q_\alpha)\right]=\alpha-\frac{1}{EK_{\mathbf{i}}}E\left[K\left(\frac{d,x-X}{a_{\mathbf{n}}}\right)F^X(q_\alpha)\right]+O\left(b_{\mathbf{n}}^{b_2}\right)$$

and

$$E\left[\alpha\widehat{F}_D^x-\widehat{F}_N^x(q_\alpha)\right]=O\left(a_{\mathbf{n}}^{b_1}+b_{\mathbf{n}}^{b_2}\right).$$

It is easy to see that, if one imposes some additional assumptions on the function $\phi_x(\cdot)$ and the bandwidth parameters ($a_{\mathbf{n}}$ and $b_{\mathbf{n}}$) we can improved our asymptotic normality by explicit asymptotic expressions of dispersion terms or by removing the bias term $C_n(x)$.

COROLLARY 1. *Under the hypotheses of Theorem* 2 *and if the bandwidth parameters* ($a_{\mathbf{n}}$ *and* $b_{\mathbf{n}}$) *and if the function* $\phi_x(a_{\mathbf{n}})$ *satisfies*

$$\lim_{n\to\infty}({a_{\mathbf{n}}}^{b_1}+{b_{\mathbf{n}}}^{b_2})\sqrt{\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})}=0\quad\text{and}\quad\lim_{n\to\infty}\frac{\phi_x(ta_{\mathbf{n}})}{\phi_x(a_{\mathbf{n}})}=\beta(t),\quad\forall t\in[0,1],$$

we have

$$\left(\frac{(f^x(t_\alpha(x)))^2\delta_1^2}{\delta_2(\alpha(1-\alpha))}\right)^{(1/2)}\sqrt{n\phi_x(a_{\mathbf{n}})}\left(\widehat{t_\alpha}(x)-t_\alpha(x)\right)\to_{n\to+\infty}N(0,1),$$

where $\delta_j=-\int_0^1(K^j)'(s)\beta(s)\mathrm{d}s$, *for*, $j=1,\,2$.

Remark 2. If we assume that (5) is satisfied instead of (4) then it is simple to have the results of Theorems 1 and 2, the only thing that changes is condition (H5) which will be replaced by some assumption that depend on $\widetilde{\beta}$.

5. APPENDIX

We first state the following lemmas which are due to Carbon et al. [6]. They are needed for the convergence of our estimates. Their proofs will then be omitted.

LEMMA 7. *Suppose* $E_1,\dots,E_r$ *be sets containing* m *sites each with* $dist(E_i,\ E_j)\geq\gamma$ *for all* $i\neq j$, *where* $1\leq i\leq r$ *and* $1\leq j\leq r$. *Suppose* $Z_1,\dots,Z_r$ *is a sequence of real-valued r.v.'s measurable with respect to* $\mathcal{B}(E_1),\dots,\mathcal{B}(E_r)$ *respectively, and* Z_i *takes values in* $[a,b]$. *Then, there exists a sequence of independent r.v.'s* $Z_1^*,\dots,Z_r^*$ *independent of* $Z_1,\dots,Z_r$ *such* *that Z_i^* has the same distribution as Z_i and satisfies*

$$\sum_{i=1}^{r} E|Z_i - Z_i^*| \leq 2r(b-a)\psi((r-1)m, m)\varphi(\gamma).$$

LEMMA 8. (i) *Suppose that* (3) *holds. Denote by $\mathcal{L}_r(\mathcal{F})$ the class of $\mathcal{F}$-measurable r.v.'s X satisfying $\|X\|_r = (E|X|^r)^{1/r} < \infty$. Suppose $X \in \mathcal{L}_r(\mathcal{B}(E))$ and $Y \in \mathcal{L}_s(\mathcal{B}(E'))$. Assume also that $1 \leq r, s, t < \infty$ and $r^{-1} + s^{-1} + t^{-1} = 1$. Then,*

$$|EXY - EXEY| \leq \tag{9}$$
$$\leq C\|X\|_r\|Y\|_s\{\psi(Card(E), Card(E'))\varphi(dist(E, E'))\}^{1/t}.$$

(ii) *For r.v.'s bounded with probability* 1, *the right-hand side of* (9) *can be replaced by $C\psi(Card(E), Card(E'))\varphi(dist(E, E'))$.*

Proof of Lemma 1. We have

$$E[\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x))] = \alpha - \frac{1}{EK_{\mathbf{1}}}\left(E\left[K_{\mathbf{1}}E\left[H_{\mathbf{1}}(q_\alpha(x)) \mid X_{\mathbf{1}}\right]\right]\right).$$

We shall use the integration by parts and the usual change of variables $t = \frac{y-z}{b_{\mathbf{n}}}$, to show that

$$E[\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x))] = \alpha - \frac{1}{EK_{\mathbf{1}}}\left(EK_{\mathbf{1}}\int K_2^{(1)}(t)F^{X_{\mathbf{1}}}((q_\alpha(x)) - b_{\mathbf{n}}t)\mathrm{d}t\right).$$

Hypotheses (H2) and (H4) allow to get

$$E[\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x))] \leq$$
$$\frac{1}{EK_{\mathbf{1}}}E\left[K_{\mathbf{1}}\int K_2^{(1)}(t)\left|F^x((q_\alpha(x)) - F^{X_{\mathbf{1}}}((q_\alpha(x)) - b_{\mathbf{n}}t)\mathrm{d}t\right|\right] \leq C\left(a_{\mathbf{n}}^{b_1} + b_{\mathbf{n}}^{b_2}\right).$$

Proof of Lemma 2. We have

$$\left\|\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x)) - E\left[\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha(x))\right]\right\|_p = \frac{1}{n\,EK_1}\left\|\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\theta_{\mathbf{i}}\right\|_p,$$

where

$$\theta_{\mathbf{i}} = K_{\mathbf{i}}(\alpha - H_{\mathbf{i}}(q_\alpha(x))) - E\left[K_{\mathbf{i}}(\alpha - H_{\mathbf{i}}(q_\alpha(x)))\right].$$

We have $EK_{\mathbf{1}} = O(\phi_x(h_K))$, (because of H3), so it remains to show that

$$\left\|\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\theta_{\mathbf{i}}\right\|_p = O(\sqrt{\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})}).$$

The evaluation of this quantity is based on ideas similar to that used by Gao *et al.* (2008), see also Abdi *et al.* (2010). More preciously, we prove the case where $p = 2m$ (for all $m \in N^*$) and we use the Hölder inequality for lower values of p.

First of all, let us notice that the notations $\theta_{\mathbf{i}}$ and $\xi_{\mathbf{i}}$, deliberately introduced above, are the same as those used in Lemma 2.2 of Gao *et al.* (2008) or Abdi *et al.* (2010a). The proof of the lemma is completely modeled on that of Lemma 2.2 of Gao *et al.*. To make easier the understanding of the effect of the boundedness of $\theta_{\mathbf{i}}$ on the results, we opt to run along the lines of Gao *et al.*'s proof (keeping the same notations) and give the moment results in a simpler form. To start, note that

$$E\left[\left(\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\theta_{\mathbf{i}}\right)^{2r}\right] = \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} E\left[\theta_{\mathbf{i}}^{2m}\right] + \sum_{s=1}^{2m-1}\sum_{\nu_0+\nu_1+\cdots+\nu_s=2r} V_s(\nu_0,\nu_1,\ldots,\nu_s),$$

where $\sum_{\nu_0+\nu_1+\cdots+\nu_s=2m}$ is the summation over $(\nu_0,\nu_1,\ldots,\nu_s)$ with positive integer components satisfying $\nu_0+\nu_1+\cdots+\nu_s = 2m$ and

$$V_s(\nu_0,\nu_1,\ldots,\nu_s) = \sum_{\mathbf{i}_0\neq\mathbf{i}_1\neq\cdots\neq\mathbf{i}_s} E\left[\theta_{\mathbf{i}_0}^{\nu_0}\theta_{\mathbf{i}_1}^{\nu_1}\ldots\theta_{\mathbf{i}_s}^{\nu_s}\right],$$

where the summation $\sum_{\mathbf{i}_0\neq\mathbf{i}_1\neq\cdots\neq\mathbf{i}_s}$ is over indexes $(\mathbf{i}_0,\mathbf{i}_1,\ldots,\mathbf{i}_s)$ with each index $\mathbf{i}_j$ taking value in $\mathcal{I}_{\mathbf{n}}$ and satisfying $\mathbf{i}_j \neq \mathbf{i}_l$ for any $j\neq l$, $0\le j,l\le s$. By stationarity and the fact that K_2 is a distribution function, we have

$$\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} E\left(\theta_{\mathbf{i}}\right)^{2m} \le C\widehat{\mathbf{n}}E\left(|\theta_{\mathbf{i}}|\right)^{2m} \le \widehat{\mathbf{n}}E\left(K_{\mathbf{i}}\right)^{2m} \le C\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}}).$$

To control the term $V_s(\nu_0,\nu_1,\ldots,\nu_s)$, we need to prove, for any positive integers $\nu_0,\nu_1,\nu_2,\ldots\nu_s$, the following results:

i) $E\left|\theta_{\mathbf{i}_1}^{\nu_1}\theta_{\mathbf{i}_2}^{\nu_2}\ldots\theta_{\mathbf{i}_s}^{\nu_s}\right| \le C\phi_x(a_{\mathbf{n}})^{1+v_s}$;

ii) $V_s(\nu_0,\nu_1,\ldots,\nu_s)=O\big(\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{s+1}\big)$, for $s=1,2,\ldots,m-1$ and $m>1$;

iii) $V_s(\nu_0,\nu_1,\ldots,\nu_s) = O\big(\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{m}\big)$, for $m\le s\le 2m-1$.

To show the result i), remark that the boundness of K_2 and (2) yield

$$E\left|\theta_{\mathbf{i}_1}^{\nu_1}\theta_{\mathbf{i}_2}^{\nu_2}\ldots\theta_{\mathbf{i}_s}^{\nu_s}\right| \le C\phi_x^{1+v_s}(a_{\mathbf{n}}).$$

Proof of ii). Note that we can write

$$V_s(\nu_0,\nu_1,\ldots,\nu_s) =$$

$$= \sum_{\mathbf{i}_0\neq\mathbf{i}_1\neq\cdots\neq\mathbf{i}_s}\left[E\left(\prod_{j=0}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right) - \prod_{j=0}^{s}E\theta_{\mathbf{i}_j}^{\nu_j}\right] + \sum_{\mathbf{i}_0\neq\mathbf{i}_1\neq\cdots\neq\mathbf{i}_s}\prod_{j=0}^{s}E\theta_{\mathbf{i}_j}^{\nu_j} =: V_{s1}+V_{s2}.$$

Clearly, we have $|V_{s2}| \le C\sum_{\mathbf{i}_0\neq\mathbf{i}_1\neq\cdots\neq\mathbf{i}_s}\left(\phi_x(a_{\mathbf{n}})\right)^{s+1} \le C\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{s+1}$.

For the term V_{s1}, notice that

$$E\left(\prod_{j=0}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right)-\prod_{j=0}^{s}E\theta_{\mathbf{i}_j}^{\nu_j}=$$
$$=\sum_{l=0}^{s-1}\left(\prod_{j=0}^{l-1}E\theta_{\mathbf{i}_j}^{\nu_j}\right)\left(E\left[\prod_{j=l}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]-E\left[\theta_{\mathbf{i}_l}^{\nu_l}\right]E\left[\prod_{j=l+1}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]\right),$$

where we define $\prod_{j=l}^{s}=1$ if $l>s$. Then, we obtain

$$|V_{s1}|\leq\sum_{l=0}^{s-1}\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^l\sum_{\mathbf{i}_l\neq\cdots\neq\mathbf{i}_s}\left|E\left[\prod_{j=l}^{s}\xi_{\mathbf{i}_j}^{\nu_j}\right]-E\left[\xi_{\mathbf{i}_l}^{\nu_l}\right]E\left[\prod_{j=l+1}^{s}\xi_{\mathbf{i}_j}^{\nu_j}\right]\right|=$$
$$=\sum_{l=0}^{s-1}\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^l V_{ls1}.$$

Let P be some positive real, we have

$$V_{ls1}=\sum_{0<dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1},\ldots,\mathbf{i}_s\})\leq P}\left|E\left[\prod_{j=l}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]-E\left[\theta_{\mathbf{i}_l}^{\nu_l}\right]E\left[\prod_{j=l+1}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]\right|+$$
$$+\sum_{0<dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1},\ldots,\mathbf{i}_s\})>P}\left|E\left[\prod_{j=l}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]-E\left[\theta_{\mathbf{i}_l}^{\nu_l}\right]E\left[\prod_{j=l+1}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]\right|:=V_{ls11}+V_{ls12}.$$

Using the result i) above leads to

$$\left|E\left[\prod_{j=l}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]-E\left[\theta_{\mathbf{i}_l}^{\nu_l}\right]E\left[\prod_{j=l+1}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]\right|\leq\left|E\left[\prod_{j=l}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]\right|+\left|E\left[\theta_{\mathbf{i}_l}^{\nu_l}\right]E\left[\prod_{j=l+1}^{s}\theta_{\mathbf{i}_j}^{\nu_j}\right]\right|\leq$$
$$\leq C\phi_x(a_{\mathbf{n}})^{1+v_{s+1-l}}.$$

Thus, we have

$$V_{ls11}\leq\sum_{0<dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1},\ldots,\mathbf{i}_s\})\leq P}C\phi_x(a_{\mathbf{n}})^{1+v_{s+1-l}}\leq$$
$$\leq C\phi_x(a_{\mathbf{n}})^{1+v_{s+1-l}}\sum_{k=1}^{P}\sum_{k\leq dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1},\ldots,\mathbf{i}_s\})=t<k+1}1.$$

Since $dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1},\dots,\mathbf{i}_s\}) = t$, it follows that there exits some $\mathbf{i}_j \in \{\mathbf{i}_{l+1},\dots,\mathbf{i}_s\}$, say $\mathbf{i}_{l+1}$, such that $dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1}\}) = t$, and therefore,

$$\sum_{k=1}^{P} \sum_{k\le dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1},\dots,\mathbf{i}_s\})=t<k+1} 1 \le \sum_{k=1}^{P} \sum_{\substack{\mathbf{i}_j\in\mathcal{I}_\mathbf{n}\\ j=l+2,\dots,s}} \sum_{k\le dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1}\})=t<k+1} 1 \le$$

$$\le \widehat{\mathbf{n}}^{(s-1-l)} \sum_{k=1}^{P} \sum_{k\le dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1}\})=t<k+1} 1.$$

Thus,

$$V_{ls11} \le C\phi_x(a_\mathbf{n})^{1+v_{s+1-l}}\widehat{\mathbf{n}}^{(s-l)} \sum_{k=1}^{P} \sum_{k\le\|\mathbf{i}\|=t<k+1} 1 \le$$

$$\le C\phi_x(a_\mathbf{n})^{1+v_{s+1-l}}\widehat{\mathbf{n}}^{(s-l)} \sum_{t=1}^{P} t^{N-1} \le C\phi_x(a_\mathbf{n})^{1+v_{s+1-l}}\widehat{\mathbf{n}}^{(s-l)}P^N.$$

For the term V_{ls12}, notice that since the variables $\theta_\mathbf{i}$ are bounded, we have (see Lemma 8)

$$\left| E\Big[\prod_{j=l}^{s} \theta_{\mathbf{i}_j}^{\nu_j}\Big] - E\big[\theta_{\mathbf{i}_l}^{\nu_l}\big] E\Big[\prod_{j=l+1}^{s} \theta_{\mathbf{i}_j}^{\nu_j}\Big] \right| \le C\psi(1,s-l)\varphi(t),$$

where $t = dist\,(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1},\dots,\mathbf{i}_s\})$. Then, under (4) or (5), we have

$$V_{ls12} \le C \sum_{k=P+1}^{\infty} \sum_{k\le dist(\{\mathbf{i}_l\},\{\mathbf{i}_{l+1}\})=t<k+1} \varphi(t) \le$$

$$\le C \sum_{k=P+1}^{\infty} \widehat{\mathbf{n}}^{(s-l)} \sum_{k\le\|\mathbf{i}\|=t<k+1} \varphi(t) \le C\widehat{\mathbf{n}}^{(s-l)} \sum_{t=P+1}^{\infty} t^{N-1}\varphi(t).$$

Combining the upper bounds of V_{ls11} and V_{ls12}, we have

$$|V_{s1}| \le C \sum_{l=0}^{s-1} (\widehat{\mathbf{n}}\phi_x(a_\mathbf{n}))^l \left[\phi_x(a_\mathbf{n})^{1+v_{s+1-l}}\widehat{\mathbf{n}}^{(s-l)}P^N + \widehat{\mathbf{n}}^{(s-l)} \sum_{t=P+1}^{\infty} t^{N-1}\varphi(t)\right] \le$$

$$\le C\,(\widehat{\mathbf{n}}\phi_x(a_\mathbf{n}))^{(s+1)} \sum_{l=0}^{s-1} (\widehat{\mathbf{n}}\phi_x(a_\mathbf{n}))^{l-s-1} \Big[\phi_x(a_\mathbf{n})^{1+v_{s+1-l}}\widehat{\mathbf{n}}^{(s-l)}P^N +$$

$$+ \widehat{\mathbf{n}}^{(s-l)} \sum_{t=P+1}^{\infty} t^{N-1}\varphi(t)\Big] =$$

$$= C\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{(s+1)}\sum_{l=0}^{s-1}\left[\widehat{\mathbf{n}}^{-1}P^N\phi_x(a_{\mathbf{n}})^{1+v_{s+1-l}}\phi_x(a_{\mathbf{n}})^{l-s-1}+\right.$$
$$\left.+\,\widehat{\mathbf{n}}^{-1}\phi_x(a_{\mathbf{n}})^{(l-s-1)}\sum_{t=P+1}^{\infty}t^{N-1}\varphi(t)\right]\leq$$
$$\leq C\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{(s+1)}\sum_{l=0}^{s-1}\left[\widehat{\mathbf{n}}^{-1}P^N\phi_x(a_{\mathbf{n}})^{1+v_{s+1-l}}\phi_x(a_{\mathbf{n}})^{l-s-1}+\right.$$
$$\left.+\,\widehat{\mathbf{n}}^{-1}\phi_x(a_{\mathbf{n}})^{(l-s-1)}P^{-sN}\sum_{t=P+1}^{\infty}t^{sN+N-1}\varphi(t)\right].$$

Taking $P=\phi_x(a_{\mathbf{n}})^{-1/N}$, we obtain

$$|V_{s1}|\leq C\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{(s+1)}\sum_{l=0}^{s-1}\left[\frac{1}{\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})}\phi_x(a_{\mathbf{n}})^{1+v_{s+1-l}}\phi_x(a_{\mathbf{n}})^{l-s-1}+\right.$$
$$\left.+\frac{1}{\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})}\left(\phi_x(a_{\mathbf{n}})^{l}\right)\sum_{t=P+1}^{\infty}t^{sN+N-1-\delta}\right]\leq C\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{(s+1)}$$

since $\delta > N(p+2)$.

Proof of iii). As indicated in Gao *et al.* (2008), since the arguments are similar for any $m \leq s \leq 2m-1$, the proof is given only for $s = 2m-1$ and $N = 2$ for simplicity. In this case, $V_{2m-1}(\nu_0, \nu_1, \ldots, \nu_{2m-1})$ is denoted W. To simplify the notations, we write $\mathbf{i} = (i, j) \in \mathbb{Z}^2$ and $\mathbf{i}_k = (i_k, j_k) \in \mathbb{Z}^2$. The main difficulty is to cope with the summation

$$\sum_{\mathbf{i}_0\neq\mathbf{i}_1\neq\ldots\mathbf{i}_{2m-1}}E\left[\theta_{\mathbf{i}_0}\theta_{\mathbf{i}_1}\ldots\theta_{\mathbf{i}_{2m-1}}\right]=$$
$$=\sum_{(i_0,j_0)\neq(i_1,j_1)\neq\cdots\neq(i_{2m-1},j_{2m-1})}E\left[\theta_{i_0j_0}\theta_{i_1j_1}\ldots\theta_{i_{2m-1}j_{2m-1}}\right].$$

To this end, a novel ordering in $\mathbb{Z}^2$ (see Gao *et al.* (2008)), makes possible to separate the indexes into two (or more) sets whose distance is greater or smaller than P (usually larger than 1) is considered. Arrange each of the index sets $\{i_0, i_1, \ldots, i_{2m-1}\}$ and $\{j_0, j_1, \ldots, j_{2m-1}\}$ in ascending orders as (retaining the same notation for the first ordered index set for simplicity) $i_0 \leq i_1 \leq \cdots \leq i_{2m-1}$ and $j_{l_0} \leq j_{l_1} \leq \cdots \leq j_{l_{2m-1}}$, where l_k is to indicate that l_k may not be equal to k. The number of such arrangements is at most $(2m)!$. Let $\Delta i_k = i_k - i_{k-1}$ and $\Delta j_k = j_{l_k} - j_{l_k-1}$ and arrange $\{\Delta i_1, \ldots, \Delta i_{2m-1}\}$ and $\{\Delta j_1, \ldots, \Delta j_{2m-1}\}$ in decreasing orders, respectively as $\Delta i_{a_1} \geq \cdots \geq \Delta i_{a_{2m-1}}$ and $\Delta j_{b_1} \geq \cdots \geq \Delta j_{b_{2m-1}}$. Let $t_1 = \Delta i_{a_m}$, $t_2 = \Delta j_{b_m}$ and $t = \max\{t_1, t_2\}$. If

$t_1 \geq t_2$ then $t = t_1$, and $0 \leq i_{a_k} - i_{a_{k-1}} \leq t_1 \leq n_1$ for $k = m+1, \dots, 2m-1$, $0 \leq j_{l_{b_k}} - j_{l_{b_{k-1}}} \leq t \leq n_2$ for $k = m, m+1, \dots, 2m-1$. Therefore,

(10) $i_{a_{k-1}} \leq i_{a_k} \leq i_{a_{k-1}} + t,\ j_{l_{b_{k-1}}} \leq j_{b_k} \leq j_{l_{b_{k-1}}} + t$ for $k = m+1, \dots, 2m-1$.

We arrange $\mathbf{i}_0 \neq \mathbf{i}_1 \neq \cdots \neq \mathbf{i}_{2m-1}$ according to the order of $i_0 \leq i_1 \leq \cdots \leq i_{2m-1}$. If $t_1 < t_2$, arrange according to the order of $j_{l_0} \leq j_{l_1} \leq \cdots \leq j_{l_{2m-1}}$ and the proof is similar.

Let $\mathcal{I} = \{i_1, \dots, i_{2m-1}\}$, $\mathcal{I}_a = \{i_{a_1}, \dots, i_{a_m}\}$, $\mathcal{I}_a^c = \mathcal{I} - \mathcal{I}_a = \{i_{a_{m+1}}, \dots, i_{a_{2m-1}}\}$, $\mathcal{J} = \{j_{l_1}, \dots, j_{l_{2m-1}}\}$, $\mathcal{J}_b = \{j_{l_{b_1}}, \dots, j_{l_{b_m}}\}$ and $\mathcal{J}_b^c = \mathcal{J} - \mathcal{J}_b = \{j_{l_{b_{m+1}}}, \dots, j_{l_{b_{2m-1}}}\}$. Remark that $(i_{a_1}, \dots, i_{a_{2m-1}})$ and $(j_{l_{b_1}}, \dots, j_{l_{b_{2m-1}}})$ are permutations of respectively $\mathcal{I}$ and $\mathcal{J}$. Then, from (10), $t = i_{a_m} - i_{a_{m-1}}$ and $t \geq j_{l_{b_m}} - j_{l_{b_{m-1}}}$, we deduce that

$$
\begin{aligned}
W &= \sum_{\mathbf{i}_0 \neq \mathbf{i}_1 \neq \dots \mathbf{i}_{2m-1}} E\left[\theta_{\mathbf{i}_0}\theta_{\mathbf{i}_1}\dots\theta_{\mathbf{i}_{2m-1}}\right] \\
&\leq C \sum_{1 \leq i_0 \leq i_1 \leq \cdots \leq i_{2m-1} \leq n_1} \ \sum_{1 \leq j_{l_0} \leq j_{l_1} \leq \cdots \leq j_{l_{2m-1}} \leq n_2} \left| E\left[\theta_{i_0 j_0}\theta_{i_1 j_1}\dots\theta_{i_{2m-1}j_{2m-1}}\right]\right| \\
&\leq C \sum_{t=1}^{\max(n_1,n_2)} \sum_{i_0=1}^{n_1} \sum_{\substack{i=1 \\ i \in \mathcal{I}_a - \{i_{a_m}\}}}^{n_1} \sum_{\substack{i_{a_k} = i_{a_{k-1}} \\ k=m+1,\dots,2m-1}}^{i_{a_{k-1}}+t} \sum_{j_{l_0}=1}^{n_2} \sum_{\substack{j=1 \\ j \in \mathcal{J}_b - \{j_{l_{b_m}}\}}}^{n_2} \sum_{\substack{j_{l_{b_k}} = j_{l_{b_{k-1}}} \\ k=m,m+1,\dots,2m-1}}^{j_{l_{b_{k-1}}}+t} \cdot \\
&\quad \cdot \left| E\left[\theta_{\mathbf{i}_0}\theta_{\mathbf{i}_1}\dots\theta_{\mathbf{i}_{2m-1}}\right]\right|.
\end{aligned}
$$

Take a positive constant P such that $1 \leq P \leq \max(n_1, n_2)$ and divide the right hand side of the previous inequality into two parts denoted by W_1 and W_2 according to $1 \leq t \leq P$ and $t > P$. Then, $W \leq W_1 + W_2$. In one hand, use the result i) with $s = 2m$, and get

$$
\begin{aligned}
W_1 &= C \sum_{t=1}^{P} \sum_{i_0=1}^{n_1} \sum_{\substack{i=1 \\ i \in \mathcal{I}_a - \{i_{a_m}\}}}^{n_1} \sum_{\substack{i_{a_k} = i_{a_{k-1}} \\ k=m+1,\dots,2m-1}}^{i_{a_{k-1}}+t} \sum_{j_{l_0}=1}^{n_2} \sum_{\substack{j=1 \\ j \in \mathcal{J}_b - \{j_{l_{b_m}}\}}}^{n_2} \sum_{\substack{j_{l_{b_k}} = j_{l_{b_{k-1}}} \\ k=m,m+1,\dots,2m-1}}^{j_{l_{b_{k-1}}}+t} \cdot \\
&\quad \cdot \left| E\left[\theta_{\mathbf{i}_0}\theta_{\mathbf{i}_1}\dots\theta_{\mathbf{i}_{2m-1}}\right]\right| \\
&\leq C \sum_{t=1}^{P} \sum_{i_0=1}^{n_1} \sum_{\substack{i=1 \\ i \in \mathcal{I}_a - \{i_{a_m}\}}}^{n_1} \sum_{\substack{i_{a_k} = i_{a_{k-1}} \\ k=m+1,\dots,2m-1}}^{i_{a_{k-1}}+t} \sum_{j_{l_0}=1}^{n_2} \sum_{\substack{j=1 \\ j \in \mathcal{J}_b - \{j_{l_{b_m}}\}}}^{n_2} \sum_{\substack{j_{l_{b_k}} = j_{l_{b_{k-1}}} \\ k=m,m+1,\dots,2m-1}}^{j_{l_{b_{k-1}}}+t} \phi_x(a_{\mathbf{n}})^{1+v_{2m}} \\
&\leq C(n_1 n_2)^m \sum_{t=1}^{P} t^{2m-1} \phi_x(a_{\mathbf{n}})^{1+v_{2m}} \leq C(n_1 n_2)^m P^{2m} \phi_x(a_{\mathbf{n}})^{1+v_{2m}}.
\end{aligned}
$$

In another hand, assume that neither i_1 nor i_{2m-1} belongs to $\mathcal{I}_a$ (if i_1 or i_{2m-1} is in $\mathcal{I}_a$ the proof is similar). In this case, $\mathcal{I}_a$ is a subset of size m chosen from the $2m-3$ remaining indexes (besides i_0, i_1 and i_{2m-1}). As raised by Gao *et al.*, this is due to the fact that there must be two successive indexes because there are not enough elements in the set of remaining indices to allow a gap between every two elements of $\mathcal{I}_a$. The first components of $\mathbf{i}_j$'s are ordered as $i_0 \leq i_1 \leq \cdots \leq i_{k^*-1} \leq i_{k^*} \leq i_{k^*+1} \leq \cdots \leq i_{2m-1}$ for some $k^* \geq 1$ and $\Delta i_j = i_j - i_{j-1}$. Then, we have, either $dist(\{\mathbf{i}_0, \ldots, \mathbf{i}_{k^*-1}\}, \{\mathbf{i}_{k^*}\}) \geq \Delta i_{k^*} \geq t$, $dist(\{\mathbf{i}_{k^*}\}, \{\mathbf{i}_{k^*+1}, \ldots, \mathbf{i}_{2m-1}\}) \geq \Delta i_{k^*+1} \geq t$ and $dist(\{\mathbf{i}_0, \ldots, \mathbf{i}_{k^*-1}\}, \{\mathbf{i}_{k^*}, \ldots, \mathbf{i}_{2m-1}\}) \geq \Delta i_{k^*+1} \geq t$, or $dist(\{\mathbf{i}_0\}, \{\mathbf{i}_1, \ldots, \mathbf{i}_{2m-1}\}) \geq \Delta i_1 \geq t$ or $dist(\{\mathbf{i}_0, \ldots, \mathbf{i}_{2m-2}\}, \{\mathbf{i}_{2m-1}\}) \geq \Delta i_{2m-1} \geq t$. Let $A_{\mathbf{i}_{k^*-1}} = \theta_{\mathbf{i}_0}\theta_{\mathbf{i}_1} \ldots \theta_{\mathbf{i}_{k^*-1}}$ and $B_{\mathbf{i}_{k^*+1}} = \theta_{\mathbf{i}_{k^*+1}} \ldots \theta_{\mathbf{i}_{2m-1}}$, then, for the case of i_{k^*} and i_{k^*+1} in $\mathcal{I}_a$, we have

$$\begin{aligned}
|E[\theta_{\mathbf{i}_0}\theta_{\mathbf{i}_1} \ldots \theta_{\mathbf{i}_{2m-1}}]| &= |E[A_{\mathbf{i}_{k^*-1}}\theta_{\mathbf{i}_{k^*}} B_{\mathbf{i}_{k^*+1}}]| \\
&\leq |E[(A_{\mathbf{i}_{k^*-1}} - EA_{\mathbf{i}_{k^*-1}})(\theta_{\mathbf{i}_{k^*}} B_{\mathbf{i}_{k^*+1}} - E\theta_{\mathbf{i}_{k^*}} B_{\mathbf{i}_{k^*+1}})]| \\
&\quad + |E[A_{\mathbf{i}_{k^*-1}}]E[(\theta_{\mathbf{i}_{k^*}} B_{\mathbf{i}_{k^*+1}})]| \\
&= |Cov(A_{\mathbf{i}_{k^*-1}}, \theta_{\mathbf{i}_{k^*}} B_{\mathbf{i}_{k^*+1}})| + |E[A_{\mathbf{i}_{k^*-1}}]|\,|Cov(\theta_{\mathbf{i}_{k^*}}, B_{\mathbf{i}_{k^*+1}})| \\
&\leq C\varphi(t) + C\phi_x(a_{\mathbf{n}})^{1+v_{k^*}}\varphi(t) \leq C\varphi(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
W_2 = C &\sum_{t=P+1}^{\max(n_1,n_2)} \sum_{i_0=1}^{n_1} \sum_{\substack{i=1 \\ i\in\mathcal{I}_a-\{i_{a_m}\}}}^{n_1} \sum_{\substack{i_{a_k}=i_{a_{k-1}} \\ k=m+1,\ldots,2m-1}}^{i_{a_{k-1}}+t} \sum_{j_{l_0}=1}^{n_2} \sum_{\substack{j=1 \\ j\in\mathcal{J}_b-\{j_{l_{b_m}}\}}}^{n_2} \sum_{\substack{j_{l_{b_k}}=j_{l_{b_{k-1}}} \\ k=m,m+1,\ldots,2m-1}}^{j_{l_{b_{k-1}}}+t} \\
&\left|E\left[\theta_{\mathbf{i}_0}\theta_{\mathbf{i}_1} \ldots \theta_{\mathbf{i}_{2m-1}}\right]\right| \leq C(n_1n_2)^m \sum_{t=P+1}^{\infty} t^{2m-1}\varphi(t).
\end{aligned}$$

It follows that

$$\begin{aligned}
W \leq W_1 + W_2 &\leq C(n_1n_2)^m P^{2m}\phi_x(a_{\mathbf{n}})^{1+v_{2m}} + C(n_1n_2)^m \sum_{t=P+1}^{\infty} t^{2m-1}\varphi(t) \\
&\leq (n_1n_2)^m \left(P^{2m}\phi_x(a_{\mathbf{n}})^{(1+v_{2m})} + P^{2m-1-\delta}\right).
\end{aligned}$$

For general N, we obtain by similar arguments

$$W \leq C(\widehat{\mathbf{n}})^m \left(P^{Nm}\phi_x(a_{\mathbf{n}})^{(1+v_{Nm})} + P^{Nm-1-\delta}\right).$$

Taking $P=\phi_x(a_{\mathbf{n}})^{-(1+v_{Nm})/1+\delta)}$, we get

$$W \le C(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}}))^m\left(\widehat{\mathbf{n}}^{-1}\phi_x(a_{\mathbf{n}})^{-Nm-1+(1+v_{Nm})}+(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}}))^{-1}\sum_{t=P+1}^{\infty}t^{Nm-1-\delta}\right)$$
$$\le C(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}}))^m$$

because $\delta > N(p+2)$. This ends the proof of the lemma.

Proof of Lemma 3. We have for all $\epsilon < 1$,

$$P\left(\widehat{F}_D^x = 0\right) \le P\left(\widehat{F}_D^x \le 1-\epsilon\right) \le P\left(|\widehat{F}_D^x - E[\widehat{F}_D^x]| \ge \epsilon\right).$$

Markov's inequality allows to get, for any $p > 0$,

$$P\left(|\widehat{F}_D^x - E[\widehat{F}_D^x]| \ge \epsilon\right) \le \frac{E\left[|\widehat{F}_D^x - E[\widehat{F}_D^x]|^p\right]}{\epsilon^p}.$$

So,

$$\left(P(\widehat{F}_D^x = 0)\right)^{1/p} = O\left(\left\|\widehat{F}_D^x - E[\widehat{F}_D^x]\right\|_p\right).$$

The computation of $\left\|\widehat{F}_D^x - E[\widehat{F}_D^x]\right\|_p$ can be done by following the same arguments as those used to prove Lemma 2. This yields the proof.

Proof of Lemma 4. Let us calculate the variance $Var(\Delta_{\mathbf{i}})$. We have

$$Var(\Delta_{\mathbf{i}}) = \frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}^2\left(\alpha - H_{\mathbf{i}}(q_\alpha)\right)^2 - \left(EK_{\mathbf{i}}\left(\alpha - H_{\mathbf{i}}(q_\alpha)\right)\right)^2\right]$$
$$= \frac{1}{EK_{\mathbf{i}}^2}EK_{\mathbf{i}}^2\left(\alpha - H_{\mathbf{i}}(q_\alpha)\right)^2 - \frac{(EK_{\mathbf{i}})^2}{EK_{\mathbf{i}}^2}\left[E\frac{K_{\mathbf{i}}\left(H_{\mathbf{i}}(q_\alpha)-\alpha\right)}{EK_{\mathbf{i}}}\right]^2 = A_1 - A_2.$$

Let us first consider A_2. We deduce from the hypothesis H_3 that there exist two positive constants C and C' such that $C\phi_x(a_{\mathbf{n}}) \le EK_{\mathbf{i}}^r \le C'\phi_x(a_{\mathbf{n}})$, $r > 1$, thus, $\frac{(EK_{\mathbf{i}})^2}{EK_{\mathbf{i}}^2} = o(1)$. If we take the conditional expectation with respect to X, we get

$$\left|E\left[\frac{K_{\mathbf{i}}(H_{\mathbf{i}}(q_\alpha)-\alpha)}{EK_{\mathbf{i}}}\right]\right| = \left|E\frac{K_{\mathbf{i}}}{EK_{\mathbf{i}}}\left[E(H_{\mathbf{i}}(q_\alpha)|X)-\alpha\right]\right| \le E\frac{K_{\mathbf{i}}}{EK_{\mathbf{i}}}\left|E(H_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}})-\alpha\right|.$$

It is easy to see that by hypothesis H_4' (ii)

$$|E\left(H_{\mathbf{i}}(q_\alpha)|X\right) - \alpha| = |E\left(H_{\mathbf{i}}(q_\alpha)|X\right) - F^x(q_\alpha)|$$
$$\le C\left(a_{\mathbf{n}}^{b_1} + b_{\mathbf{n}}^{b_2}\int_{\mathbb{R}}|t|^{b_2}\left|K_2^{(1)}(t)\right|\mathrm{d}t\right),$$

$$\left|E\frac{K_{\mathbf{i}}\left(H_{\mathbf{i}}(q_\alpha)-\alpha\right)}{EK_{\mathbf{i}}}\right| = O\left(a_{\mathbf{n}}^{b_1} + b_{\mathbf{n}}^{b_2}\right).$$

Then, we deduce that A_2 tends to 0. Concerning A_1, we have

$$(\alpha - H_{\mathbf{i}}(q_\alpha))^2 = \left(H^2_{\mathbf{i}}(q_\alpha) - \alpha\right) - 2\alpha\left(H_{\mathbf{i}}(q_\alpha) - \alpha\right) + \alpha - \alpha^2.$$

Then, we can write

$$A_1 = \frac{1}{EK^2_{\mathbf{i}}}\left[EK^2_{\mathbf{i}}\left(H^2_{\mathbf{i}}(q_\alpha) - \alpha\right) - 2\alpha EK^2_{\mathbf{i}}\left(H_{\mathbf{i}}(q_\alpha) - \alpha\right)\right] + \alpha\left(1 - \alpha\right).$$

The conditional expectation with respect to $X_{\mathbf{i}}$, permits to obtain

$$A_1 = E\frac{K^2_{\mathbf{i}}}{EK^2_{\mathbf{i}}}\left[E\left(H^2_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}}\right) - \alpha\right] - 2\alpha E\frac{K^2_{\mathbf{i}}}{EK^2_{\mathbf{i}}}\left[E\left(H_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}}\right) - \alpha\right] + \alpha\left(1 - \alpha\right).$$

The same argument as above, gives

$$\left|E\frac{K^2_{\mathbf{i}}}{EK^2_{\mathbf{i}}}\left[E\left(H_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}}\right) - \alpha\right]\right| = O\left(a^{b_1}_{\mathbf{n}} + b^{b_2}_{\mathbf{n}}\right).$$

It remains to show that

$$\left|E\left(H^2_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}}\right) - \alpha\right| = O\left(a^{b_1}_{\mathbf{n}} + b^{b_2}_{\mathbf{n}}\right).$$

By an integration by part and hypotheses H'_2 and H'_4, we have

$$\begin{aligned}
\left|E\left(H^2_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}}\right) - \alpha\right| &= \left|\int_{\mathbb{R}} K^2_2\left(\frac{q_\alpha - z}{b_{\mathbf{n}}}\right) f^{X_{\mathbf{i}}}(z)dz - F^x(q_\alpha)\right| \\
&= \left|\int_{\mathbb{R}} 2K_2(t)K^{(1)}_2(t)\left(F^{X_{\mathbf{i}}}(q_\alpha - b_{\mathbf{n}}t) - F^x(q_\alpha)\right)\mathrm{d}t\right| \\
&\le a^{b_1}_{\mathbf{n}}\int_{\mathbb{R}} 2K_2(t)K^{(1)}_2(t)\mathrm{d}t + b^{b_2}_{\mathbf{n}}\int_{\mathbb{R}} 2K_2(t)|t|^{b_2}K^{(1)}_2(t)\mathrm{d}t \\
&\le Ca^{b_1}_{\mathbf{n}} + b^{b_2}_{\mathbf{n}}\int_{\mathbb{R}} 2|t|^{b_2}K^{(1)}_2(t)\mathrm{d}t = O\left(a^{b_1}_{\mathbf{n}} + b^{b_2}_{\mathbf{n}}\right).
\end{aligned}$$

We deduce from above that A_1 converges to $\alpha\left(1 - \alpha\right)$; then,

$$Var\left(\Delta_{\mathbf{i}}\right) \to \alpha\left(1 - \alpha\right).$$

Let us focus now on the covariance term. We consider

$$E_1 = \{\mathbf{i}, \mathbf{j} \in \mathcal{I}_{\mathbf{n}} : 0 < \|\mathbf{i} - \mathbf{j}\| \le c_{\mathbf{n}}\},$$
$$E_2 = \{\mathbf{i}, \mathbf{j} \in \mathcal{I}_{\mathbf{n}} : \|\mathbf{i} - \mathbf{j}\| > c_{\mathbf{n}}\}.$$

We have

$$Cov\,(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}) = E\Delta_{\mathbf{i}}\Delta_{\mathbf{j}} =$$
$$= \frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}K_{\mathbf{j}}\,(\alpha - H_{\mathbf{i}}(q_\alpha))\,(\alpha - H_{\mathbf{j}}(q_\alpha)) - (EK_{\mathbf{i}}\,(\alpha - H_{\mathbf{i}}(q_\alpha)))^2\right] \le$$
$$\le \frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}K_{\mathbf{j}}\,(\alpha - H_{\mathbf{i}}(q_\alpha))\,(\alpha - H_{\mathbf{j}}(q_\alpha))\right] + \frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}\,(\alpha - H_{\mathbf{i}}(q_\alpha))\right]^2.$$

The conditional expectation with respect to $X_{\mathbf{i}}$, gives

$$\frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}\,(\alpha - H_{\mathbf{i}}(q_\alpha))\right]^2 =$$
$$= \frac{1}{EK_{\mathbf{i}}^2}\left|EK_{\mathbf{i}}\,(\alpha - E(H_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}}))\right|^2 \le \frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}\left|E\left(H_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}}\right) - \alpha\right|\right]^2.$$

Recall that

$$|E\left(H_{\mathbf{i}}(q_\alpha)|X\right) - \alpha| = O\left(a_{\mathbf{n}}^{b_1} + b_{\mathbf{n}}^{b_2}\right);\ \text{then, } |E\left(H_{\mathbf{i}}(q_\alpha)|X\right) - \alpha| \le C.$$

So,

$$\frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}\,(\alpha - H_{\mathbf{i}}(q_\alpha))\right]^2 \le C\phi_x(a_{\mathbf{n}}).$$

Since K_1 is bounded, we get

$$\frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}K_{\mathbf{j}}\,(\alpha - H_{\mathbf{i}}(q_\alpha))\,(\alpha - H_{\mathbf{j}}(q_\alpha))\right] \le C\frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}K_{\mathbf{j}}\right] \le$$
$$\le C\frac{1}{EK_{\mathbf{i}}^2}P\left[(X_{\mathbf{i}}, X_{\mathbf{j}}) \in B(x, a_{\mathbf{n}}) \times B(x, a_{\mathbf{n}})\right].$$

Then, we deduce from (2)

$$\frac{1}{EK_{\mathbf{i}}^2}\left[EK_{\mathbf{i}}K_{\mathbf{j}}\,(\alpha - H_{\mathbf{i}}(q_\alpha))\,(\alpha - H_{\mathbf{j}}(q_\alpha))\right] \le$$
$$\le C\frac{1}{EK_{\mathbf{i}}^2}(\phi_x(a_{\mathbf{n}}))^{1+v_2} \le (\phi_x(a_{\mathbf{n}}))^{v_2}.$$

Then, we have since $v_2 > 1$: $Cov\,(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}) \le C(\phi_x(a_{\mathbf{n}}) + (\phi_x(a_{\mathbf{n}}))^{v_2}) \le C(\phi_x(a_{\mathbf{n}}))$ and $\sum_{E_1} Cov\,(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}) \le C\widehat{\mathbf{n}}c_{\mathbf{n}}^N\phi_x(a_{\mathbf{n}})$.

Lemma 8 and $|\Delta_{\mathbf{i}}| \le C\phi_x(a_{\mathbf{n}})^{-1/2}$, permit to write that

$$|Cov\,(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}})| \le C\phi_x(a_{\mathbf{n}})^{-1}\varphi\left(\|\mathbf{i}-\mathbf{j}\|\right)$$

and

$$\sum_{E_2} Cov\left(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}\right) \leq C\phi_x(a_{\mathbf{n}})^{-1} \sum_{(\mathbf{i},\mathbf{j})\in E_2} \varphi\left(\|\mathbf{i}-\mathbf{j}\|\right) \leq C\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})^{-1} \sum_{\mathbf{i}:\|\mathbf{i}\|>c_{\mathbf{n}}} \varphi\left(\|\mathbf{i}\|\right)$$
$$\leq C\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})^{-1} c_{\mathbf{n}}^{-\delta} \sum_{\mathbf{i}:\|\mathbf{i}\|>c_{\mathbf{n}}} \|\mathbf{i}\|^{\delta}\varphi\left(\|\mathbf{i}\|\right).$$

Finally, for $\delta > 0$ we have

$$\sum Cov\left(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}\right) \leq \left(C\widehat{\mathbf{n}}c_{\mathbf{n}}^{N}\phi_x(a_{\mathbf{n}}) + C\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})^{-1}c_{\mathbf{n}}^{-\delta} \sum_{\mathbf{i}:\|\mathbf{i}\|>c_{\mathbf{n}}} \|\mathbf{i}\|^{\delta}\varphi\left(\|\mathbf{i}\|\right)\right).$$

Let $c_{\mathbf{n}} = \phi_x(a_{\mathbf{n}})^{-1/N}$, then, we have

$$\sum Cov\left(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}\right) \leq \left(C\widehat{\mathbf{n}} + C\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})^{\delta/N-1} \sum_{\mathbf{i}:\|\mathbf{i}\|>c_{\mathbf{n}}} \|\mathbf{i}\|^{\delta}\varphi\left(\|\mathbf{i}\|\right)\right).$$

Hence, we obtain that

$$\sum Cov\left(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}\right) = o\left(\widehat{\mathbf{n}}\right).$$

In conclusion, we have

$$\frac{1}{\widehat{\mathbf{n}}} var\left(\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} \Delta_{\mathbf{i}}\right) = \left(var\left(\Delta_{\mathbf{i}}\right) + \frac{1}{\widehat{\mathbf{n}}} \sum_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{\mathbf{n}}} Cov\left(\Delta_{\mathbf{i}},\Delta_{\mathbf{j}}\right)\right) \to \alpha(1-\alpha) \text{ when } \mathbf{n}\to\infty.$$

This yields the proof.

Proof of Lemma 5. Let

$$S_{\mathbf{n}} = \sum_{\substack{j_k=1\\ k=1,\dots,N}}^{n_k} \Delta_{\mathbf{j}}$$

with

$$\Delta_{\mathbf{j}} = \frac{1}{\sqrt{EK_{\mathbf{i}}}}\left[\alpha K_{\mathbf{i}} - K_{\mathbf{i}}H_{\mathbf{i}}(q_\alpha) - E\left(\alpha K_{\mathbf{i}} - K_{\mathbf{i}}H_{\mathbf{i}}(q_\alpha)\right)\right].$$

Then, we can write

$$\left[\frac{\widehat{\mathbf{n}}E^2K_{\mathbf{i}}}{\alpha(1-\alpha)EK_{\mathbf{i}}^2}\right]^{1/2} \left(\left[\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha)\right] - E\left[\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha)\right]\right) =$$
$$= \left(\widehat{\mathbf{n}}\alpha(1-\alpha)\right)^{-1/2} S_{\mathbf{n}}.$$

Consider the same spatial block decomposition (due to Tran (1990)) as Lemma 2, with $q_{\mathbf{n}} = o\left(\left[\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})^{(1+2N)}\right]^{1/(2N)}\right)$, $m_{\mathbf{n}} = \left[\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{1/(2N)}/s_{\mathbf{n}}\right]$

where $s_{\mathbf{n}} = o\left(\left[\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})^{1+2N}\right]^{1/(2N)} q_{\mathbf{n}}^{-1}\right)$. Then, we have

$$S_{\mathbf{n}} = \sum_{i=1}^{2^N} T(\mathbf{n}, x, i),$$

where

$$T(\mathbf{n}, x, i) = \sum_{\mathbf{j}\in\mathcal{J}} U(i, \mathbf{n}, x, \mathbf{j}).$$

Hence,

$$S_{\mathbf{n}}/\left(\widehat{\mathbf{n}}\alpha(1-\alpha)\right)^{1/2} = T\left(\mathbf{n}, x, 1\right)/\left(\widehat{\mathbf{n}}\alpha(1-\alpha)\right)^{1/2} + \\ + \sum_{i=2}^{2N} T\left(\mathbf{n}, x, i\right)/\left(\widehat{\mathbf{n}}\alpha(1-\alpha)\right)^{1/2}.$$

Thus, the proof of the asymptotic normality of $\left(\widehat{\mathbf{n}}\alpha(1-\alpha)\right)^{-1/2} S_{\mathbf{n}}$ is reduced to the proofs of the following results

$$Q_1 \equiv \left| E\exp\left[iuT(\mathbf{n}, x, 1)\right] - \prod_{\substack{j_k=0\\ k=1,\ldots,N}}^{r_k-1} E\exp\left[iuU(1, \mathbf{n}, x, \mathbf{j})\right]\right| \to 0 \tag{11}$$

$$Q_2 \equiv \widehat{\mathbf{n}}^{-1} E\left(\sum_{i=2}^{2^N} T(\mathbf{n}, x, i)\right)^2 \to 0 \tag{12}$$

$$Q_3 \equiv \widehat{\mathbf{n}}^{-1} \sum_{\mathbf{j}\in\mathcal{J}} E\Big[U(1, \mathbf{n}, x, \mathbf{j})\Big]^2 \to \alpha(1-\alpha) \tag{13}$$

(14)
$$Q_4 \equiv \widehat{\mathbf{n}}^{-1} \sum_{\mathbf{j}\in\mathcal{J}} E\Big[(U(1, \mathbf{n}, x, \mathbf{j}))^2 \mathbf{1}_{\{|U(1,\mathbf{n},x,\mathbf{j})|>\epsilon(\alpha(1-\alpha)\widehat{\mathbf{n}})^{1/2}\}}\Big] \to 0, \text{ for all } \epsilon > 0.$$

Proof of (11). Let us numerate the $M = \prod_{k=1}^{N} r_k = \widehat{\mathbf{n}}(m_{\mathbf{n}}+q_{\mathbf{n}})^{-N} \leq \widehat{\mathbf{n}} m_{\mathbf{n}}^{-N}$ random variables $U(1, \mathbf{n}, x, \mathbf{j})$; $\mathbf{j} \in \mathcal{J}$ in the arbitrary way $\widetilde{U}_1, \ldots, \widetilde{U}_M$. For $\mathbf{j} \in \mathcal{J}$, let

$$I(1, \mathbf{n}, x, \mathbf{j}) = \{\mathbf{i} : j_k(m_{\mathbf{n}} + q_{\mathbf{n}}) + 1 \leq i_k \leq j_k(m_{\mathbf{n}} + q_{\mathbf{n}}) + m_{\mathbf{n}};\ k = 1, \ldots, N\}$$

then, we have $U(1, \mathbf{n}, x, \mathbf{j}) = \sum_{\mathbf{i}\in I(1,\mathbf{n},x,\mathbf{j})} \Delta_{\mathbf{i}}$. Note that each of the sets of site $I(1, \mathbf{n}, x, \mathbf{j})$ contains $m_{\mathbf{n}}^N$, these sets are distant of $m_{\mathbf{n}}$ at least.

Let us apply the lemma of Volkonski and Rozanov (1959) to the variable $\big(\exp(iu\widetilde{U}_1),\ldots,\exp(iu\widetilde{U}_M)\big)$. The fact that $\Big|\prod_{s=j+1}^{M}\exp[iu\widetilde{U}_s]\Big| \le 1$, implies

$$\begin{aligned}
Q_1 &= \Big| E\exp\big[iuT(\mathbf{n},x,1)\big] - \prod_{\substack{j_k=0\\ k=1,\ldots,N}}^{r_k-1} E\exp\big[iuU(1,\mathbf{n},x,\mathbf{j})\big]\Big| \\
&= \Big| E\prod_{\substack{j_k=0\\ k=1,\ldots,N}}^{r_k-1} \exp\left[iuU(1,\mathbf{n},x,\mathbf{j})\right] - \prod_{\substack{j_k=0\\ k=1,\ldots,N}}^{r_k-1} E\exp\big[iuU(1,\mathbf{n},x,\mathbf{j})\big]\Big| \\
&\le \sum_{k=1}^{M-1}\sum_{j=k+1}^{M}\Big| E\big(\exp[iu\widetilde{U}_k]-1\big)\big(\exp[iu\widetilde{U}_j]-1\big)\prod_{s=j+1}^{M}\exp[iu\widetilde{U}_s] \\
&\qquad - E\big(\exp[iu\widetilde{U}_k]-1\big)E\big(\exp[iu\widetilde{U}_j]-1\big)\prod_{s=j+1}^{M}\exp[iu\widetilde{U}_s]\Big| \\
&= \sum_{k=1}^{M-1}\sum_{j=k+1}^{M}\Big| E\big(\exp[iu\widetilde{U}_k]-1\big)\big(\exp[iu\widetilde{U}_j]-1\big) \\
&\qquad - E\big(\exp[iu\widetilde{U}_k]-1\big)E\big(\exp[iu\widetilde{U}_j]-1\big)\Big|\,\Big|\prod_{s=j+1}^{M}\exp[iu\widetilde{U}_s]\Big| \\
&\le \sum_{k=1}^{M-1}\sum_{j=k+1}^{M}\Big| E\big(\exp[iu\widetilde{U}_k]-1\big)\big(\exp[iu\widetilde{U}_j]-1\big) \\
&\qquad - E\big(\exp[iu\widetilde{U}_k]-1\big)E\big(\exp[iu\widetilde{U}_j]-1\big)\Big|.
\end{aligned}$$

Let $\widetilde{I}_j$ be the set of sites among the $I(1,\mathbf{n},x,\mathbf{j})$ such that $\widetilde{U}_j = \sum_{\mathbf{i}\in\widetilde{I}(\mathbf{j})}\Delta_{\mathbf{i}}$. The lemma of Carbon *et al.* (1997) and assumption (3), give

$$\begin{aligned}
&\Big| E\big(\exp[iu\widetilde{U}_k]-1\big)\big(\exp[iu\widetilde{U}_j]-1\big) - E\big(\exp[iu\widetilde{U}_k]-1\big)E\big(\exp[iu\widetilde{U}_j]-1\big)\Big| \\
&\qquad\le C\varphi\big(d(\widetilde{I}_j,\widetilde{I}_k)\big)m_{\mathbf{n}}^{N}.
\end{aligned}$$

Then,

$$Q_1 \le C m_{\mathbf{n}}^N \sum_{k=1}^{M-1} \sum_{j=k+1}^{M} \varphi\left(d(\widetilde{I}_j, \widetilde{I}_k)\right) \le C m_{\mathbf{n}}^N M \sum_{k=2}^{M} \varphi\left(d(\widetilde{I}_1, \widetilde{I}_k)\right)$$
$$\le C m_{\mathbf{n}}^N M \sum_{i=1}^{\infty} \sum_{k: i q_{\mathbf{n}} \le d(\widetilde{I}_1, \widetilde{I}_k) < (i+1) q_{\mathbf{n}}} \varphi\left(d(\widetilde{I}_1, \widetilde{I}_k)\right)$$
$$\le C m_{\mathbf{n}}^N M \sum_{i=1}^{\infty} i^{N-1} \varphi(i q_{\mathbf{n}}) \le C \widehat{\mathbf{n}} q_{\mathbf{n}}^{-N\delta} \sum_{i=1}^{\infty} i^{N-1-N\delta},$$

by (6). This last tends to zero by the fact that $\widehat{\mathbf{n}} q_{\mathbf{n}}^{-N\delta} \to 0$ (see (H5)).

Proof of (12). We have

$$Q_2 \equiv \widehat{\mathbf{n}}^{-1} E\left(\sum_{i=2}^{2^N} T(\mathbf{n}, x, i)\right)^2 =$$
$$= \widehat{\mathbf{n}}^{-1}\left(\sum_{i=2}^{2^N} E\left[T(\mathbf{n}, x, i)\right]^2 + \sum_{\substack{i,j=2,\ldots,2^N \\ i \ne j}} E\left[T(\mathbf{n}, x, i)\right]\left[T(\mathbf{n}, x, j)\right]\right).$$

By Cauchy-Schwartz inequality, we get $\forall\, 2 \le i \le 2^N$:

$$\widehat{\mathbf{n}}^{-1} E\left[T(\mathbf{n}, x, i)\right]\left[T(\mathbf{n}, x, j)\right] \le \left(\widehat{\mathbf{n}}^{-1} E\left[T(\mathbf{n}, x, i)\right]^2\right)^{1/2}\left(\widehat{\mathbf{n}}^{-1} E\left[T(\mathbf{n}, x, j)\right]^2\right)^{1/2}.$$

Then, it suffices to prove that

$$\widehat{\mathbf{n}}^{-1} E\left[T(\mathbf{n}, x, i)\right]^2 \to 0, \quad \forall\, 2 \le i \le 2^N.$$

We will prove this for $i = 2$, the case where $i \ne 2$ is similar. We have $T(\mathbf{n}, x, 2) = \sum_{\mathbf{j} \in \mathcal{J}} U(2, \mathbf{n}, x, \mathbf{j}) = \sum_{j=1}^{M} \widehat{U}_j$, where we enumerate the $U(2, \mathbf{n}, x, \mathbf{j})$ in the arbitrary way $\widehat{U}_1, \ldots, \widehat{U}_M$. Then,

$$E\left[T(\mathbf{n}, x, 2)\right]^2 = \sum_{i=1}^{M} Var\big(\widehat{U}_i\big) + \sum_{i=1}^{M} \sum_{\substack{j=1 \\ i \ne j}}^{M} Cov\big(\widehat{U}_i, \widehat{U}_j\big) = A_1 + A_2.$$

The stationarity of the process $(X_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^N}$, implies that

$$Var(\widehat{U}_i) = Var\Big(\sum_{\substack{i_k=1\\k=1,\dots,N-1}}^{m_{\mathbf{n}}} \sum_{i_N=1}^{q_{\mathbf{n}}} \Delta_{\mathbf{i}}\Big)^2$$

$$= m_{\mathbf{n}}^{N-1} q_{\mathbf{n}}\, Var\left(\Delta_{\mathbf{i}}\right) + \sum_{\substack{i_k=1\\k=1,\dots,N-1}}^{m_{\mathbf{n}}} \sum_{i_N=1}^{q_{\mathbf{n}}} \sum_{\substack{j_k=1\\k=1,\dots,N-1\\\mathbf{i}\neq\mathbf{j}}}^{m_{\mathbf{n}}} \sum_{j_N=1}^{q_{\mathbf{n}}} E\Delta_{\mathbf{i}}\Delta_{\mathbf{j}}.$$

We proved above that $Var\left(\Delta_{\mathbf{i}}\right) < C$. By Lemma 8, we have

$$|E\Delta_{\mathbf{i}}(x)\Delta_{\mathbf{j}}(x)| \le C\phi_x(a_{\mathbf{n}})^{-1}\varphi\left(\|\mathbf{i}-\mathbf{j}\|\right). \tag{15}$$

Then, we deduce that

$$Var(\widehat{U}_i) \le C m_{\mathbf{n}}^{N-1} q_{\mathbf{n}}\Big(1 + \phi_x(a_{\mathbf{n}})^{-1} \sum_{\substack{i_k=1\\k=1,\dots,N-1}}^{m_{\mathbf{n}}} \sum_{i_N=1}^{q_{\mathbf{n}}} \left(\varphi\left(\|\mathbf{i}\|\right)\right)\Big)$$

$$\le C m_{\mathbf{n}}^{N-1} q_{\mathbf{n}} \phi_x(a_{\mathbf{n}})^{-1} \sum_{\substack{i_k=1\\k=1,\dots,N-1}}^{m_{\mathbf{n}}} \sum_{i_N=1}^{q_{\mathbf{n}}} (\varphi(\|\mathbf{i}\|).$$

Consequently, we have

$$A_1 \le CM m_{\mathbf{n}}^{N-1} q_{\mathbf{n}} \phi_x(a_{\mathbf{n}})^{-1} \sum_{i=1}^{\infty} i^{N-1}\left(\varphi(i)\right).$$

Let

$$I\left(2,\mathbf{n},x,\mathbf{j}\right) = \big\{\mathbf{i} : j_k(m_{\mathbf{n}}+q_{\mathbf{n}})+1 \le i_k \le j_k(m_{\mathbf{n}}+q_{\mathbf{n}})+m_{\mathbf{n}},\ 1\le k\le N-1;$$
$$+j_N(m_{\mathbf{n}}+q_{\mathbf{n}})+m_{\mathbf{n}}+1 \le i_N \le (J_N+1)(m_{\mathbf{n}}+q_{\mathbf{n}})\big\}.$$

The variable $U\left(2,\mathbf{n},x,\mathbf{j}\right)$ is the sum of the $\Delta_{\mathbf{i}}$ such that $\mathbf{i}$ is in $I\left(2,\mathbf{n},x,\mathbf{j}\right)$. Since $m_{\mathbf{n}} > q_{\mathbf{n}}$, if $\mathbf{i}$ and $\mathbf{i}'$ are respectively in the two different sets $I\left(2,\mathbf{n},x,\mathbf{j}\right)$ and $I\left(2,\mathbf{n},x,\mathbf{j}'\right)$; then, $i_k \neq i'_k$, for a certain k such that $1\le k\le N$ and $\|\mathbf{i}-\mathbf{i}'\| > q_{\mathbf{n}}$.

By using the definition of A_2, the stationarity of the process and (15), we have

$$A_2 \le \sum_{\substack{j_k=1\\k=1,\dots,N\\\|\mathbf{i}-\mathbf{j}\|>q_{\mathbf{n}}}}^{n_k} \sum_{\substack{i_k=1\\k=1,\dots,N}}^{n_k} E\Delta_{\mathbf{i}}\Delta_{\mathbf{j}} \le C\phi_x(a_{\mathbf{n}})^{-1}\widehat{\mathbf{n}} \sum_{\substack{i_k=1\\k=1,\dots,N\\\|\mathbf{i}\|>q_{\mathbf{n}}}}^{n_k} (\varphi(\|\mathbf{i}\|))$$

and

$$A_2 \le C\phi_x(a_{\mathbf{n}})^{-1}\widehat{\mathbf{n}}\sum_{i=q_{\mathbf{n}}}^{\infty} i^{N-1}\left(\varphi(i)\right).$$

We deduce that

$$\widehat{\mathbf{n}}^{-1}E\left[T(\mathbf{n},x,2)\right]^2 \le CMm_{\mathbf{n}}^{N-1}q_{\mathbf{n}}\widehat{\mathbf{n}}^{-1}\phi_x(a_{\mathbf{n}})^{-1}\sum_{i=1}^{\infty} i^{N-1-\delta}+$$
$$+C\phi_x(a_{\mathbf{n}})^{-1}\sum_{i=q_{\mathbf{n}}}^{\infty} i^{N-1-\delta}.$$

From $(m_{\mathbf{n}}+q_{\mathbf{n}})^{-N}m_{\mathbf{n}}^{N-1}q_{\mathbf{n}} = (m_{\mathbf{n}}+q_{\mathbf{n}})^{-N}m_{\mathbf{n}}^{N}\left(\frac{q_{\mathbf{n}}}{m_{\mathbf{n}}}\right) \le \frac{q_{\mathbf{n}}}{m_{\mathbf{n}}}$, we get

$$CMm_{\mathbf{n}}^{N-1}q_{\mathbf{n}}\widehat{\mathbf{n}}^{-1}\phi_x(a_{\mathbf{n}})^{-1} = \widehat{\mathbf{n}}(m_{\mathbf{n}}+q_{\mathbf{n}})^{-N}m_{\mathbf{n}}^{N-1}q_{\mathbf{n}}\widehat{\mathbf{n}}^{-1}\phi_x(a_{\mathbf{n}})^{-1} \le$$
$$\le \left(\frac{q_{\mathbf{n}}}{m_{\mathbf{n}}}\right)\phi_x(a_{\mathbf{n}})^{-1} = q_{\mathbf{n}}s_{\mathbf{n}}\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})\right)^{\frac{-1}{2N}}\phi_x(a_{\mathbf{n}})^{-1} = q_{\mathbf{n}}s_{\mathbf{n}}\left(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}})^{(1+2N)}\right)^{\frac{-1}{2N}}.$$

By the hypothesis on $q_{\mathbf{n}}s_{\mathbf{n}}$, this last term converges to $\to 0$. Finally, we have

$$C\phi_x(a_{\mathbf{n}})^{-1}\sum_{i=q_{\mathbf{n}}}^{\infty} i^{N-1-\delta} \le C\phi_x(a_{\mathbf{n}})^{-1}\int_{q_{\mathbf{n}}}^{\infty} t^{N-1-\delta}\mathrm{d}t = C\phi_x(a_{\mathbf{n}})^{-1}q_{\mathbf{n}}^{N-\delta}.$$

This last term converges to zero by (8) and ends the proof of (12).

Proof of (13). Let us use the following decomposition of small and big blocks

$$S'_{\mathbf{n}} = T(\mathbf{n},x,1), \quad S''_{\mathbf{n}} = \sum_{i=2}^{2^N} T(\mathbf{n},x,i).$$

Then, we can write

$$\widehat{\mathbf{n}}^{-1}E\left(S'_{\mathbf{n}}\right)^2 = \widehat{\mathbf{n}}^{-1}ES_{\mathbf{n}}^2 + \widehat{\mathbf{n}}^{-1}E\left(S''_{\mathbf{n}}\right)^2 - 2\widehat{\mathbf{n}}^{-1}ES_{\mathbf{n}}S''_{\mathbf{n}}.$$

Lemma 4(iii) and (12) imply, respectively, that $\widehat{\mathbf{n}}^{-1}E\left(S_{\mathbf{n}}\right)^2 = \widehat{\mathbf{n}}^{-1}var\left(S_{\mathbf{n}}\right) \to \alpha(1-\alpha)$ and $\widehat{\mathbf{n}}^{-1}E\left(S''_{\mathbf{n}}\right)^2 \to 0$. Then, to show that $\widehat{\mathbf{n}}^{-1}E\left(S'_{\mathbf{n}}\right)^2 \to \alpha(1-\alpha)$, it suffices to remark that $\widehat{\mathbf{n}}^{-1}ES_{\mathbf{n}}S''_{\mathbf{n}} \to 0$ because, by Cauchy-Schwartz's inequality, we can write

$$\left|\widehat{\mathbf{n}}^{-1}ES_{\mathbf{n}}S''_{\mathbf{n}}\right| \le \widehat{\mathbf{n}}^{-1}E\left|S_{\mathbf{n}}S''_{\mathbf{n}}\right| \le \left(\widehat{\mathbf{n}}^{-1}ES_{\mathbf{n}}^2\right)^{1/2}\left(\widehat{\mathbf{n}}^{-1}ES''^2_{\mathbf{n}}\right)^{1/2}.$$

Recall that $T(\mathbf{n},x,1) = \sum_{\mathbf{j}\in\mathcal{J}} U(1,\mathbf{n},x,\mathbf{j})$, so

$$\widehat{\mathbf{n}}^{-1}E\left(S'_{\mathbf{n}}\right)^2 = \widehat{\mathbf{n}}^{-1} \sum_{\substack{j_k=0\\k=1,\dots,N}}^{r_k-1} E\left[U(1,\mathbf{n},x,\mathbf{j})\right]^2 +$$

$$+\widehat{\mathbf{n}}^{-1} \times \sum_{\substack{j_k=0\\k=1,\dots,N}}^{r_k-1} \sum_{\substack{i_k=0\\k=1,\dots,N\\ i_k\neq j_k \text{ for some } k}}^{r_k-1} cov[U(1,\mathbf{n},x,\mathbf{j}),U(1,\mathbf{n},x,\mathbf{i})].$$

By similar arguments used above for A_2, this last term is not greater than

$$C\phi_x(a_{\mathbf{n}})^{-1} \sum_{\substack{i_k=1\\k=1,\dots,N\\ \|\mathbf{i}\|>q_{\mathbf{n}}}}^{r_k-1} (\varphi(\|\mathbf{i}\|)) \le C\phi_x(a_{\mathbf{n}})^{-1} \sum_{i=q_{\mathbf{n}}}^{\infty} i^{N-1}(\varphi(i)) \le C\phi_x(a_{\mathbf{n}})^{-1} q_{\mathbf{n}}^{N-\delta} \to 0.$$

So, $Q_3 \to \alpha(1-\alpha)$. This ends the proof.

Proof of (14). Since $|\Delta_{\mathbf{i}}| \le C\phi_x(a_{\mathbf{n}})^{-1/2}$, we have $|U(1,\mathbf{n},x,\mathbf{j})| \le Cm_{\mathbf{n}}^N\phi_x(a_{\mathbf{n}})^{-1/2}$. Then, we deduce that

$$Q_4 \le Cm_{\mathbf{n}}^{2N}\phi_x(a_{\mathbf{n}})^{-1}\widehat{\mathbf{n}}^{-1} \sum_{\substack{j_k=0\\k=1,\dots,N}}^{r_k-1} P\left[|U(1,\mathbf{n},x,\mathbf{j})| > \epsilon\left(\alpha(1-\alpha)\widehat{\mathbf{n}}\right)^{1/2}\right].$$

We have $|U(1,\mathbf{n},x,\mathbf{j})|/((\alpha(1-\alpha)\widehat{\mathbf{n}})^{1/2}) \le Cm_{\mathbf{n}}^N(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}}))^{-1/2} = C(s_{\mathbf{n}})^{-N} \to 0$, because $m_{\mathbf{n}} = [(\widehat{\mathbf{n}}\phi_x(a_{\mathbf{n}}))^{1/(2N)}/s_{\mathbf{n}}]$ and $s_{\mathbf{n}} \to \infty$. So, for all ϵ and $\mathbf{j} \in \mathcal{J}$; if $\widehat{\mathbf{n}}$ is great enough, then $P[U(1,\mathbf{n},x,\mathbf{j}) > \epsilon(\alpha(1-\alpha)\widehat{\mathbf{n}})^{1/2}] = 0$. Then, $Q_4 = 0$ for $\widehat{\mathbf{n}}$ great enough. This yields the proof.

Proof of Lemma 6. By change of variables, using the stationarity of the process, we have

$$\begin{aligned}
E\Big[\alpha\widehat{F}_D^x - \widehat{F}_N^x(q_\alpha)\Big] &= \alpha - \frac{1}{EK_{\mathbf{i}}}E\left[K_{\mathbf{i}}H_{\mathbf{i}}(q_\alpha)\right] = \alpha - \frac{1}{EK_{\mathbf{i}}}EK_{\mathbf{i}}E\left[H_{\mathbf{i}}(q_\alpha)|X_{\mathbf{i}}\right]\\
&= \alpha - \frac{1}{EK_{\mathbf{i}}}E\left(K_{\mathbf{i}}\int_{\mathbb{R}} K_2\left(\frac{q_\alpha - y}{b_{\mathbf{n}}}\right) f^{X_{\mathbf{i}}}(y)\mathrm{d}y\right)\\
&= \alpha - \frac{1}{EK_{\mathbf{i}}}E\left(K_{\mathbf{i}}\int_{\mathbb{R}} b_{\mathbf{n}}{}^{(-1)}K_2^{(1)}\left(\frac{q_\alpha - y}{b_{\mathbf{n}}}\right) F^{X_{\mathbf{i}}}(y)\mathrm{d}y\right)\\
&= \alpha - \frac{1}{EK_{\mathbf{i}}}E\left(K_{\mathbf{i}}\int_{\mathbb{R}} K_2^{(1)}(t)F^{X_{\mathbf{i}}}\left(q_\alpha - b_{\mathbf{n}}t\right)\mathrm{d}t\right)\\
&= \alpha + \beta_1 + \beta_2,
\end{aligned}$$

where

$$\beta_1 = -\frac{1}{EK_{\mathbf{i}}} E\left(K_{\mathbf{i}} \int_{\mathbb{R}} K_2^{(1)}(t) F^{X_{\mathbf{i}}}(q_\alpha) \mathrm{d}t\right) = -\frac{1}{EK_{\mathbf{i}}} E\left(K_1\left(\frac{d(x,X)}{a_{\mathbf{n}}}\right) F^X(q_\alpha)\right)$$

and

$$\begin{aligned}\beta_2 &= \frac{1}{EK_{\mathbf{i}}} E\left(K_{\mathbf{i}} \int_{\mathbb{R}} K_2^{(1)}(t) \left[F^{X_{\mathbf{i}}}(q_\alpha) - F^{X_{\mathbf{i}}}(q_\alpha - b_{\mathbf{n}} t)\right] \mathrm{d}t\right) \\ &\leq \frac{1}{EK_{\mathbf{i}}} E\left(K_{\mathbf{i}} \int_{\mathbb{R}} K_2^{(1)}(t) \left|F^{X_{\mathbf{i}}}(q_\alpha) - F^{X_{\mathbf{i}}}(q_\alpha - b_{\mathbf{n}} t)\right| \mathrm{d}t\right) \\ &\leq C \frac{1}{EK_{\mathbf{i}}} E\left(K_{\mathbf{i}} (b_{\mathbf{n}})^{b_2} \int_{\mathbb{R}} |t|^{b_2} K_2^{(1)}(t)\right) \mathrm{d}t \leq C (b_{\mathbf{n}})^{b_2}.\end{aligned}$$

This yields the proof of the first result of the lemma. The following result ends the proof of the second result

$$\begin{aligned}\beta_1 &= \frac{1}{EK_{\mathbf{i}}} E\left(K_1\left(\frac{d(x,X)}{a_{\mathbf{n}}}\right) \left[F^x(q_\alpha) - F^X(q_\alpha)\right]\right) - \\ &\quad - \frac{1}{EK_{\mathbf{i}}} E\left(K_1\left(\frac{d(x,X)}{a_{\mathbf{n}}}\right) F^x(q_\alpha)\right) \\ &\leq \frac{1}{EK_{\mathbf{i}}} E\left(K_1\left(\frac{d(x,X)}{a_{\mathbf{n}}}\right) \left|F^x(q_\alpha) - F^X(q_\alpha)\right|\right) - \alpha \\ &\leq C (a_{\mathbf{n}})^{b_1} \frac{1}{EK_{\mathbf{i}}} E\left(K_1\left(\frac{d(x,X)}{a_{\mathbf{n}}}\right)\right) - \alpha \leq C (a_{\mathbf{n}})^{b_1} - \alpha.\end{aligned}$$

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Received 14 April 2011

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