SOME VECTOR-VALUED SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION

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In the present paper we introduce some vector-valued sequence spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$. We also study some topological properties and some inclusion relations between these spaces.

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1. INTRODUCTION

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [15] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \right\}$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \right\}.$$

It is shown in [15] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$, for all values of $x \geq 0$, and for $L > 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)\mathrm{d}t,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

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A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function, see ([18], [26]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \big\{x \in w : I_{\mathcal{M}}(cx) < \infty, \text{ for some } c > 0\big\},$$
$$h_{\mathcal{M}} = \big\{x \in w : I_{\mathcal{M}}(cx) < \infty, \text{ for all } c > 0\big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

The Cesaro summable sequence spaces defined by an Orlicz function were studied by Parashar and Choudhary [27], Bhardwaj and Singh [1], Mursaleen et al. [23], Et et. al. [6] and many others.

Let q_1 and q_2 be seminorms on a vector space X. Then, q_1 is said to be stronger than q_2 if whenever (x_n) is a sequence such that $q_1(x_n) \to 0$, then $q_2(x_n) \to 0$ also. If each is stronger than the others q_1 and q_2 are said to be equivalent see [38].

Let l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers. Throughout the paper, $w(X)$, $c(X)$, $c_0(X)$ and $l_\infty(X)$ will represent the spaces of all, convergent, null and bounded X valued sequence spaces. For $X = \mathbb{C}$, the field of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\theta = (0, 0, \ldots, 0)$, where θ is the zero element of X.

The studies on vector valued sequence spaces are done by Rath and Srivastava [28], Das and Choudhary [5], Leonard [14], Srivastava and Srivastava [36], Tripathy and Sen [37], Et et. al. [6] and many others.

Let $u = (u_k)$ be a sequences of non-zero scalar. Then, for a sequence space E, the multiplier sequence space $E(u)$, associated with the multiplier sequence u is defined as

$$E(u) = \{(x_k) \in w : (u_k x_k) \in E\}.$$

The studies on the multiplier sequence spaces are done by Çolak [4], Srivastava and Srivastava [36] and many others.

The notion of difference sequence spaces was introduced by Kızmaz [12], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let m, n be non-negative integers, then, for $Z = l_\infty$, c and c_0, we have sequence spaces,

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+n})$ and $\Delta_n^0 x_k = x_k$, for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [12].

A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r)$ and $q_r = \frac{k_r}{k_{r-1}}$. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [9] as

$$N_\theta = \left\{ x \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \text{ for some } l \right\}.$$

The space N_θ is a BK-space with norm

$$\|x\|_\theta = \sup_r \left(h_r^{-1} \sum_{k \in I_r} |(x_k)| \right).$$

Freedman et al. [9] also gave some relation between N_θ and the space $|\sigma_1|$ of strongly Cesaro summable sequences, which is defined by

$$|\sigma_1| = \left\{ x = (x_k) : \lim_{n\to\infty} \sum_{k=1}^{n} |x_k - l| = 0, \text{ for some } l \right\}.$$

Strongly almost convergent sequence was introduced and studied by Maddox [16] and Freedman [9]. Parashar and Choudhary [27] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence spaces $[C, 1, p]$, $[C, 1, p]_0$ and $[C, 1, p]_\infty$. It may be noted here that the space of strongly summable sequences were discussed by Maddox [17]. Subsequently, difference sequence spaces have been discussed by several authors see ([2], [20], [21], [22], [23], [29], [30], [31], [32]) and references therein.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$;

(2) $p(-x) = p(x)$, for all $x \in X$;

(3) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$;

(4) if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then, $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [38], Theorem 10.4.2, p. 183).

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive reals such that $u_k \neq 0$ for all k, X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q_k, for each $k \in \mathbb{N}$. We define the following sequence spaces in the present paper:

$$(w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) =$$
$$= \Big\{ x = (x_k) \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(\frac{u_k \Delta_n^m x_k}{\rho} \Big)\Big)\Big]^{p_k} \to 0,$$
$$\text{as } r \to \infty, \text{ for some } \rho > 0 \Big\},$$

$$(w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) =$$
$$= \Big\{ x = (x_k) \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(\frac{u_k \Delta_n^m x_k - l}{\rho} \Big)\Big)\Big]^{p_k} \to 0,$$
$$\text{as } r \to \infty, \text{ for some } \rho > 0 \text{ and } l \in X \Big\},$$

and

$$(w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) =$$
$$= \Big\{ x = (x_k) \in w(X) : \sup_r \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(\frac{u_k \Delta_n^m x_k}{\rho} \Big)\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0 \Big\}.$$

If we take $\mathcal{M}(x) = x$, we get

$$(w_0, \theta, \Delta_n^m, Q, u, p) = \Big\{x = (x_k) \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big(q_k\Big(\frac{u_k\Delta_n^m x_k}{\rho}\Big)\Big)^{p_k} \to 0, \text{ as } r \to \infty, \text{ for some } \rho > 0\Big\},$$

$$(w, \theta, \Delta_n^m, Q, u, p) = \Big\{x = (x_k) \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big(q_k\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big)\Big)^{p_k} \to 0, \text{ as } r \to \infty, \text{ for some } \rho > 0 \text{ and } l \in X\Big\},$$

and

$$(w_\infty, \theta, \Delta_n^m, Q, u, p) = \\ = \Big\{x = (x_k) \in w(X) : \sup_r \frac{1}{h_r} \sum_{k\in I_r} \Big(q_k\Big(\frac{u_k\Delta_n^m x_k}{\rho}\Big)\Big)^{p_k} < \infty, \text{for some } \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$, we have

$$(w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u) = \Big\{x = (x_k) \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k}{\rho}\Big)\Big) \to 0, \text{ as } r \to \infty, \text{ for some } \rho > 0\Big\},$$

$$(w, \mathcal{M}, \theta, \Delta_n^m, Q, u) = \Big\{x = (x_k) \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big)\Big) \to 0, \text{ as } r \to \infty, \text{ for some } \rho > 0 \text{ and } l \in X\Big\},$$

and

$$(w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u) = \\ = \Big\{x = (x_k) \in w(X) : \sup_r \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k}{\rho}\Big)\Big) < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$ and $u = e = (1, 1, 1, \ldots, 1)$ then, these spaces reduces to

$$(w_0, \theta, \Delta_n^m, Q, p) = \Big\{x = (x_k) \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big(q_k\Big(\frac{\Delta_n^m x_k}{\rho}\Big)\Big)^{p_k} \to 0, \text{ as } r \to \infty, \text{ for some } \rho > 0\Big\},$$

$$(w,\theta,\Delta_n^m,Q,p)=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big(q_k\Big(\frac{\Delta_n^m x_k-l}{\rho}\Big)\Big)^{p_k}\to 0,$$
$$\text{as } r\to\infty, \text{ for some } \rho>0 \text{ and } l\in X\Big\},$$

and

$$(w_\infty,\theta,\Delta_n^m,Q,p)=$$
$$=\Big\{x=(x_k)\in w(X):\sup_r\frac{1}{h_r}\sum_{k\in I_r}\Big(q_k\Big(\frac{\Delta_n^m x_k}{\rho}\Big)\Big)^{p_k}<\infty, \text{ for some } \rho>0\Big\}.$$

The following inequality will be used throughout the paper. If $0\le p_k\le \sup p_k=H$, $K=\max(1,2^{H-1})$ then,

$$|a_k+b_k|^{p_k}\le K\{|a_k|^{p_k}+|b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k,b_k\in\mathbb{C}$. Also, $|a|^{p_k}\le\max(1,|a|^H)$, for all $a\in\mathbb{C}$.

The main motive of this paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

2. MAIN RESULTS

THEOREM 2.1. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of strictly positive real numbers then, the classes of sequences $(w_0,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$, $(w,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$ and $(w_\infty,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x=(x_k),y=(y_k)\in(w_0,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$ and $\alpha,\beta\in\mathbb{C}$. In order to prove the result we need to find some ρ_3 such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m(\alpha x_k+\beta y_k)}{\rho_3}\Big)\Big)\Big]^{p_k}=0.$$

Since $x=(x_k),y=(y_k)\in(w_0,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$, there exist positive numbers $\rho_1,\rho_2>0$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k}{\rho_1}\Big)\Big)\Big]^{p_k}=0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m y_k}{\rho_2}\Big)\Big)\Big]^{p_k}=0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_k is non-decreasing convex function and so by using inequality (1), we have

$$
\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m(\alpha x_k+\beta y_k)}{\rho_3}\Big)\Big)\Big]^{p_k}\\
&\le \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(\frac{\alpha u_k\Delta_n^m x_k}{\rho_3}+\frac{\beta u_k\Delta_n^m y_k}{\rho_3}\Big)\Big)\Big]^{p_k}\\
&\le K\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k}{\rho_1}\Big)\Big)\Big]^{p_k}+K\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m y_k}{\rho_2}\Big)\Big)\Big]^{p_k}\\
&\le K\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k}{\rho_1}\Big)\Big)\Big]^{p_k}+K\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m y_k}{\rho_2}\Big)\Big)\Big]^{p_k}\\
&\to 0 \text{ as } r\to\infty.
\end{aligned}
$$

Thus, we have $\alpha x+\beta y\in(w_0,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$. Hence, $(w_0,\mathcal{M},\theta,\Delta_n^m,$ $Q,u,p)$ is a linear space. Similarly, we can prove that $(w,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$ and $(w_\infty,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$ are linear spaces. □

THEOREM 2.2. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then, $(w_0,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$ is a topological linear space paranormed by*

$$
g(x)=\sum_{i=1}^m q(x_i)+\inf_{\rho>0,\,s\ge1}\Big\{\rho^{\frac{p_s}{H}}:\sup_k\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k}{\rho}\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\le 1\Big\},
$$

where $H=\max(1,\sup_k p_k)<\infty$.

Proof. (i) Clearly, $g(x)\ge 0$, for $x=(x_k)\in(w_0,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$. Since $M_k(0)=0$, we get $g(0)=0$.

(ii) $g(-x)=g(x)$.

(iii) Let $x=(x_k), y=(y_k)\in(w_0,\mathcal{M},\theta,\Delta_n^m,Q,u,p)$ then, there exist $\rho_1,\rho_2>0$ such that

$$
\sup_k\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\big(\frac{u_k\Delta_n^m x_k}{\rho_1}\big)\Big)\Big]^{p_k}\Big)\le 1
$$

and

$$
\sup_k\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q_k\big(\frac{u_k\Delta_n^m y_k}{\rho_2}\big)\Big)\Big]^{p_k}\Big)\le 1.
$$

Let $\rho = \rho_1 + \rho_2$, then by Minkowski's inequality, we have

$$\begin{aligned}
&\sup_k \Big(\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_n^m (x_k + y_k)}{\rho} \big) \Big) \Big]^{p_k} \Big) \\
&= \sup_k \Big(\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_n^m (x_k + y_k)}{\rho_1 + \rho_2} \big) \Big) \Big]^{p_k} \Big) \\
&\le \Big(\frac{\rho_1}{\rho_1 + \rho_2} \Big) \sup_k \Big(\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_n^m x_k}{\rho_1} \big) \Big) \Big]^{p_k} \Big) \\
&+ \Big(\frac{\rho_2}{\rho_1 + \rho_2} \Big) \sup_k \Big(\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_n^m y_k}{\rho_2} \big) \Big) \Big]^{p_k} \Big)
\end{aligned}$$

and thus,

$$\begin{aligned}
&g(x+y) = \sum_{i=1}^{m} q(x_i + y_i) \\
&+ \inf \Big\{ (\rho_1 + \rho_2)^{\frac{p_s}{H}} : \sup_n \Big(\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_m^n x_k + u_k \Delta_m^n y_k}{\rho} \big) \Big) \Big]^{p_k} \Big)^{\frac{1}{H}} \\
&\le 1, \ \rho > 0 \Big\} \\
&\le \sum_{i=1}^{m} q(x_i) + \inf \Big\{ (\rho_1)^{\frac{p_s}{H}} : \sup_n \Big(\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_m^n x_k}{\rho_1} \big) \Big) \Big]^{p_k} \Big)^{\frac{1}{H}} \\
&\le 1, \ \rho_1 > 0 \Big\} \\
&+ \sum_{i=1}^{m} q(y_i) + \inf \Big\{ (\rho_2)^{\frac{p_s}{H}} : \sup_n \Big(\frac{1}{n} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_m^n y_k}{\rho_2} \big) \Big) \Big]^{p_k} \Big)^{\frac{1}{H}} \\
&\le 1, \ \rho_2 > 0 \Big\} \le g(x) + g(y).
\end{aligned}$$

(iv) Finally, we prove that scalar multiplication is continuous. Let λ be any complex number by definition

$$\begin{aligned}
g(\lambda x) &= \sum_{i=1}^{m} q(\lambda x_i) + \\
&+ \inf \Big\{ (\rho)^{\frac{p_s}{H}} : \sup_n \Big(\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_m^n \lambda x_k}{\rho} \big) \Big) \Big]^{p_k} \Big)^{\frac{1}{H}} \le 1, \ \rho > 0 \Big\}
\end{aligned}$$

$$= |\lambda| \sum_{i=1}^{m} q(x_i) +$$

$$+ \inf \Big\{ (|\lambda| r)^{\frac{p_s}{H}} : \sup_n \Big(\frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \big(\frac{u_k \Delta_m^n x_k}{t} \big) \Big) \Big]^{p_k} \Big)^{\frac{1}{H}} \leq 1,\ \rho > 0 \Big\},$$

where $t = \frac{\rho}{|\lambda|}$. Hence, $(w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$ is a paranormed space. □

THEOREM 2.3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$, for all fixed $x > 0$, then*

$$(w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) \subseteq (w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p).$$

Proof. Let $x = (x_k) \in (w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$, then, there exists positive number ρ_1 such that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \Big(\frac{u_k \Delta_n^m x_k}{\rho_1} \Big) \Big) \Big]^{p_k} = 0.$$

Define $\rho = 2\rho_1$. Since M_k is non-decreasing and convex and so, by using inequality (1), we have

$$\begin{aligned}
&\sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \Big(\frac{u_k \Delta_n^m x_k}{\rho} \Big) \Big) \Big]^{p_k} \\
&= \sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \Big(\frac{u_k \Delta_n^m x_k + L - L}{\rho} \Big) \Big) \Big]^{p_k} \\
&\leq K \sup_r \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} \Big[M_k \Big(q_k \Big(\frac{u_k \Delta_n^m x_k - L}{\rho_1} \Big) \Big) \Big]^{p_k} \\
&+ K \sup_r \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} \Big[M_k \Big(q_k \Big(\frac{L}{\rho_1} \Big) \Big) \Big]^{p_k} \\
&\leq K \sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \Big(\frac{u_k \Delta_n^m x_k - L}{\rho_1} \Big) \Big) \Big]^{p_k} \\
&+ K \sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(q_k \Big(\frac{L}{\rho_1} \Big) \Big) \Big]^{p_k} < \infty.
\end{aligned}$$

Hence, $x = (x_k) \in (w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$. □

THEOREM 2.4. *Let $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ and $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ be two Musielak-Orlicz functions satisfying Δ_2-condition, then we have*

(i) $(w_0, \mathcal{M}', \theta, \Delta_n^m, Q, u, p) \subset (w_0, \mathcal{M} \circ \mathcal{M}', \theta, \Delta_n^m, Q, u, p)$;

(ii) $(w, \mathcal{M}', \theta, \Delta_n^m, Q, u, p) \subset (w, \mathcal{M} \circ \mathcal{M}', \theta, \Delta_n^m, Q, u, p)$;

(iii) $(w_\infty, \mathcal{M}', \theta, \Delta_n^m, Q, u, p) \subset (w_\infty, \mathcal{M} \circ \mathcal{M}', \theta, \Delta_n^m, Q, u, p)$.

Proof. Let $x = (x_k) \in (w_0, \mathcal{M}', \theta, \Delta_n^m, Q, u, p)$ then, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k'\Big(q_k\Big(\frac{u_k \Delta_n^m x_k}{\rho}\Big)\Big)\Big]^{p_k} = 0.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$, for $0 \le t \le \delta$. Let $y_k = M_k'\Big(q_k\Big(\frac{u_k \Delta_n^m x_k}{\rho}\Big)\Big)$, for all $k \in \mathbb{N}$. We can write

$$\frac{1}{h_r} \sum_{k\in I_r} M_k[y_k]^{p_k} = \frac{1}{h_r} \sum_{k\in I_r, y_k \le \delta} M_k[y_k]^{p_k} + \frac{1}{h_r} \sum_{k\in I_r, y_k \ge \delta} M_k[y_k]^{p_k}.$$

So, we have

$$\begin{aligned}(2.1)\qquad \frac{1}{h_r} \sum_{k\in I_r, y_k\le\delta} M_k[y_k]^{p_k} &\le [M_k(1)]^H \frac{1}{h_r} \sum_{k\in I_r, y_k\le\delta} M_k[y_k]^{p_k} \\ &\le [M_k(2)]^H \frac{1}{h_r} \sum_{k\in I_r, y_k\le\delta} M_k[y_k]^{p_k}.\end{aligned}$$

For $y_k > \delta$, $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Since $M_k's$ are non-decreasing and convex, it follows that

$$M_k(y_k) < M_k\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2} M_k(2) + \frac{1}{2} M_k\left(\frac{2y_k}{\delta}\right).$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we can write

$$M_k(y_k) < \frac{1}{2} T \frac{y_k}{\delta} M_k(2) + \frac{1}{2} T \frac{y_k}{\delta} M_k(2) = T \frac{y_k}{\delta} M_k(2).$$

Hence,

$$(2.2)\qquad \frac{1}{h_r} \sum_{k\in I_r, y_k\ge\delta} M_k[y_k]^{p_k} \le \max\left(1, \left(T \frac{M_k(2)}{\delta}\right)^H\right) \frac{1}{h_r} \sum_{k\in I_r, y_k\ge\delta} [y_k]^{p_k}.$$

From equation (2) and (3), we have $x = (x_k) \in (w_0, \mathcal{M} \circ \mathcal{M}', \theta, \Delta_n^m, Q, u, p)$. This completes the proof of (i). Similarly, we can prove that

$$(w, \mathcal{M}', \theta, \Delta_n^m, Q, u, p) \subset (w, \mathcal{M} \circ \mathcal{M}', \theta, \Delta_n^m, Q, u, p)$$

and

$$(w_\infty, \mathcal{M}', \theta, \Delta_n^m, Q, u, p) \subset (w_\infty, \mathcal{M} \circ \mathcal{M}', \theta, \Delta_n^m, Q, u, p). \quad \square$$

THEOREM 2.5. *Let* $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. *Then, for a Musielak-Orlicz function* $\mathcal{M} = (M_k)$ *which satisfies* Δ_2*-condition, we have*

(i) $(w_0, \theta, \Delta_n^m, Q, u, p) \subset (w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$;

(ii) $(w, \theta, \Delta_n^m, Q, u, p) \subset (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$;

(iii) $(w_\infty, \theta, \Delta_n^m, Q, u, p) \subset (w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$.

Proof. It is easy to prove so, we omit the details. $\square$

THEOREM 2.6. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $0 < h = \inf p_k$. Then, $(w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) \subset (w_0, \theta, \Delta_n^m, Q, u, p)$ if and only if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k(t)^{p_k} = \infty, \tag{2.3}$$

for some $t > 0$.

Proof. Let $(w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) \subset (w_0, \theta, \Delta_n^m, Q, u, p)$. Suppose that (4) does not hold. Therefore, there are subinterval $I_{r(j)}$ of the set of interval I_r and a number $t_0 > 0$, where

$$t_0 = q_k\Big(\frac{u_k \Delta_n^m x_k}{\rho}\Big) \quad \text{for all } k,$$

such that

$$\frac{1}{h_{r(j)}} \sum_{k\in I_{r(j)}} M_k(t_0)^{p_k} \le K < \infty, \quad m = 1, 2, 3, \ldots. \tag{2.4}$$

Let us define $x = (x_k)$ as follows

$$\Delta_n^m x_k = \begin{cases} \rho t_0 & k \in I_{r(j)}, \\ 0 & k \notin I_{r(j)}. \end{cases}$$

Thus, by (5), $x \in (w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$. But $x \notin (w_0, \theta, \Delta_n^m, Q, u, p)$. Hence, (4) must hold.

Conversely, suppose that (4) holds and let $x \in (w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$. Then, for each r,

$$\frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(\frac{u_k \Delta_n^m x_k}{\rho}\Big)\Big)\Big]^{p_k} \le K < \infty. \tag{2.5}$$

Suppose that $x \notin (w_0, \theta, \Delta_n^m, Q, u, p)$. Then, for some number $\epsilon > 0$, there is a number k_0 such that for a subinterval $I_{r(j)}$, of the set of interval I_r,

$$q_k\Big(\frac{u_k \Delta_n^m x_k}{\rho}\Big) > \epsilon, \quad \text{for } k \ge k_0.$$

From properties of sequence of Orlicz function, we obtain

$$M_k\Big(q_k\Big(\frac{u_k \Delta_n^m x_k}{\rho}\Big)\Big)^{p_k} \ge M_k(\epsilon)^{p_k}$$

which contradicts (4), by using (6). Hence, we get

$$(w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) \subset (w_0, \theta, \Delta_n^m, Q, u, p).$$

This completes the proof. $\square$

THEOREM 2.7. *Let* $0 \leq p_k \leq s_k$, *for all* k *and let* $(\frac{s_k}{p_k})$ *be bounded. Then,*

$$(w, \mathcal{M}, \theta, \Delta_n^m, Q, u, s) \subseteq (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p).$$

Proof. Let $x = (x_k) \in (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, s)$, write

$$t_k = M_k\Big(q_k\Big(\frac{u_k \Delta_n^m x_k - l}{\rho}\Big)\Big)^{s_k}$$

and $\mu_k = \frac{p_k}{s_k}$, for all $k \in \mathbb{N}$. Then, $0 < \mu_k \leq 1$, for all $k \in \mathbb{N}$. Take $0 < \mu \leq \mu_k$, for $k \in \mathbb{N}$. Define sequences (u_k) and (v_k) as follows:

For $t_k \geq 1$, let $u_k = t_k$ and $v_k = 0$ and, for $t_k < 1$, let $u_k = 0$ and $v_k = t_k$. Then, clearly, for all $k \in \mathbb{N}$, we have

$$t_k = u_k + v_k, \quad t_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}$$

Now, it follows that $u_k^{\mu_k} \leq u_k \leq t_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Therefore,

$$\frac{1}{h_r} \sum_{k \in I_r} t_k^{\mu_k} = \frac{1}{h_r} \sum_{k \in I_r} (u_k^{\mu_k} + v_k^{\mu_k}) \leq \frac{1}{h_r} \sum_{k \in I_r} t_k + \frac{1}{h_r} \sum_{k \in I_r} v_k^{\mu}.$$

Now, for each k,

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} v_k^{\mu} &= \sum_{k \in I_r} \Big(\frac{1}{h_r} v_k\Big)^{\mu} \Big(\frac{1}{h_r}\Big)^{1-\mu} \\ &\leq \Big(\sum_{k \in I_r} \Big[\Big(\frac{1}{h_r} v_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu} \Big(\sum_{k \in I_r} \Big[\Big(\frac{1}{h_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu} = \Big(\frac{1}{h_r} \sum_{k \in I_r} v_k\Big)^{\mu} \end{aligned}$$

and so,

$$\frac{1}{h_r} \sum_{k \in I_r} t_k^{\mu_k} \leq \frac{1}{h_r} \sum_{k \in I_r} t_k + \Big(\frac{1}{h_r} \sum_{k \in I_r} v_k\Big)^{\mu}.$$

Hence, $x = (x_k) \in (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$. This completes the proof of the theorem. □

THEOREM 2.8. (i) *If* $0 < \inf p_k \leq p_k \leq 1$, *for all* $k \in \mathbb{N}$, *then*

$$(w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) \subseteq (w, \mathcal{M}, \theta, \Delta_n^m, Q, u).$$

(ii) *If* $1 \leq p_k \leq \sup p_k = H < \infty$, *for all* $k \in \mathbb{N}$, *then*

$$(w, \mathcal{M}, \theta, \Delta_n^m, Q, u) \subseteq (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p).$$

Proof. (i) Let $x = (x_k) \in (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$, then

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(q_k\Big(\frac{u_k \Delta_n^m x_k - L}{\rho}\Big)\Big)\Big]^{p_k} = 0.$$

Since $0 < \inf p_k \leq p_k \leq 1$. This implies that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k - L}{\rho}\Big)\Big) \leq \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k - L}{\rho}\Big)\Big)^{p_k},$$

therefore,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k - L}{\rho}\Big)\Big) = 0.$$

Therefore,

$$(w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) \subseteq (w, \mathcal{M}, \theta, \Delta_n^m, Q, u).$$

(ii) Let $p_k \geq 1$, for each k and $\sup p_k < \infty$. Let $x = (x_k) \in (w, \mathcal{M}, \theta, \Delta_n^m, Q, u)$, then, for each $\rho > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k - L}{\rho}\Big)\Big)\Big] = 0 < 1.$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k - L}{\rho}\Big)\Big)\Big]^{p_k} &\leq \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} M_k\Big(q_k\Big(\frac{u_k\Delta_n^m x_k - L}{\rho}\Big)\Big) \\ &= 0 < 1.\end{aligned}$$

Therefore, $x = (x_k) \in (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$, for each $\rho > 0$. Hence,

$$(w, \mathcal{M}, \theta, \Delta_n^m, Q, u) \subseteq (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p).$$

This completes the proof of the theorem. □

THEOREM 2.9. *If* $0 < \inf p_k \leq p_k \leq \sup p_k = H < \infty$, *for all* $k \in \mathbb{N}$, *then*

$$(w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) = (w, \mathcal{M}, \theta, \Delta_n^m, Q, u).$$

Proof. It is easy to prove so, we omit the details. □

3. STATISTICAL CONVERGENCE

The notion of statistical convergence was introduced by Fast [8] and Schoenberg [35] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [10], Connor [3], Salat [33], Isik [11], Savaş [34], Malkosky and Savas [19], Kolk [13], Maddox [16], Tripathy and Sen [37], Mursaleen et. al. ([24], [25]) and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists: $\delta(E) = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k)$, where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

In this section, we introduce lacunary $\Delta_n^m u_q$-statistical convergent sequences and give some relations between lacunary $\Delta_n^m u_q$-statistical convergent sequences and $(w, \mathcal{M}, \theta, \Delta_n^m, q, u, p)$-summable sequences.

A sequence $x = (x_k)$ is said to be lacunary $\Delta_n^m u_q$-statistically convergent to l, if for every $\epsilon > 0 \lim\limits_r \frac{1}{h_r}\big|\big\{k \in I_r : q(u_k \Delta_n^m x_k - l) \geq \epsilon\big\}\big| = 0$. In this case, we write $x_k \to l\big(S_\theta\big(\Delta_n^m u_q\big)\big)$. The set of all lacunary $\Delta_n^m u_q$-statistically convergent sequences is denoted by $S_\theta\big(\Delta_n^m u_q\big)$.

Theorem 3.1. *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then, $(w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p) \subset S_\theta(\Delta_n^m u_q)$.*

Proof. Let $x \in (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$ and $\epsilon > 0$ be given. Then,

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big)\Big)\Big]^{p_k}\\
&\geq \frac{1}{h_r}\sum_{k\in I_r, q\left(\frac{u_k\Delta_n^m x_k - l}{\rho}\right)\geq\epsilon}\Big[M_k\Big(q\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big)\Big)\Big]^{p_k}\\
&\geq \frac{1}{h_r}\sum_{k\in I_r, q\left(\frac{u_k\Delta_n^m x_k - l}{\rho}\right)\geq\epsilon}\big[M_k(\epsilon)\big]^{p_k}\\
&\geq \frac{1}{h_r}\sum_{k\in I_r, q\left(\frac{u_k\Delta_n^m x_k - l}{\rho}\right)\geq\epsilon}\min\Big(\big[M_k(\epsilon)\big]^h, \big[M_k(\epsilon)\big]^H\Big)\\
&\geq \frac{1}{h_r}\Big|\Big\{k \in I_r : q\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big) \geq \epsilon\Big\}\Big|\min\Big(\big[M_k(\epsilon)\big]^h, \big[M_k(\epsilon)\big]^H\Big).
\end{aligned}$$

Hence, $x \in S_\theta(\Delta_n^m u_q)$. $\square$

Theorem 3.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then, $S_\theta(\Delta_n^m u_q) \subset (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$.*

Proof. Suppose that $M = (M_k)$ is bounded. Then, there exists an integer K such that $M_k(t) < K$, for all $t \geq 0$. Then,

$$
\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big)\Big)\Big]^{p_k}\\
&= \frac{1}{h_r}\sum_{k\in I_r, q\left(\frac{u_k\Delta_n^m x_k - l}{\rho}\right)\geq\epsilon}\Big[M_k\Big(q\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big)\Big)\Big]^{p_k}\\
&+ \frac{1}{h_r}\sum_{k\in I_r, q\left(\frac{u_k\Delta_n^m x_k - l}{\rho}\right)<\epsilon}\Big[M_k\Big(q\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big)\Big)\Big]^{p_k}\\
&\leq \frac{1}{h_r}\sum_{k\in I_r, q\left(\frac{u_k\Delta_n^m x_k - l}{\rho}\right)\geq\epsilon}\max(K^h, K^H) + \frac{1}{h_r}\sum_{k\in I_r, q\left(\frac{u_k\Delta_n^m x_k - l}{\rho}\right)<\epsilon}[M_k(\epsilon)]^{p_k}\\
&\leq \max(K^h, K^H)\frac{1}{h_r}\Big|\Big\{k\in I_r : q\Big(\frac{u_k\Delta_n^m x_k - l}{\rho}\Big)\geq\epsilon\Big\}\Big|\\
&+ \max\Big(\big[M_k(\epsilon)\big]^h, \big[M_k(\epsilon)\big]^H\Big).
\end{aligned}
$$

Hence, $x \in (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$. $\square$

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