SOME VECTOR-VALUED SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION

KULDIP RAJ and SUNIL K. SHARMA

In the present paper we introduce some vector-valued sequence spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$. We also study some topological properties and some inclusion relations between these spaces.

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1. INTRODUCTION

An Orlicz function $M$ is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [15] used the idea of Orlicz function to define the following sequence space. Let $w$ be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called an Orlicz sequence space. The space $\ell_M$ is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$ 

It is shown in [15] that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p$ ($p \geq 1$). The $\Delta_2$-condition is equivalent to $M(Lx) \leq kLM(x)$, for all values of $x \geq 0$, and for $L > 1$. An Orlicz function $M$ can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where $\eta$ is known as the kernel of $M$, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, $\eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

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A sequence $M = (M_k)$ of Orlicz function is called a Musielak-Orlicz function, see ([18], [26]). A sequence $N = (N_k)$ defined by

\[ N_k(v) = \sup \{|u - M_k(u)| : u \geq 0\}, \quad k = 1, 2, \ldots \]

is called the complementary function of a Musielak-Orlicz function $M$. For a given Musielak-Orlicz function $M$, the Musielak-Orlicz sequence space $t_M$ and its subspace $h_M$ are defined as follows

\[ t_M = \{ x \in w : I_M(cx) < \infty, \text{ for some } c > 0 \}, \]
\[ h_M = \{ x \in w : I_M(cx) < \infty, \text{ for all } c > 0 \}, \]

where $I_M$ is a convex modular defined by

\[ I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_M. \]

We consider $t_M$ equipped with the Luxemburg norm

\[ \|x\| = \inf \left\{ k > 0 : I_M\left(\frac{x}{k}\right) \leq 1 \right\} \]

or equipped with the Orlicz norm

\[ \|x\|^0 = \inf \left\{ \frac{1}{k} \left(1 + I_M(kx)\right) : k > 0 \right\}. \]

The Cesaro summable sequence spaces defined by an Orlicz function were studied by Parashar and Choudhary [27], Bhardwaj and Singh [1], Mursaleen et al. [23], Et et al. [6] and many others.

Let $q_1$ and $q_2$ be seminorms on a vector space $X$. Then, $q_1$ is said to be stronger than $q_2$ if whenever $(x_n)$ is a sequence such that $q_1(x_n) \to 0$, then $q_2(x_n) \to 0$ also. If each is stronger than the others $q_1$ and $q_2$ are said to be equivalent see [38].

Let $l_\infty$, $c$ and $c_0$ be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers. Throughout the paper, $w(X)$, $c(X)$, $c_0(X)$ and $l_\infty(X)$ will represent the spaces of all, convergent, null and bounded $X$ valued sequence spaces. For $X = \mathbb{C}$, the field of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\theta = (0, 0, \ldots, 0)$, where $\theta$ is the zero element of $X$.

The studies on vector valued sequence spaces are done by Rath and Srivastava [28], Das and Choudhary [5], Leonard [14], Srivastava and Srivastava [36], Tripathy and Sen [37], Et et. al. [6] and many others.

Let $u = (u_k)$ be a sequence of non-zero scalar. Then, for a sequence space $E$, the multiplier sequence space $E(u)$, associated with the multiplier
sequence \( u \) is defined as

\[ E(u) = \{ (x_k) \in w : (u_k x_k) \in E \} \]

The studies on the multiplier sequence spaces are done by Çolak [4], Srivastava and Srivastava [36] and many others.

The notion of difference sequence spaces was introduced by Kızmaz [12], who studied the difference sequence spaces \( l_\infty(\Delta), c(\Delta) \) and \( c_0(\Delta) \). The notion was further generalized by Et and Çolak [7] by introducing the spaces \( l_\infty(\Delta^n) \), \( c(\Delta^n) \) and \( c_0(\Delta^n) \). Let \( m, n \) be non-negative integers, then, for \( Z = l_\infty, c \) and \( c_0 \), we have sequence spaces,

\[ Z(\Delta^m) = \{ x = (x_k) \in w : (\Delta^m x_k) \in Z \}, \]

where \( \Delta^m x_k = (\Delta^m x_k) = (\Delta^m x_k - \Delta^{m-1} x_{k+n}) \) and \( \Delta^0 x_k = x_k \), for all \( k \in \mathbb{N} \), which is equivalent to the following binomial representation

\[ \Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}. \]

Taking \( m = n = 1 \), we get the spaces \( l_\infty(\Delta), c(\Delta) \) and \( c_0(\Delta) \) introduced and studied by Kızmaz [12].

A sequence of positive integers \( \theta = (k_r) \) is called lacunary if \( k_0 = 0 \), \( 0 < k_r < k_{r+1} \) and \( h_r = k_r - k_{r-1} \to \infty \) as \( r \to \infty \). The intervals determined by \( \theta \) will be denoted by \( I_r = (k_{r-1}, k_r) \) and \( q_r = \frac{k_r - k_{r-1}}{h_{r-1}} \). The space of lacunary strongly convergent sequences \( N_\theta \) was defined by Freedman et al. [9] as

\[ N_\theta = \left\{ x \in w : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \text{ for some } l \right\}. \]

The space \( N_\theta \) is a BK-space with norm

\[ \| x \|_\theta = \sup_{r} \left( \frac{1}{h_r} \sum_{k \in I_r} |x_k| \right). \]

Freedman et al. [9] also gave some relation between \( N_\theta \) and the space \( |\sigma_1| \) of strongly Cesaro summable sequences, which is defined by

\[ |\sigma_1| = \left\{ x = (x_k) : \lim_{n \to \infty} \sum_{k=1}^{n} |x_k - l| = 0, \text{ for some } l \right\}. \]

Strongly almost convergent sequence was introduced and studied by Maddox [16] and Freedman [9]. Parashar and Choudhary [27] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function \( M \), which generalized the well-known Orlicz sequence spaces \( [C, 1, p], [C, 1, p]_0 \) and \( [C, 1, p]_\infty \). It may be noted here that the space of strongly summable sequences were discussed by Maddox [17]. Subsequently,
difference sequence spaces have been discussed by several authors see ([2], [20], [21], [22], [23], [29], [30], [31], [32]) and references therein.

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$;
2. $p(-x) = p(x)$, for all $x \in X$;
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$;
4. if $(\sigma_n)$ is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then, $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [38], Theorem 10.4.2, p. 183).

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive reals such that $u_k \neq 0$ for all $k$, $X$ be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm $q_k$, for each $k \in \mathbb{N}$. We define the following sequence spaces in the present paper:

$$(w_0, \mathcal{M}, \theta, \Delta^m_n, Q, u, p) = \left\{ x = (x_k) \in w(X) : \lim_{r \to \infty} \frac{1}{H_r} \sum_{k \in I_r} [M_k(q_k(u_k \Delta^m_n x_k \rho))]^{p_k} \to 0, \right.$$

as $r \to \infty$, for some $\rho > 0$ $\left. \right\}$,

$$(w, \mathcal{M}, \theta, \Delta^m_n, Q, u, p) = \left\{ x = (x_k) \in w(X) : \lim_{r \to \infty} \frac{1}{H_r} \sum_{k \in I_r} [M_k(q_k(u_k \Delta^m_n x_k - l \rho))]^{p_k} \to 0, \right.$$

as $r \to \infty$, for some $\rho > 0$ and $l \in X \left. \right\}$,

and

$$(w_\infty, \mathcal{M}, \theta, \Delta^m_n, Q, u, p) = \left\{ x = (x_k) \in w(X) : \sup_{r} \frac{1}{H_r} \sum_{k \in I_r} [M_k(q_k(u_k \Delta^m_n x_k \rho))]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$
If we take $M(x) = x$, we get

$$(w_0, \theta, \Delta^m_n, Q, u, p) = \left\{ x = (x_k) \in w(X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left( q_k \left( \frac{u_k \Delta^m_n x_k}{\rho} \right) \right)^{p_k} \to 0, \right. \left. \text{ as } r \to \infty, \text{ for some } \rho > 0 \right\},$$

$$(w, \theta, \Delta^m_n, Q, u, p) = \left\{ x = (x_k) \in w(X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left( q_k \left( \frac{u_k \Delta^m_n x_k - l}{\rho} \right) \right)^{p_k} \to 0, \right. \left. \text{ as } r \to \infty, \text{ for some } \rho > 0 \text{ and } l \in X \right\},$$

and

$$(w_\infty, \theta, \Delta^m_n, Q, u, p) = \left\{ x = (x_k) \in w(X) : \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} \left( q_k \left( \frac{u_k \Delta^m_n x_k}{\rho} \right) \right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$, we have

$$(w_0, M, \theta, \Delta^m_n, Q, u) = \left\{ x = (x_k) \in w(X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k}{\rho} \right) \right) \to 0, \right. \left. \text{ as } r \to \infty, \text{ for some } \rho > 0 \right\},$$

$$(w, M, \theta, \Delta^m_n, Q, u) = \left\{ x = (x_k) \in w(X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - l}{\rho} \right) \right) \to 0, \right. \left. \text{ as } r \to \infty, \text{ for some } \rho > 0 \text{ and } l \in X \right\},$$

and

$$(w_\infty, M, \theta, \Delta^m_n, Q, u) = \left\{ x = (x_k) \in w(X) : \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k}{\rho} \right) \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $M(x) = x$ and $u = e = (1, 1, 1, \ldots, 1)$ then, these spaces reduces to

$$(w_0, \theta, \Delta^m_n, Q, p) = \left\{ x = (x_k) \in w(X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left( q_k \left( \frac{\Delta^m_n x_k}{\rho} \right) \right)^{p_k} \to 0, \right. \left. \text{ as } r \to \infty, \text{ for some } \rho > 0 \right\},$$
\((w, \theta, \Delta_n^m, Q, p) = \{ x = (x_k) \in w(X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left( q_k \left( \frac{\Delta_n^m x_k}{\rho} \right) \right)^{p_k} \to 0, \quad \text{as } r \to \infty, \text{ for some } \rho > 0 \text{ and } l \in X \}\),

and

\[(w_\infty, \theta, \Delta_n^m, Q, p) = \{ x = (x_k) \in w(X) : \sup_r \frac{1}{h_r} \sum_{k \in I_r} \left( q_k \left( \frac{\Delta_n^m x_k}{\rho} \right) \right)^{p_k} < \infty, \text{ for some } \rho > 0 \}\.

The following inequality will be used throughout the paper. If \(0 \leq p_k \leq \sup_p p_k = H\), \(K = \max(1, 2^{H-1})\) then,

\[|a_k + b_k|^{p_k} \leq K \{ |a_k|^{p_k} + |b_k|^{p_k} \}\]

for all \(k\) and \(a_k, b_k \in \mathbb{C}\). Also, \(|a|^{p_k} \leq \max(1, |a|^H)\), for all \(a \in \mathbb{C}\).

The main motive of this paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

2. MAIN RESULTS

**Theorem 2.1.** Let \(\mathcal{M} = (M_k)\) be a Musielak-Orlicz function, \(p = (p_k)\) be a bounded sequence of positive real numbers and \(u = (u_k)\) be any sequence of strictly positive real numbers then, the classes of sequences \((w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)\), \((w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)\) and \((w_\infty, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)\) are linear spaces over the field of complex number \(\mathbb{C}\).

**Proof.** Let \(x = (x_k), y = (y_k) \in (w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)\) and \(\alpha, \beta \in \mathbb{C}\). In order to prove the result we need to find some \(\rho_3\) such that

\[\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{\Delta_n^m (\alpha x_k + \beta y_k)}{\rho_3} \right) \right) \right]^{p_k} = 0.\]

Since \(x = (x_k), y = (y_k) \in (w_0, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)\), there exist positive numbers \(\rho_1, \rho_2 > 0\) such that

\[\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{\Delta_n^m x_k}{\rho_1} \right) \right) \right]^{p_k} = 0\]

and

\[\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{\Delta_n^m y_k}{\rho_2} \right) \right) \right]^{p_k} = 0.\]
Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M_k$ is non-decreasing convex function and so by using inequality (1), we have

$$
\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta_n^m (\alpha x_k + \beta y_k) \rho_3}{\rho_3} \right) \right) \right]^{p_k}
$$

$$
\leq \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{\alpha u_k \Delta_n^m x_k}{\rho_3} + \frac{\beta u_k \Delta_n^m y_k}{\rho_3} \right) \right) \right]^{p_k}
$$

$$
\leq K \frac{1}{h_r} \sum_{k \in I_r} \left[ \frac{1}{2^{p_k}} M_k \left( q_k \left( \frac{u_k \Delta_n^m x_k}{\rho_1} \right) \right) \right]^{p_k} + K \frac{1}{h_r} \sum_{k \in I_r} \left[ \frac{1}{2^{p_k}} M_k \left( q_k \left( \frac{u_k \Delta_n^m y_k}{\rho_2} \right) \right) \right]^{p_k}
$$

$$
\rightarrow 0 \text{ as } r \rightarrow \infty.
$$

Thus, we have $\alpha x + \beta y \in (w_0, M, \theta, \Delta_n^m, Q, u, p)$. Hence, $(w_0, M, \theta, \Delta_n^m, Q, u, p)$ is a linear space. Similarly, we can prove that $(w, M, \theta, \Delta_n^m, Q, u, p)$ and $(w_\infty, M, \theta, \Delta_n^m, Q, u, p)$ are linear spaces. □

**Theorem 2.2.** Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then, $(w_0, M, \theta, \Delta_n^m, Q, u, p)$ is a topological linear space paranormed by

$$
g(x) = \sum_{i=1}^{m} q(x_i) + \inf_{\rho > 0, s > 1} \left\{ \rho \sup_{\|x\|} \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta_n^m x_k}{\rho} \right) \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \leq 1 \right\},
$$

where $H = \max(1, \sup_k p_k) < \infty$.

**Proof.** (i) Clearly, $g(x) \geq 0$, for $x = (x_k) \in (w_0, M, \theta, \Delta_n^m, Q, u, p)$. Since $M_k(0) = 0$, we get $g(0) = 0$.

(ii) $g(-x) = g(x)$.

(iii) Let $x = (x_k), y = (y_k) \in (w_0, M, \theta, \Delta_n^m, Q, u, p)$ then, there exist $\rho_1, \rho_2 > 0$ such that

$$
\sup_k \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta_n^m x_k}{\rho_1} \right) \right) \right]^{p_k} \right) \leq 1
$$

and

$$
\sup_k \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta_n^m y_k}{\rho_2} \right) \right) \right]^{p_k} \right) \leq 1.
$$
Let $\rho = \rho_1 + \rho_2$, then by Minkowski’s inequality, we have

$$
\sup_k \left( \frac{1}{h} \sum_{k \in I} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n (x_k + y_k)}{\rho} \right) \right) \right]^{p_k} \right)
= \sup_k \left( \frac{1}{h} \sum_{k \in I} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n (x_k + y_k)}{\rho_1 + \rho_2} \right) \right) \right]^{p_k} \right)
\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_k \left( \frac{1}{h} \sum_{k \in I} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k}{\rho_1} \right) \right) \right]^{p_k} \right)
+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_k \left( \frac{1}{h} \sum_{k \in I} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n y_k}{\rho_2} \right) \right) \right]^{p_k} \right)
$$

and thus,

$$
g(x + y) = \sum_{i=1}^{m} q(x_i + y_i)
+ \inf \left\{ (\rho_1 + \rho_2)^{\frac{p}{n}} : \sup_n \left( \frac{1}{h} \sum_{k \in I} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k + u_k \Delta^m_n y_k}{\rho} \right) \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1, \enspace \rho > 0 \right\}
\leq \sum_{i=1}^{m} q(x_i) + \inf \left\{ (\rho_1)^{\frac{p}{n}} : \sup_n \left( \frac{1}{h} \sum_{k \in I} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k}{\rho_1} \right) \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1, \enspace \rho_1 > 0 \right\}
\leq g(x) + g(y).
$$

(iv) Finally, we prove that scalar multiplication is continuous. Let $\lambda$ be any complex number by definition

$$
g(\lambda x) = \sum_{i=1}^{m} q(\lambda x_i) +
+ \inf \left\{ (\rho)^{\frac{p}{n}} : \sup_n \left( \frac{1}{h} \sum_{k \in I} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n \lambda x_k}{\rho} \right) \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1, \enspace \rho > 0 \right\}.$$
\[= |\lambda| \sum_{i=1}^{m} q(x_i) +
\]
\[+ \inf \left\{ (|\lambda| r)^{\frac{m}{r}} : \sup_n \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k}{t} \right) \right) \right] \right)^{\frac{1}{r}} \leq 1, \ \rho > 0 \right\},
\]
where \( t = \frac{\rho}{|\lambda|} \). Hence, \((w_0, M, \theta, \Delta^m_n, Q, u, p)\) is a paranormed space. \(\square\)

**Theorem 2.3.** Let \( M = (M_k) \) be a Musielak-Orlicz function. If \( \sup_k [M_k(x)]^{p_k} < \infty \), for all fixed \( x > 0 \), then
\[(w_0, M, \theta, \Delta^m_n, Q, u, p) \subseteq (w_\infty, M, \theta, \Delta^m_n, Q, u, p)\).

**Proof.** Let \( x = (x_k) \in (w_0, M, \theta, \Delta^m_n, Q, u, p) \), then, there exists positive number \( \rho_1 \) such that
\[\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k}{\rho_1} \right) \right) \right]^{p_k} = 0.
\]
Define \( \rho = 2 \rho_1 \). Since \( M_k \) is non-decreasing and convex and so, by using inequality (1), we have
\[\sup_r \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k}{\rho} \right) \right) \right]^{p_k}
\]
\[= \sup_r \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k + L - L}{\rho} \right) \right) \right]^{p_k}
\]
\[\leq K \sup_r \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2p_k} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - L}{\rho_1} \right) \right) \right]^{p_k}
\]
\[+ K \sup_r \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2p_k} \left[ M_k \left( q_k \left( \frac{L}{\rho_1} \right) \right) \right]^{p_k}
\]
\[\leq K \sup_r \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - L}{\rho_1} \right) \right) \right]^{p_k}
\]
\[+ K \sup_r \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{L}{\rho_1} \right) \right) \right]^{p_k} < \infty.
\]
Hence, \( x = (x_k) \in (w_\infty, M, \theta, \Delta^m_n, Q, u, p) \). \(\square\)

**Theorem 2.4.** Let \( 0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty \) and \( M = (M_k), M' = (M'_k) \) be two Musielak-Orlicz functions satisfying \( \Delta_2 \)-condition, then we have
(i) \((w_0, M', \theta, \Delta^m_n, Q, u, p) \subseteq (w_0, M \circ M', \theta, \Delta^m_n, Q, u, p)\);
(ii) \((w, M', \theta, \Delta^m_n, Q, u, p) \subseteq (w, M \circ M', \theta, \Delta^m_n, Q, u, p)\);
Hence, we have

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k^r \left( \frac{y_k \Delta_m^x x_k}{\rho} \right) \right]^p = 0.
\]

Let \( \epsilon > 0 \) and choose \( \delta > 0 < \delta < 1 \) such that \( M_k(t) < \epsilon \), for \( 0 \leq t \leq \delta \).

Let \( y_k = M_k^r \left( \frac{u_k \Delta_m^x x_k}{\rho} \right) \), for all \( k \in \mathbb{N} \). We can write

\[
\frac{1}{h_r} \sum_{k \in I_r} M_k[y_k]^p = \frac{1}{h_r} \sum_{k \in I_r, y_k \leq \delta} M_k[y_k]^p + \frac{1}{h_r} \sum_{k \in I_r, y_k \geq \delta} M_k[y_k]^p.
\]

So, we have

\[
(2.1) \quad \frac{1}{h_r} \sum_{k \in I_r, y_k \leq \delta} M_k[y_k]^p \leq [M_k(1)]^H \frac{1}{h_r} \sum_{k \in I_r, y_k \leq \delta} M_k[y_k]^p
\]

\[
\leq [M_k(2)]^H \frac{1}{h_r} \sum_{k \in I_r, y_k \leq \delta} M_k[y_k]^p.
\]

For \( y_k > \delta \), \( y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta} \). Since \( M_k^r \)'s are non-decreasing and convex, it follows that

\[
M_k(y_k) < M_k \left( 1 + \frac{y_k}{\delta} \right) < \frac{1}{2} M_k(2) + \frac{1}{2} M_k \left( \frac{2y_k}{\delta} \right).
\]

Since \( \mathcal{M} = (M_k) \) satisfies \( \Delta_2 \)-condition, we can write

\[
M_k(y_k) < \frac{1}{2} T \frac{y_k}{\delta} M_k(2) + \frac{1}{2} T \frac{y_k}{\delta} M_k(2) = T \frac{y_k}{\delta} M_k(2).
\]

Hence,

\[
(2.2) \quad \frac{1}{h_r} \sum_{k \in I_r, y_k \geq \delta} M_k[y_k]^p \leq \max \left( 1, \left( T \frac{M_k(2)}{\delta} \right)^H \right) \frac{1}{h_r} \sum_{k \in I_r, y_k \geq \delta} [y_k]^p.
\]

From equation (2) and (3), we have \( x = (x_k) \in (w_0, \mathcal{M} \circ \mathcal{M'}, \theta, \Delta_m^x, Q, u, p) \).

This completes the proof of (i). Similarly, we can prove that

\[
(w, \mathcal{M}, \theta, \Delta_m^x, Q, u, p) \subset (w, \mathcal{M} \circ \mathcal{M'}, \theta, \Delta_m^x, Q, u, p)
\]

and

\[
(w_\infty, \mathcal{M}, \theta, \Delta_m^x, Q, u, p) \subset (w_\infty, \mathcal{M} \circ \mathcal{M'}, \theta, \Delta_m^x, Q, u, p).
\]

**Theorem 2.5.** Let \( 0 < h = \inf p_k = p_k < \sup p_k = H < \infty \). Then, for a Musielak-Orlicz function \( \mathcal{M} = (M_k) \) which satisfies \( \Delta_2 \)-condition, we have

(i) \((w_0, \theta, \Delta_m^x, Q, u, p) \subset (w_0, \mathcal{M}, \theta, \Delta_m^x, Q, u, p)\);

(ii) \((w, \theta, \Delta_m^x, Q, u, p) \subset (w, \mathcal{M}, \theta, \Delta_m^x, Q, u, p)\);

(iii) \((w_\infty, \theta, \Delta_m^x, Q, u, p) \subset (w_\infty, \mathcal{M}, \theta, \Delta_m^x, Q, u, p)\).
Theorem 2.6. Let $M = (M_k)$ be a Musielak-Orlicz function and $0 < h = \inf p_k$. Then, $(w_\infty, M, \theta, \Delta_n^m, Q, u, p) \subset (w_0, \theta, \Delta_n^m, Q, u, p)$ if and only if
\begin{equation}
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k(t)^{p_k} = \infty,
\end{equation}
for some $t > 0$.

Proof. Let $(w_\infty, M, \theta, \Delta_n^m, Q, u, p) \subset (w_0, \theta, \Delta_n^m, Q, u, p)$. Suppose that (4) does not hold. Therefore, there are subinterval $I_{r(j)}$ of the set of interval $I_r$ and a number $t_0 > 0$, where
\begin{equation}
t_0 = q_k \left( \frac{u_k \Delta_n^m x_k}{\rho} \right) \quad \text{for all } k,
\end{equation}
such that
\begin{equation}
\frac{1}{h_{r(j)}} \sum_{k \in I_{r(j)}} M_k(t_0)^{p_k} \leq K < \infty, \quad m = 1, 2, 3, \ldots.
\end{equation}

Let us define $x = (x_k)$ as follows
\begin{equation*}
\Delta_n^m x_k = \begin{cases}
\rho t_0 & k \in I_{r(j)}, \\
0 & k \notin I_{r(j)}.
\end{cases}
\end{equation*}
Thus, by (5), $x \in (w_\infty, M, \theta, \Delta_n^m, Q, u, p)$. But $x \notin (w_0, \theta, \Delta_n^m, Q, u, p)$. Hence, (4) must hold.

Conversely, suppose that (4) holds and let $x \in (w_\infty, M, \theta, \Delta_n^m, Q, u, p)$. Then, for each $r$,
\begin{equation}
\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta_n^m x_k}{\rho} \right) \right) \right]^{p_k} \leq K < \infty.
\end{equation}
Suppose that $x \notin (w_0, \theta, \Delta_n^m, Q, u, p)$. Then, for some number $\epsilon > 0$, there is a number $k_0$ such that for a subinterval $I_{r(j)}$, of the set of interval $I_r$,
\begin{equation*}
q_k \left( \frac{u_k \Delta_n^m x_k}{\rho} \right) > \epsilon, \quad \text{for } k \geq k_0.
\end{equation*}
From properties of sequence of Orlicz function, we obtain
\begin{equation*}
M_k \left( q_k \left( \frac{u_k \Delta_n^m x_k}{\rho} \right) \right)^{p_k} \geq M_k(\epsilon)^{p_k}
\end{equation*}
which contradicts (4), by using (6). Hence, we get
\begin{equation*}
(w_\infty, M, \theta, \Delta_n^m, Q, u, p) \subset (w_0, \theta, \Delta_n^m, Q, u, p).
\end{equation*}
This completes the proof.
Theorem 2.7. Let \( 0 \leq p_k \leq s_k \), for all \( k \) and let \( \frac{2s_k}{p_k} \) be bounded. Then,
\[
(w, \mathcal{M}, \theta, \Delta^m_n, Q, u, s) \subseteq (w, \mathcal{M}, \theta, \Delta^m_n, Q, u, p).
\]

Proof. Let \( x = (x_k) \in (w, \mathcal{M}, \theta, \Delta^m_n, Q, u, s) \), write
\[
t_k = M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - l}{\rho} \right)^s \right)
\]
and \( \mu_k = \frac{p_k}{s_k} \), for all \( k \in \mathbb{N} \). Then, \( 0 < \mu_k \leq 1 \), for all \( k \in \mathbb{N} \). Take \( 0 < \mu \leq \mu_k \), for \( k \in \mathbb{N} \). Define sequences \( (u_k) \) and \( (v_k) \) as follows:

For \( t_k \geq 1 \), let \( u_k = t_k \) and \( v_k = 0 \) and, for \( t_k < 1 \), let \( u_k = 0 \) and \( v_k = t_k \). Then, clearly, for all \( k \in \mathbb{N} \), we have
\[
t_k = u_k + v_k, \quad t_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}
\]
Now, it follows that \( u_k^{\mu_k} \leq u_k \leq t_k \) and \( v_k^{\mu_k} \leq v_k \). Therefore,
\[
\frac{1}{h_r} \sum_{k \in I_r} t_k^{\mu_k} = \frac{1}{h_r} \sum_{k \in I_r} (u_k^{\mu_k} + v_k^{\mu_k}) \leq \frac{1}{h_r} \sum_{k \in I_r} t_k + \frac{1}{h_r} \sum_{k \in I_r} v_k^{\mu_k}.
\]
Now, for each \( k \),
\[
\frac{1}{h_r} \sum_{k \in I_r} v_k^{\mu_k} = \sum_{k \in I_r} \left( \frac{1}{h_r} v_k \right)^\mu \left( \frac{1}{h_r} \right)^{1-\mu} \leq \left( \sum_{k \in I_r} \left( \frac{1}{h_r} v_k \right)^{\frac{\mu}{\mu}} \right)^\mu \left( \sum_{k \in I_r} \left( \frac{1}{h_r} \right)^{1-\mu} \right)^{1-\mu} = \left( \frac{1}{h_r} \sum_{k \in I_r} v_k \right)^\mu
\]
and so,
\[
\frac{1}{h_r} \sum_{k \in I_r} t_k^{\mu_k} \leq \frac{1}{h_r} \sum_{k \in I_r} t_k + \left( \frac{1}{h_r} \sum_{k \in I_r} v_k \right)^\mu.
\]
Hence, \( x = (x_k) \in (w, \mathcal{M}, \theta, \Delta^m_n, Q, u, p) \). This completes the proof of the theorem. \( \square \)

Theorem 2.8. (i) If \( 0 < \inf p_k \leq p_k \leq 1 \), for all \( k \in \mathbb{N} \), then
\[
(w, \mathcal{M}, \theta, \Delta^m_n, Q, u, p) \subseteq (w, \mathcal{M}, \theta, \Delta^m_n, Q, u).
\]

(ii) If \( 1 \leq p_k \leq \sup p_k = H < \infty \), for all \( k \in \mathbb{N} \), then
\[
(w, \mathcal{M}, \theta, \Delta^m_n, Q, u) \subseteq (w, \mathcal{M}, \theta, \Delta^m_n, Q, u, p).
\]

Proof. (i) Let \( x = (x_k) \in (w, \mathcal{M}, \theta, \Delta^m_n, Q, u, p) \), then
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - l}{\rho} \right) \right)^{p_k} \right] = 0.
\]
Since $0 < \inf p_k \leq p_k \leq 1$. This implies that
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - L}{\rho} \right) \right) \leq \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - L}{\rho} \right) \right)^{p_k},
\]
therefore,
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - L}{\rho} \right) \right) = 0.
\]

Therefore,
\[(w, \mathcal{M}, \theta, \Delta^m_n, Q, u) \subseteq (w, \mathcal{M}, \theta, \Delta^m_n, Q, u).
\]

(ii) Let $p_k \geq 1$, for each $k$ and $\sup p_k < \infty$. Let $x = (x_k) \in (w, \mathcal{M}, \theta, \Delta^m_n, Q, u)$, then, for each $\rho > 0$, we have
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - L}{\rho} \right) \right) \right] = 0 < 1.
\]

Since $1 \leq p_k \leq \sup p_k < \infty$, we have
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - L}{\rho} \right) \right) \right]^{p_k} \leq \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( q_k \left( \frac{u_k \Delta^m_n x_k - L}{\rho} \right) \right)
\]
\[= 0 < 1.
\]

Therefore, $x = (x_k) \in (w, \mathcal{M}, \theta, \Delta^m_n, Q, u)$, for each $\rho > 0$. Hence,
\[(w, \mathcal{M}, \theta, \Delta^m_n, Q, u) \subseteq (w, \mathcal{M}, \theta, \Delta^m_n, Q, u).
\]

This completes the proof of the theorem. □

**Theorem 2.9.** If $0 < \inf p_k \leq p_k \leq \sup p_k = H < \infty$, for all $k \in \mathbb{N}$, then

\[(w, \mathcal{M}, \theta, \Delta^m_n, Q, u) = (w, \mathcal{M}, \theta, \Delta^m_n, Q, u).
\]

**Proof.** It is easy to prove so, we omit the details. □

### 3. STATISTICAL CONVERGENCE

The notion of statistical convergence was introduced by Fast [8] and Schoenberg [35] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [10], Connor [3], Salat [33], Isik [11], Savaš [34], Malkosky and Savaš [19], Kolk [13], Maddox [16], Tripathy and Sen [37], Mursaleen et. al. ([24], [25]) and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure...
of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set \( \mathbb{N} \) of natural numbers.

A subset \( E \) of \( \mathbb{N} \) is said to have the natural density \( \delta(E) \) if the following limit exists: 
\[
\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),
\]
where \( \chi_E \) is the characteristic function of \( E \). It is clear that any finite subset of \( \mathbb{N} \) has zero natural density and \( \delta(E^c) = 1 - \delta(E) \).

In this section, we introduce lacunary \( \Delta^m \)-statistical convergent sequences and give some relations between lacunary \( \Delta^m \)-statistical convergent sequences and \( (w, M, \theta, \Delta^m, q, u, p) \)-summable sequences.

A sequence \( x = (x_k) \) is said to be lacunary \( \Delta^m \)-statistically convergent to \( l \), if for every \( \epsilon > 0 \) \( \lim \frac{1}{n} \left| \left\{ k \in I_r : q(u_k \Delta^m x_k - l) \geq \epsilon \right\} \right. \) = 0. In this case, we write \( x_k \to l(S_\theta(\Delta^m u_q)) \). The set of all lacunary \( \Delta^m \)-statistically convergent sequences is denoted by \( S_\theta(\Delta^m u_q) \).

**Theorem 3.1.** Let \( \mathcal{M} = (M_k) \) be Musielak-Orlicz function and \( 0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty \). Then, \( (w, \mathcal{M}, \theta, \Delta^m, Q, u, p) \subset S_\theta(\Delta^m u_q) \).

**Proof.** Let \( x \in (w, \mathcal{M}, \theta, \Delta^m, Q, u, p) \) and \( \epsilon > 0 \) be given. Then,
\[
\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{u_k \Delta^m x_k - l}{\rho} \right) \right) \right]^{p_k}
\geq \frac{1}{h_r} \sum_{k \in I_r, q \left( \frac{u_k \Delta^m x_k - l}{\rho} \right) \geq \epsilon} \left[ M_k \left( q \left( \frac{u_k \Delta^m x_k - l}{\rho} \right) \right) \right]^{p_k}
\geq \frac{1}{h_r} \sum_{k \in I_r, q \left( \frac{u_k \Delta^m x_k - l}{\rho} \right) \geq \epsilon} \left[ M_k(\epsilon) \right]^{p_k}
\geq \frac{1}{h_r} \sum_{k \in I_r, q \left( \frac{u_k \Delta^m x_k - l}{\rho} \right) \geq \epsilon} \min \left( \left[ M_k(\epsilon) \right]^h, \left[ M_k(\epsilon) \right]^H \right)
\geq \frac{1}{h_r} \left\{ k \in I_r : q \left( \frac{u_k \Delta^m x_k - l}{\rho} \right) \geq \epsilon \right\} \min \left( \left[ M_k(\epsilon) \right]^h, \left[ M_k(\epsilon) \right]^H \right).
\]

Hence, \( x \in S_\theta(\Delta^m u_q) \). \( \square \)

**Theorem 3.2.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function and \( 0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty \). Then, \( S_\theta(\Delta^m u_q) \subset (w, \mathcal{M}, \theta, \Delta^m, Q, u, p) \).
Proof. Suppose that $M = (M_k)$ is bounded. Then, there exists an integer $K$ such that $M_k(t) < K$, for all $t \geq 0$. Then,

$$\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta_m x_k - l}{\rho} \right) \right) \right]^{p_k}$$

$$= \frac{1}{h_r} \sum_{k \in I_r, q \left( \frac{u_k \Delta_m x_k - l}{\rho} \right) \geq \epsilon} \left[ M_k \left( q \left( \frac{u_k \Delta_m x_k - l}{\rho} \right) \right) \right]^{p_k}$$

$$+ \frac{1}{h_r} \sum_{k \in I_r, q \left( \frac{u_k \Delta_m x_k - l}{\rho} \right) < \epsilon} \left[ M_k \left( q \left( \frac{u_k \Delta_m x_k - l}{\rho} \right) \right) \right]^{p_k}$$

$$\leq \frac{1}{h_r} \sum_{k \in I_r, q \left( \frac{u_k \Delta_m x_k - l}{\rho} \right) \geq \epsilon} \text{max}(K^h, K^H) + \frac{1}{h_r} \sum_{k \in I_r, q \left( \frac{u_k \Delta_m x_k - l}{\rho} \right) < \epsilon} \left[ M_k(\epsilon) \right]^{p_k}$$

$$\leq \text{max}(K^h, K^H) \frac{1}{h_r} \left\{ k \in I_r : q \left( \frac{u_k \Delta_m x_k - l}{\rho} \right) \geq \epsilon \right\}$$

$$+ \text{max} \left( \left[ M_k(\epsilon) \right]^h, \left[ M_k(\epsilon) \right]^H \right).$$

Hence, $x \in (w, \mathcal{M}, \theta, \Delta_n^m, Q, u, p)$. □

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Shri Mata Vaishno Devi University
School of Mathematics
Katra-182220, J&K, India
kuldiraj68@gmail.com
sunilksharma42@yahoo.co.in