We study optimality conditions and generalized Mond-Weir duality for multiobjective fractional programming involving \( n \)-set functions which satisfy appropriate generalized \( V \)-type-I univexity conditions. Our results generalize to fractional programming those obtained by Preda et al. [6].

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1. INTRODUCTION

Problems of multiobjective optimization are widespread in mathematical modelling of real world systems for a very broad range of applications. In particular, several classes of multiobjective problems with set and \( n \)-set functions have been the subject of several papers in the last few decades.

For a historical survey of optimality conditions and duality for programming problems involving set and \( n \)-set functions the reader is referred to Stancu-Minasian and Preda’s review paper [8].

General theory for optimizing \( n \)-set functions was first developed by Morris [5] who, for fractions of a single set, obtained results that are similar to the standard mathematical programming problem. Corley [1] gave the concept of derivative of a real-valued \( n \)-set function and generalized the results of Morris [5] to \( n \)-set functions and established optimality conditions and Lagrangian duality.

Along the lines of Jeyakumar and Mond [3], and Mishra et al. [4], Preda et al. [6] defined new classes of \( n \)-set functions, called \((\rho, \rho')\)-\( V \)-univex type-I, \((\rho, \rho')\)-quasi-\( V \)-univex type-I, \((\rho, \rho')\)-pseudo-\( V \)-univex type-I, \((\rho, \rho')\)-quasi pseudo-\( V \)-univex type-I and \((\rho, \rho')\)-pseudo quasi-\( V \)-univex type-I. They obtained necessary and sufficient optimality criteria and some duality results.
Fractional programming have been the subject of many research papers, survey papers and books. Fractional programming models have been successfully developed for various branches of human activity and especially to economics.

In this paper we generalize to fractional programming the results obtained by Preda et al. [6]. We study optimality conditions and generalized Mond-Weir duality for multiobjective fractional programming involving n-set functions which satisfy appropriate generalized V-type-I univexity conditions.

In Section 2 of this paper we give some definitions introduced in [6] for multi-objective programming and state necessary optimality condition for a multi-objective optimization problem involving set-functions. In Sections 3 and 4 we study optimality conditions and generalized Mond-Weir duality for multiobjective fractional programming involving n-set functions which satisfy appropriate generalized V-type-I univexity conditions.

2. DEFINITIONS AND PRELIMINARIES

For any vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, we use the following notations: $x < y$ iff $x_i < y_i$, $i = 1, 2, \ldots, n$; $x \leq y$ iff $x_i \leq y_i$, $i = 1, 2, \ldots, n$; $x \leq y$ iff $x_i \leq y_i$, but $x \neq y$; $x \cdot y = \sum_{i=1}^{n} x_i y_i$.

For an arbitrary vector $x \in \mathbb{R}^n$ and a subset $J$ of the index set $\{1, 2, \ldots, n\}$, we denote by $x_J$ the vector with the components $x_j$, $j \in J$. Let $\mathbb{R}_+^n$ be the nonnegative orthant of $\mathbb{R}^n$.

Let $(X, \Gamma, \mu)$ be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ a separable space. For $h \in L_1(X, \Gamma, \mu)$ and $Z \in \Gamma$ with indicator (characteristic) function $I_Z \in L_\infty(X, \Gamma, \mu)$, the general integral $\int_Z h \, d\mu$ will be denoted by $\langle h, I_Z \rangle$.


A set function $\varphi : \Gamma \rightarrow \mathbb{R}$ is said to be differentiable at $T$ if there exists $D\varphi_T \in L_1(X, \Gamma, \mu)$, called the derivative of $\varphi$ at $T$, such that for each $S \in \Gamma$,

$$\varphi(S) = \varphi(T) + \langle D\varphi_T, I_S - I_T \rangle + \psi(S, T),$$

where $\psi : \Gamma \times \Gamma \rightarrow \mathbb{R}$ and has the property that $\psi(S, T)$ is $o(d(S, T))$, that is

$$\lim_{d(S, T) \to 0} \psi(S, T)/d(S, T) = 0,$$

and $d$ is a pseudometric on $\Gamma$.

A function $h : \Gamma^n \rightarrow \mathbb{R}$ is said to have a partial derivative at $S^0 = (S_1^0, \ldots, S_n^0)$ with respect to its $k$-th argument (1 \(\leq k \leq n\)), if the function

$$\varphi(S_k) = h(S_1^0, \ldots, S_{k-1}^0, S_k, S_{k+1}^0, \ldots, S_n^0)$$
has derivative $D\varphi_{S_k^0}$, and we define $D_k\varphi(S^0) = D\varphi_{S_k^0}$. If there exist all $D_k\varphi(S^0)$,
$1 \leq k \leq n$, then we put $D\varphi(S^0) = (D_1\varphi(S^0), \ldots, D_n\varphi(S^0))$. If $H : \Gamma^n \to \mathbb{R}^m$, $H = (H_1, \ldots, H_m)$, we put $D_kH(S^0) = (D_kH_1(S^0), \ldots, D_kH_m(S^0))$.

The function $h : \Gamma^n \to \mathbb{R}$ is differentiable at $S^0 \in \Gamma^n$ if there exist $Dh(S^0)$ and $\psi : \Gamma^n \times \Gamma^n \to \mathbb{R}$ such that

$$h(S) = h(S^0) + \sum_{k=1}^n (D_kh(S^0), I_{S_k^0} - I_{S_k^0}) + \psi(S, S^0),$$

where $\psi(S, S^0) = o(d(S, S^0))$, and $d$ is a pseudometric on $\Gamma^n$.

A vector valued set function $f = (f_1, \ldots, f_p) : \Gamma \to \mathbb{R}^p$ is differentiable
on $\Gamma$ if all component functions $f_i$, $1 \leq i \leq p$, are differentiable on $\Gamma$.

Consider the following $n$-set function multiobjective optimization problem

$$\text{(VP)} \quad \text{minimize } \{f(S) = (f_1(S), \ldots, f_p(S)) \mid g(S) \leq 0, S \in \Gamma^n\},$$

where $f : \Gamma^n \to \mathbb{R}^p$ and $g : \Gamma^n \to \mathbb{R}^m$ are differentiable $n$-set functions on $\Gamma^n$.

The problem is to find the collection of (properly) efficient sets defined below.

We denote $\mathcal{P} = \{S \in \Gamma^n \mid g(S) \leq 0\}$ the set of all feasible solutions of

problem (VP).

\textbf{Definition 2.1.} A set $S^0 \in \mathcal{P}$ is an efficient solution (Pareto solution) of

problem (VP) if there exists no $S \in \mathcal{P}$ such that $f(S) \leq f(S^0)$.

\textbf{Definition 2.2.} An efficient solution $S^0$ of (VP) is called properly efficient, if

there exists a positive number $M$ with the property that, if $f_i(S) < f_i(S^0)$ for any $i$ and for $S \in \mathcal{P}$, then $\frac{f_i(S) - f_i(S^0)}{f_j(S) - f_j(S^0)} \leq M$ for some $j$ for which $f_j(S) \leq f_j(S^0)$.

With respect to the constraints of the problem (VP), we consider the
partition $\{J_0, J_1, \ldots, J_k\}$ of the index set $\{1, 2, \ldots, m\}$, that is, $\bigcup_{s=0}^k J_s = \{1, 2, \ldots, m\}$, and for any $s \neq t$, we have $J_s \cap J_t = \emptyset$.

According to the partition $\{J_0, J_1, \ldots, J_k\}$ associated to problem (VP), we introduce the notation

$$\psi_i(S, \lambda, \beta_0) = f_i(S) + \lambda_{\beta_0}^T g_{\beta_0}(S)$$

for any $i$, $1 \leq i \leq p$, where $\lambda \in \mathbb{R}_+^m$ is a given vector. Also let the vectors
$ho = (\rho_1, \ldots, \rho_p) \in \mathbb{R}^p$, $\rho' = (\rho'_1, \ldots, \rho'_k) \in \mathbb{R}^k$, and the real numbers $\rho_0, \rho'_0 \in \mathbb{R}$.

The following definitions were introduced by Preda \textit{et al.} \cite{6} and extend similar concepts defined by Jeyakumar and Mond \cite{3} and Mishra \textit{et al.} \cite{4}.

\textbf{Definition 2.3.} We say that problem (VP) is $(\rho, \rho')$-$V$-univex type I at
$S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\lambda \in \mathbb{R}_+^m$, if there
exist positive real functions \( \alpha_1, \ldots, \alpha_p \) and \( \beta_1, \ldots, \beta_k \) defined on \( \Gamma^n \times \Gamma^n \), nonnegative functions \( b_0 \) and \( b_1 \), also defined on \( \Gamma^n \times \Gamma^n \), \( \varphi_0 : \mathbb{R} \to \mathbb{R} \), \( \varphi_1 : \mathbb{R} \to \mathbb{R} \), such that

\[
b_0(S, S^0) \varphi_0 \left[ \sum_{t=1}^{n} \left( D_t \psi_i(S^0, \lambda_j) - I_{S_t} - I_{S^0_t} \right) \right] \geq \\
\geq \alpha_i(S, S^0) \sum_{t=1}^{n} (D_t \psi_i(S^0, \lambda_j), I_{S_t} - I_{S^0_t}) + \rho_d^2(S, S^0)
\]

(2.1)

and

\[
-b_1(S, S^0) \varphi_1 \left[ \sum_{j \in J_s} \lambda_j g_j(S^0) \right] \geq \\
\geq \beta_s(S, S^0) \sum_{j \in J_s} \lambda_j \sum_{t=1}^{n} (D_t g_j(S^0), I_{S_t} - I_{S^0_t}) + \rho_d^2(S, S^0),
\]

(2.2)

for any \( S \in \mathcal{P} \), \( i \in \{1, \ldots, p\} \) and \( s \in \{1, \ldots, k\} \).

If (VP) is \((\rho, \rho')\)-\(V\)-univex type I at all \( S^0 \in \mathcal{P} \), according to the partition \( \{J_0, J_1, \ldots, J_k\} \) relative to \( \lambda \in \mathbb{R}^m_+ \), then we say that (VP) is \((\rho, \rho')\)-\(V\)-univex type I on \( \mathcal{P} \) according to the partition \( \{J_0, J_1, \ldots, J_k\} \) relative to \( \lambda \in \mathbb{R}^m_+ \).

If strict inequality holds in (2.1) (whenever \( S \neq S^0 \)), then we say that (VP) is \((\rho, \rho')\)-semi strictly \(V\)-univex type I at \( S^0 \) or on \( \mathcal{P} \), according to the partition \( \{J_0, J_1, \ldots, J_k\} \) relative to \( \lambda \in \mathbb{R}^m_+ \), depending on the case.

**Definition 2.4.** We say that problem (VP) is \((\rho_0, \rho_0')\)-quasi \(V\)-univex type I at \( S^0 \in \mathcal{P} \) according to the partition \( \{J_0, J_1, \ldots, J_k\} \) relative to \( \tau \in \mathbb{R}^p_+ \) and \( \lambda \in \mathbb{R}^m_+ \), if there exist positive real functions \( \alpha_1, \ldots, \alpha_p \) and \( \beta_1, \ldots, \beta_k \) defined on \( \Gamma^n \times \Gamma^n \), nonnegative functions \( b_0 \) and \( b_1 \), also defined on \( \Gamma^n \times \Gamma^n \), \( \varphi_0 : \mathbb{R} \to \mathbb{R} \), \( \varphi_1 : \mathbb{R} \to \mathbb{R} \), such that the implications

\[
b_0(S, S^0) \varphi_0 \left[ \sum_{i=1}^{p} \tau_i \alpha_i(S, S^0)[\psi_i(S, \lambda_j) - \psi_i(S^0, \lambda_j)] \right] \leq 0
\]

(2.3)

\[
\Rightarrow \sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} (D_t \psi_i(S^0, \lambda_j), I_{S_t} - I_{S^0_t}) \leq -\rho_0 d^2(S, S^0), \quad \forall S \in \mathcal{P}
\]

and

\[
b_1(S, S^0) \varphi_1 \left[ \sum_{s=1}^{k} \beta_s(S, S^0) \sum_{j \in J_s} \lambda_j g_j(S^0) \right] \leq 0
\]

(2.4)

\[
\Rightarrow \sum_{j=1, j \neq 0}^{m} \lambda_j \sum_{t=1}^{n} (D_t g_j(S^0), I_{S_t} - I_{S^0_t}) \leq -\rho_0' d^2(S, S^0), \quad \forall S \in \mathcal{P}
\]

both hold.
If (VP) is \((\rho_0, \rho_0')\)-quasi V-univex type I at all \(S^0 \in \mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}_+^m\), then we say that (VP) is \((\rho_0, \rho_0')\)-quasi V-univex type I on \(\mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}_+^m\).

If the second (implied) inequality in (2.3) is strict \((S \neq S^0)\), then we say that (VP) is \((\rho_0, \rho_0')\)-semi strictly quasi V-univex type I at \(S^0\) on \(\mathcal{P}\), according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}_+^m\), depending on the case.

\textit{Definition 2.5.} We say that problem (VP) is \((\rho_0, \rho_0')\)-pseudo V-univex type I at \(S^0 \in \mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}_+^m\) if there exist positive real functions \(\alpha_1, \ldots, \alpha_p\) and \(\beta_1, \ldots, \beta_k\) defined on \(\Gamma^n \times \Gamma^n\), nonnegative functions \(b_0\) and \(b_1\), also defined on \(\Gamma^n \times \Gamma^n\), \(\varphi_0 : \mathbb{R} \rightarrow \mathbb{R}\), \(\varphi_1 : \mathbb{R} \rightarrow \mathbb{R}\), such that for all \(S \in \mathcal{P}\), the implications

\[
\sum_{i=1}^p \tau_i \sum_{t=1}^n \left\langle D_t \psi_i \left(S^0, \lambda_{J_0}\right), I_{S_t} - I_{S^0_t}\right\rangle \geq -\rho_0 d^2 (S, S^0) \tag{2.5}
\]

\[
\Rightarrow b_0 (S, S^0) \varphi_0 \left[ \sum_{i=1}^p \tau_i \alpha_i \left(S, S^0\right) \left[ \psi_i \left(S, \lambda_{J_0}\right) - \psi_i \left(S^0, \lambda_{J_0}\right) \right] \right] \geq 0,
\]

and

\[
\sum_{j=1, j \neq J_0}^m \lambda_j \sum_{t=1}^n \left\langle D_t g_j \left(S^0, S^0\right), I_{S_t} - I_{S^0_t}\right\rangle \geq -\rho_0' d^2 (S, S^0) \tag{2.6}
\]

\[
\Rightarrow b_1 (S, S^0) \varphi_1 \left[ \sum_{s=1}^k \beta_s \left(S, S^0\right) \sum_{j \in J_s} \lambda_j g_j \left(S^0\right) \right] \leq 0.
\]

both hold.

If (VP) is \((\rho_0, \rho_0')\)-pseudo V-univex type I at all \(S^0 \in \mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}_+^m\), then we say that (VP) is \((\rho_0, \rho_0')\)-pseudo V-univex type I on \(\mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}_+^m\).

If the second (implied) inequality in (2.5) is strict \((S \neq S^0)\), then we say that (VP) is \((\rho_0, \rho_0')\)-semi strictly pseudo V-univex type I at \(S^0\) on \(\mathcal{P}\), according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}_+^m\), depending on the case.

If the second (implied) inequalities in (2.5) and (2.6) are both strict, then we say that (VP) is \((\rho_0, \rho_0')\)-strictly pseudo V-univex type I at \(S^0\) on \(\mathcal{P}\), according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}_+^m\), depending on the case.
Definition 2.6. We say that problem (VP) is \((\rho_0, \rho'_0)\)-quasi pseudo V-univex type I at \(S^0 \in \mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}^m_+\), if there exist positive real functions \(\alpha_1, \ldots, \alpha_p\) and \(\beta_1, \ldots, \beta_k\) defined on \(\Gamma^n \times \Gamma^n\), nonnegative functions \(b_0\) and \(b_1\), also defined on \(\Gamma^n \times \Gamma^n\), \(\varphi_0 : \mathbb{R} \to \mathbb{R}, \varphi_1 : \mathbb{R} \to \mathbb{R}\), such that the implications

\[
\begin{align*}
&b_0 \left( S, S^0 \right) \varphi_0 \left[ \sum_{i=1}^{p} \tau_i \alpha_i \left( S, S^0 \right) \left[ \psi_i \left( S, \lambda_{J_0} \right) - \psi_i \left( S^0, \lambda_{J_0} \right) \right] \right] \leq 0, \\
&\Rightarrow \sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left< D_t \psi_i \left( S^0, \lambda_{J_0} \right), I_{S_t} - I_{S^0_t} \right> \leq -\rho_0 d^2 \left( S, S^0 \right), \ \forall S \in \mathcal{P}
\end{align*}
\]

(2.7)

and

\[
\begin{align*}
&b_1 \left( S, S^0 \right) \varphi_1 \left[ \sum_{s=1}^{k} \beta_s \left( S, S^0 \right) \sum_{j \in J_s} \lambda_j g_j \left( S^0 \right) \right] \leq 0, \ \forall S \in \mathcal{P}.
\end{align*}
\]

(2.8)

both hold.

If (VP) is \((\rho_0, \rho'_0)\)-quasi pseudo V-univex type I at all \(S^0 \in \mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}^m_+\), then we say that (VP) is \((\rho_0, \rho'_0)\)-quasi pseudo V-univex type I on \(\mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}^m_+\).

If the second (implied) inequality in (2.8) is strict \((S \neq S^0)\), then we say that (VP) is \((\rho_0, \rho'_0)\)-quasi strictly pseudo V-univex type I at \(S^0\) or on \(\mathcal{P}\), according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}^m_+\), depending on the case.

Definition 2.7. We say that problem (VP) is \((\rho_0, \rho'_0)\)-pseudo quasi V-univex type I at \(S^0 \in \mathcal{P}\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}_+^p\) and \(\lambda \in \mathbb{R}^m_+\), if there exist positive real functions \(\alpha_1, \ldots, \alpha_p\) and \(\beta_1, \ldots, \beta_k\) defined on \(\Gamma^n \times \Gamma^n\), nonnegative functions \(b_0\) and \(b_1\), also defined on \(\Gamma^n \times \Gamma^n\), \(\varphi_0 : \mathbb{R} \to \mathbb{R}, \varphi_1 : \mathbb{R} \to \mathbb{R}\), such that for any \(S \in \mathcal{P}\), the implications

\[
\begin{align*}
&\sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left< D_t \psi_i \left( S^0, \lambda_{J_0} \right), I_{S_t} - I_{S^0_t} \right> \geq -\rho_0 d^2 \left( S, S^0 \right),
\end{align*}
\]

(2.9)

\[
\Rightarrow b_0 \left( S, S^0 \right) \varphi_0 \left[ \sum_{i=1}^{p} \tau_i \alpha_i \left( S, S^0 \right) \left[ \psi_i \left( S, \lambda_{J_0} \right) - \psi_i \left( S^0, \lambda_{J_0} \right) \right] \right] \geq 0,
\]

and

\[
\begin{align*}
&\sum_{s=1}^{k} \beta_s \left( S, S^0 \right) \sum_{j \in J_s} \lambda_j g_j \left( S^0 \right) \leq 0, \ \forall S \in \mathcal{P}.
\end{align*}
\]
and

\[
\sum_{j=1}^{m} \sum_{t=1}^{n} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_s - I_{S_0} \right\rangle \leq -\rho_0 d^2 (S, S^0).
\]

both hold.

If (VP) is \((\rho_0, \rho_0')\)-pseudo quasi \(V\)-univex type I at all \(S^0 \in P\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}^p_+\) and \(\lambda \in \mathbb{R}^m_+\), then we say that (VP) is \((\rho_0, \rho_0')\)-pseudo quasi \(V\)-univex type I on \(P\) according to the partition \(\{J_0, J_1, \ldots, J_k\}\) relative to \(\tau \in \mathbb{R}^p_+\) and \(\lambda \in \mathbb{R}^m_+\), depending on the case.

The following necessary conditions are from Zalmai ([9]).

**Theorem 2.1** (Zalmai [9]). Suppose that:

(a1) \(S^0\) is a properly efficient solution of (VP);

(a2) there exists an \(S^* \in P\) with \(g_{M_0}(S^*) < 0\) where \(M_0 = \{ j \mid g_j(S^0) = 0 \}\) such that

\[
g_j(S^0) + \sum_{i=1}^{n} \left\langle D_t f_i(S^0), I_s - I_{S_0} \right\rangle < 0, \quad \forall j \in \{1, \ldots, m\}.
\]

Then there exist \(\tau^0 \in \mathbb{R}^p, \tau^0 > 0, \) and \(\lambda^0 \in \mathbb{R}^m_+\) such that

\[
\sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n} \left\langle D_t f_i(S^0), I_s - I_{S_0} \right\rangle + \sum_{j=1}^{m} \lambda_j^0 \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_s - I_{S_0} \right\rangle \geq 0,
\]

\(\forall S \in P, \) and

\[
\lambda_j^0 g_j(S^0) = 0, \quad j \in \{1, \ldots, m\}.
\]

### 3. OPTIMALITY CONDITIONS

Consider the multiobjective fractional programming problem

(VFP) \[
\begin{align*}
\text{Minimize} & \quad \left( f_1(S), \ldots, f_p(S) \right) \\
\text{subject to} & \quad h_j(S) \leq 0, \quad j = 1, \ldots, m, \quad S \in \Gamma^m.
\end{align*}
\]

Assume that \(g_i(S) > 0, \) \(i \in P = \{1, \ldots, p\}, \) \(S \in \Gamma^m.\)
Denote by $S_0 = \{ h_j(S) \leq 0, j = 1, \ldots, m, S \in \Gamma^n \}$ the set of all feasible solutions of problem (VFP).

In order to derive a set of necessary and sufficient conditions for (VFP), we employ a Dinkelbach-type indirect approach via the following auxiliary problem

\[(VFP_\mu) \quad \text{Minimize} \quad (f_1(S) - \mu_1 g_1(S), \ldots, f_p(S) - \mu_p g_p(S)), \]

where $\mu_i, i \in P$, are parameters. This problem is equivalent to (VFP) in the sense that for particular choices of $\mu_i, i \in P$, the two problems have the same set of efficient solutions. This equivalence is stated more precisely in the following lemma whose proof is straightforward, and hence, omitted.

**Lemma 3.1 ([7]).** An $S^* \in S_0$ is an efficient solution of (VFP) if and only if it is an efficient solution of $(VFP_\mu^*)$ with $\mu^*_i = f_i(S^*) / g_i(S^*)$, $i \in P$.

Put $\psi_i(S, \mu, \lambda_{J_0}) = f_i(S) - \mu_i g_i(S) + \lambda^T_{J_0} h_{J_0}(S)$ where $\mu_i, i \in P$, are parameters and $\lambda \in \mathbb{R}^n_+$ is a given vector.

The following theorem gives sufficient conditions for a set to be an efficient set solution of problem (VFP) under generalized type I conditions with respect to a partition of the constraints.

**Theorem 3.1 (Sufficiency).** Suppose that:

(a1) $S^0 \in S_0$;

(a2) there exist $\tau^0 \in \mathbb{R}^p_+, \sum_{i=1}^p \tau^0_i = 1, \mu^0 \in \mathbb{R}^p_+$ and $\lambda^0 \in \mathbb{R}^m_+$ such that

(a) for any $S \in S_0$ we have

$$\sum_{i=1}^p \tau^0_i \sum_{t=1}^n \left< D_t f_i(S^0) - \mu^0_i D_t g_i(S^0), I_{S_t} - I_{S^0_t} \right> + \sum_{j=1}^m \lambda^0_j \sum_{t=1}^n \left< D_t h_j(S^0), I_{S_t} - I_{S^0_t} \right> \geq 0,$$

(b) with respect to the partition $\{ J_0, J_1, \ldots, J_k \}$ we have

$$\sum_{j \in J_s} \lambda^0_j h_j(S^0) = 0 \text{ for any } s \in \{0, 1, \ldots, k\};$$

(a3) problem $(VFP_\mu^\rho_0)$ is $(\rho_0, \rho'_0)$-quasi strictly pseudo $V$-univex type I at $S^0$ with $\rho_0 + \rho'_0 \geq 0$, according to the partition $\{ J_0, J_1, \ldots, J_k \}$ with respect to $\tau^0, \lambda^0$ and for some positive functions $\alpha_i, i \in \{1, \ldots, p\}$ and $\beta_j, j \in \{1, \ldots, k\}$.
Further, suppose that, for \( r \in \mathbb{R} \), we have

\[
(3.1) \quad r \leq 0 \Rightarrow \varphi_0(r) \leq 0,
\]

\[
(3.2) \quad \varphi_1(r) < 0 \Rightarrow r < 0,
\]

\[
(3.3) \quad b_0(S, S^0) > 0, \quad b_1(S, S^0) > 0, \quad \forall S \in S_0.
\]

Then \( S^0 \) is an efficient solution for (VFP).

**Proof.** Suppose that \( S^0 \) is not an efficient solution of (VFP). According to Lemma 3.1, \( S^0 \) is not an efficient solution of (VFP, \( \mu^0 \)) with \( \mu^0_i = \frac{f_i(S^0)}{g_i(S^0)} \), \( i = 1, \ldots, p \). Then there exists an \( S' \in S_0 \) such that \( f_i(S') - \mu^0_i g_i(S') \leq f_i(S^0) - \mu^0_i g_i(S^0) \), for any \( i = 1, \ldots, p \), with strict inequality for at least one \( i \). Since \( (\lambda^0_{j_0})^T h_{j_0}(S) \leq 0 \) for \( \forall S \in S_0 \) and by hypothesis (a2)(b), \( (\lambda^0_{j_0})^T h_{j_0}(S^0) = 0 \), it follows that for any \( i = 1, \ldots, p \), we have

\[
\psi_i(S', \mu^0, \lambda^0_{j_0}) - \psi_i(S^0, \mu^0, \lambda^0_{j_0}) \leq (f_i(S') - \mu^0_i g_i(S')) - (f_i(S^0) - \mu^0_i g_i(S^0)) \leq 0.
\]

Since \( \tau^0 \in \mathbb{R}_+^p \) and \( \alpha_i(S', S^0) > 0, i = 1, \ldots, p \), it follows

\[
\sum_{i=1}^p \tau^0_i \alpha_i(S', S^0) \left[ \psi_i(S', \mu^0, \lambda^0_{j_0}) - \psi_i(S^0, \mu^0, \lambda^0_{j_0}) \right] \leq 0
\]

and using (3.1) and (3.3) we get

\[
(3.4) \quad b_0(S', S^0) \varphi_0 \left[ \sum_{i=1}^p \tau^0_i \alpha_i(S', S^0) \left( \psi_i(S', \mu^0, \lambda^0_{j_0}) - \psi_i(S^0, \mu^0, \lambda^0_{j_0}) \right) \right] \leq 0.
\]

From (2.7) and (3.4) it follows that

\[
\sum_{i=1}^p \tau^0_i \sum_{t=1}^n \left( \langle D_t \psi_i(S^0, \mu^0, \lambda^0_{j_0}), I_{S^0_t} - I_{S^0_t} \rangle \right) \leq -\rho_0 d^2 (S', S^0).
\]

Since \( \psi_i(S^0, \mu^0, \lambda^0_{j_0}) = (f_i - \mu^0_i g_i)(S^0) + \sum_{j \in J_0} \lambda^0_j h_j(S^0) \) and \( \sum_{i=1}^p \tau^0_i = 1 \), the last inequality becomes

\[
(3.5) \quad \sum_{i=1}^p \tau^0_i \sum_{t=1}^n \left( \langle D_t (f_i - \mu^0_i g_j)(S^0), I_{S^0_t} - I_{S^0_t} \rangle \right) + \sum_{j \in J_0} \lambda^0_j \sum_{t=1}^n \left( \langle D_t h_j(S^0), I_{S^0_t} - I_{S^0_t} \rangle \right) \leq -\rho_0 d^2 (S', S^0).
\]
By inequality (3.5), assumption (a2) (a) and from the assumption that $\rho_0 + \rho_0' \geq 0$, we have

$$\sum_{j=1, j \notin J_0}^m \lambda_j^0 \sum_{t=1}^n \left\langle D_t h_j(S^0), I_{S^*_t} - I_{S_t^0} \right\rangle \geq \rho_0 d^2(S', S^0) \geq -\rho_0' d^2(S', S^0).$$

From (3.6) and assumption (a3), it follows that

$$b_1(S', S^0) \varphi_1 \left[ \sum_{s=1}^k \beta_s(S', S^0) \sum_{j \in J_s} \lambda_j^0 h_j(S^0) \right] < 0.$$

By (3.7), (3.2) and (3.3), we have

$$\sum_{s=1}^k \beta_s(S', S^0) \sum_{j \in J_s} \lambda_j^0 h_j(S^0) < 0.$$

On the other hand, by hypotheses (a2) (b) we have $\sum_{j \in J_s} \lambda_j^0 h_j(S^0) = 0$ for any $s \in \{1, \ldots, k\}$, which implies

$$\sum_{s=1}^k \beta_s(S', S^0) \sum_{j \in J_s} \lambda_j^0 h_j(S^0) = 0.$$

Equation (3.9) contradicts inequality (3.8), and therefore the theorem is proved.

The next theorem gives necessary condition for a properly efficient set solution of (VFP) and results from Lemma 3.1 and Theorem 2.1.

**Theorem 3.2 (Necessity).** Suppose that:

(b1) $S^0$ is a properly efficient solution of (VFP);
(b2) there exists an $S^* \in S_0$ with $h_{M_0}(S^*) < 0$ where $M_0 = \{j \mid h_j(S^0) = 0\}$ such that

$$h_j(S^0) + \sum_{t=1}^n \left\langle D_t h_j(S^0), I_{S^*_t} - I_{S_t^0} \right\rangle < 0, \quad \forall j \in \{1, \ldots, m\}.$$

Then there exist $\tau^0 \in \mathbb{R}^p$, $\mu^0 \in \mathbb{R}^p$ and $\lambda^0 \in \mathbb{R}_+^m$ such that

$$\sum_{i=1}^p \tau_{i}^0 \sum_{t=1}^n \left\langle D_t f_i(S^0) - \mu_{i}^0 D_t g_i(S^0), I_{S_t} - I_{S_t^0} \right\rangle +$$

$$+ \sum_{j=1}^m \lambda_j^0 \sum_{t=1}^n \left\langle D_t h_j(S^0), I_{S_t} - I_{S_t^0} \right\rangle \geq 0, \quad \text{for any } S \in S_0.$$
and

\[(3.11) \quad \lambda^0_j h_j(S^0) = 0, \quad j \in \{1, \ldots, m\}.
\]

The above theorems contain two sets of parameters \(\tau^0_i\) and \(\mu^0_i\), \(i \in P\). It is possible to eliminate one of these two sets of parameters, and thus, obtain a semi-parametric version of Theorems 3.1 and 3.2. This can be accomplished by simply replacing \(\mu^0_i\) by \(f_i(S^0)/g_i(S^0), \quad i \in P\), and redefining \(\tau^0\) and \(\lambda^0\). For further reference, we state this in next theorems.

**Theorem 3.3 (Sufficiency).** Suppose that

(a1) \(S^0 \in S_0\);

(a2) there exist \(\tau^0 \in \mathbb{R}^p_+, \sum_{i=1}^p \tau^0_i = 1\), and \(\lambda^0 \in \mathbb{R}^m_+\) such that

(a) for any \(S \in S_0\); we have

\[
\sum_{i=1}^p \tau^0_i \sum_{t=1}^n g_i(S^0) D_t f_i(S^0) - f_i(S^0) D_t g_i(S^0), I_{S_t} - I_{S^0_t} \geq 0,
\]

\[
+ \sum_{j=1}^m \lambda^0_j \sum_{t=1}^n \left( D_t h_j(S^0), I_{S_t} - I_{S^0_t} \right) \geq 0,
\]

(b) with respect to the partition \(\{J_0, J_1, \ldots, J_k\}\) we have

\[
\sum_{j \in J_s} \lambda^0_j h_j(S^0) = 0, \quad \text{for any } s \in \{0, 1, \ldots, k\};
\]

(a3) problem (VFP\(_\mu\)) is \((\rho_0, \rho'_0)\)-quasi strictly pseudo \(V\)-univex type I at \(S^0\) with \(\rho_0 + \rho'_0 \geq 0\), according to the partition \(\{J_0, J_1, \ldots, J_k\}\), with respect to \(\tau^0, \lambda^0\) and for some positive functions \(\alpha_i, i \in \{1, \ldots, p\}\) and \(\beta_j, j \in \{1, \ldots, k\}\). Further, suppose that, for \(r \in \mathbb{R}\), we have

\[
r \leq 0 \Rightarrow \varphi_0(r) \leq 0,
\]

\[
\varphi_1(r) < 0 \Rightarrow r < 0
\]

and

\[
b_0(S,S^0) > 0, \quad b_1(S,S^0) > 0, \quad \forall S \in P.
\]

Then \(S^0\) is an efficient solution for (VFP).

**Theorem 3.4 (Necessity).** Suppose that:

(b1) \(S^0\) is a properly efficient solution of \((VFP)\);

(b2) there exists an \(S^* \in S_0\) with \(h_{M_0}(S^*) < 0\) where \(M_0 = \{j \mid h_j(S^0) = 0\}\) such that

\[
h_j(S^0) + \sum_{t=1}^n \left( D_t h_j(S^0), I_{S^*_t} - I_{S^0_t} \right) < 0, \quad \forall j \in \{1, \ldots, m\}.
\]
Then there exist $\tau^0 \in \mathbb{R}^p$, $\tau^0 > 0$, and $\lambda^0 \in \mathbb{R}^m_+$ such that for any $S \in \mathcal{S}_0$ we have

$$
\sum_{i=1}^p \tau_i^0 \sum_{t=1}^n \langle g_i(S^0) D_t f_i(S^0) - f_i(S^0) D_t g_i(S^0), I_{S_i} - I_{S_i}^0 \rangle + 
\sum_{j=1}^m \lambda^0_j \sum_{t=1}^n \langle D_t h_j(S^0), I_{S_i} - I_{S_i}^0 \rangle \geq 0
$$

and

$$
\lambda^0_j h_j(S^0) = 0, \quad j \in \{1, \ldots, m\}.
$$

4. GENERALIZED MOND-WEIR DUALITY

We associate to problem (VFP), with respect to the partition $\{J_0, J_1, \ldots, J_k\}$ of its constraints, the following generalized Mond-Weir dual problem:

**maximize**

$$(GMWD) \quad (f_1(T) - \mu_1 g_1(T) + \lambda^T_{J_0} h_{J_0}(T), \ldots, f_p(T) - \mu_p g_p(T) + \lambda^T_{J_0} h_{J_0}(T)),$$

subject to $(T, \tau, \mu, \lambda) \in \mathcal{D}$,

where $\mathcal{D}$ is the set of feasible solutions of problem (GMWD), that is

$$
\mathcal{D} = \left\{ (T, \tau, \mu, \lambda) \mid \begin{array}{l}
\sum_{i=1}^p \tau_i \sum_{t=1}^n \langle D_t (f_i - \mu_i g_i + \lambda^T_{J_0} h_{J_0}) (T), I_{S_i} - I_{S_i} \rangle + 
\sum_{j=1}^m \lambda_j \sum_{t=1}^n \langle D_t h_j(T), I_{S_i} - I_{S_i} \rangle \geq 0, \forall S \in \Gamma^n \n\lambda^T_{J_0} h_{J_0}(T) \geq 0, \quad s = 1, \ldots, k,
T \in \Gamma^n, \quad \tau \in \mathbb{R}^p_+, \quad e^T \tau = 1, \quad \lambda \in \mathbb{R}^m_+, \quad \mu \in \mathbb{R}^p_+
\end{array} \right\}
$$

and $e = (1, \ldots, 1)^T \in \mathbb{R}^p$.

**Theorem 4.1 (Weak duality).** Suppose that:

1. $S \in \mathcal{S}_0$;
2. $(T, \tau, \mu, \lambda) \in \mathcal{D}$ and $\tau > 0$;
3. problem $\text{VFP}_\mu$ is $(\rho_0, \rho'_0)$-pseudo quasi $V$-univex type I at $T$ with $\rho_0 + \rho'_0 \geq 0$, according to the partition $\{J_0, J_1, \ldots, J_k\}$ with respect to $\tau$, $\lambda$ and some positive functions $\alpha_i$, $i \in \{1, \ldots, p\}$, $\beta_s$, $s \in \{1, \ldots, k\}$.

Further, assume that for $r \in \mathbb{R}$ we have

$$
\begin{align*}
\varphi_0(r) \geq 0 & \Rightarrow r \geq 0, \\
r \geq 0 & \Rightarrow \varphi_1(r) \geq 0, \\
b_0(S, T) > 0 & \quad b_1(S, T) \geq 0.
\end{align*}
$$
Then, the following cannot hold

\begin{align}
(4.4) & \quad f_i(S) \leq f_i(T) - \mu_i g_i(T) + \lambda_i^T \mathbf{h}_i(T), \quad \text{for any } i \in \{1, \ldots, p\}, \\
(4.5) & \quad f_j(S) < f_j(T) - \mu_j g_j(T) + \lambda_j^T \mathbf{h}_j(T), \quad \text{for some } j \in \{1, \ldots, p\}.
\end{align}

Proof. By hypothesis (i2), we have \( \lambda_j^T \mathbf{h}_j(T) \geq 0 \), for all \( j \in \{1, \ldots, k\} \).

Since \( \beta_s, s \in \{1, \ldots, k\} \), are all positive functions, we have

\begin{equation}
(4.6) \quad \sum_{s=1}^{k} \beta_s(S) \lambda_j^T \mathbf{h}_j(T) \geq 0.
\end{equation}

By hypothesis (i3), (4.2), (4.3) and (4.6), it follows that

\begin{equation}
(4.7) \quad \sum_{j=1}^{m} \lambda_j \sum_{s=1}^{n} \langle D_t h_j(T), I_{s} - I_{T} \rangle \leq -\rho_0 d^2(S, T).
\end{equation}

From (4.7), hypothesis (i2), and by assumption that \( \rho_0 + \rho'_0 \geq 0 \), we have

\begin{equation}
(4.8) \quad \sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \langle D_t (f_i(T) - \mu_i g_i(T) + \lambda_i^T \mathbf{h}_i(T), I_{s} - I_{T} \rangle \geq -\rho_0 d^2(S, T).
\end{equation}

From (4.8) and using again hypothesis (i3), we get

\begin{equation}
(4.9) \quad b_0(S, T) \varphi_0 \left[ \sum_{i=1}^{p} \tau_i \alpha_i(S, T) \left[ (f_i(T) - \mu_i g_i(T) + \lambda_i^T \mathbf{h}_i(T), I_{s} - I_{T}) - (f_i(T) - \mu_i g_i(T) + \lambda_i^T \mathbf{h}_i(T), I_{s} - I_{T}) \right] \right] \geq 0.
\end{equation}

From (4.9), (4.1) and (4.3), it follows

\begin{equation}
(4.10) \quad \sum_{i=1}^{p} \tau_i \alpha_i(S, T) \left[ (f_i(T) - \mu_i g_i(T) + \lambda_i^T \mathbf{h}_i(T), I_{s} - I_{T}) - (f_i(T) - \mu_i g_i(T) + \lambda_i^T g_i(T), I_{s} - I_{T}) \right] \geq 0.
\end{equation}

Assume that (4.1) and (4.5) hold. For \( S \in \mathcal{S}_0 \) and \( \lambda_{0j} \geq 0 \) we have \( \lambda_{0j}^T g_j(S) \leq 0 \). Since \( \alpha_i > 0, \tau > 0 \), from (4.2) and (4.3) we get

\begin{equation}
\sum_{i=1}^{p} \tau_i \alpha_i(S, T) \left[ (f_i(T) - \mu_i g_i(T) + \lambda_i^T g_i(T) - (f_i(T) - \mu_i g_i(T) - \lambda_i^T g_i(T), I_{s} - I_{T}) \right] < 0,
\end{equation}

which contradicts (4.10). Therefore, the theorem is proved. \( \square \)

**Theorem 4.2 (Weak duality).** Suppose that assumptions (i1) and (i2) of Theorem 4.1 hold. We also assume
From (4.17) and hypothesis (i2), it follows that:

\[ \sum_{s=1}^{k} \frac{\tau_s \rho_s}{\alpha_s(S,T)} + \sum_{i=1}^{p} \frac{\rho_i}{\beta_i(S,T)} \geq 0, \]

where \( \alpha_s, \beta_i \) are some positive functions, \( s \in \{1, \ldots, k\} \) and \( i \in \{1, \ldots, p\} \), respectively. Further, assume that the functions \( \varphi_0 \) and \( \varphi_1 \) have the properties:

\[ \varphi_0(r) \geq 0 \Rightarrow r \geq 0, \]

\[ r \geq 0 \Rightarrow \varphi_1(r) \geq 0 \]

with \( \varphi_0 \) linear, and

\[ b_0(S, T) < 0, \quad b_1(S, T) \geq 0. \]

Then, the following cannot hold:

\[ f_i(S) \leq f_i(T) - \mu_i g_i(T) + \lambda_i^T h_i(T), \quad \text{for any } i \in \{1, \ldots, p\}, \]

\[ f_j(S) < f_j(T) - \mu_j g_j(T) + \lambda_j^T h_j(T), \quad \text{for some } j \in \{1, \ldots, p\}. \]

**Proof.** By hypothesis (i2) we have \( \lambda_j^T h_j(S) \geq 0, \) for all \( s \in \{1, \ldots, k\} \).

From (4.12) and since \( \beta_s^*, s = \{1, \ldots k\} \) are all positive functions, it follows that

\[ \beta_s^*(S,T) \varphi_1 \left( \sum_{j \in J_s} \lambda_j h_j(T) \right) \geq 0. \]

If we replace in Definition 2.3 the expression \( b_1(S, S^0) / \beta_s(S, S^0) \) by \( \beta_s^*(S,T) \), from (4.16) and (i3') it follows

\[ \sum_{j \in J_s} \lambda_j \sum_{t=1}^{n} \langle D_t h_j(T), I_{S_t} - I_{T_t} \rangle \leq \frac{-\rho_s d^2(S, T)}{\beta_s(S,T)}. \]

Summing these relations over \( s \in \{1, \ldots, k\} \), we get

\[ \sum_{j=1, j \not\in J_0}^{m} \lambda_j \sum_{t=1}^{n} \langle D_t h_j(T), I_{S_t} - I_{T_t} \rangle \leq \sum_{s=1}^{k} \frac{-\rho_s}{\beta_s(S,T)} d^2(S, T). \]

Since \( \sum_{i=1}^{p} \frac{\tau_i \rho_i}{\alpha_i(S,T)} + \sum_{s=1}^{k} \frac{\rho_s}{\beta_s(S,T)} \geq 0 \), we obtain

\[ \sum_{j=1, j \not\in J_0}^{m} \lambda_j \sum_{t=1}^{n} \langle D_t h_j(T), I_{S_t} - I_{T_t} \rangle \leq \sum_{i=1}^{p} \frac{\tau_i \rho_i}{\alpha_i(S,T)} d^2(S, T). \]

From (4.17) and hypothesis (i2), it follows

\[ \sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \langle D_t \left( f_i - \mu_i g_i + \lambda_i^T h_i \right)(T), I_{S_t} - I_{T_t} \rangle \geq \sum_{i=1}^{p} \frac{\tau_i \rho_i}{\alpha_i(S,T)} d^2(S, T). \]
Dividing both sides of (2.1) by \( \alpha_i(S, S^0) \) and taking \( S^0 = T \), by hypothesis (i3'), we get
\[
\frac{b_0(S, T)}{\alpha_i(S, T)} \varphi_0 \left( \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(S) - \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(T) \right) > \\
\frac{D \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(T)}{\alpha_i(S, T)} > \frac{\rho_i}{\alpha_i(S, T)} d^2(S, T), \forall i.
\]
Multiplying by \( \tau_i \) and taking \( \alpha^*_i(S, T) = 1/\alpha_i(S, T) \), we get for \( \forall i \),
\[
b_0(S, T) \tau_i \alpha^*_i(S, T) \varphi_0 \left( \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(S) - \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(T) \right) > \\
\tau_i \left( D \psi_i(T, \mu, \lambda, J_0), I_S - I_T \right) + \rho_i \alpha^*_i(S, T) d^2(S, T).
\]
Applying (4.18), using (4.11), (4.13) and then summing with respect to \( i \), we have
\[
(4.19) \sum_{i=1}^p \tau_i \alpha^*_i(S, T) \left[ \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(S) - \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(T) \right] > 0.
\]
If we assume that (4.14) and (4.15) hold, since \( \alpha^*_i > 0 \) and \( \tau > 0 \), we have
\[
\sum_{i=1}^p \tau_i \alpha^*_i(S, T) \left[ \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(S) - \left( f_i - \mu_i g_i + \lambda^*_h J_0 \right)(T) \right] < 0,
\]
which contradicts (4.19).

**Theorem 4.3 (Strong duality).** Suppose that:

1. \( S^0 \) is a properly efficient solution of \( (\text{VFP}_\mu) \);
2. there exists an \( S^* \in S_0 \) with \( h_{M_0}(S^*) < 0 \), where \( M_0 = \{ j \mid h_j(S^0) = 0 \} \), such that
\[
h_j(S^0) + \sum_{t=1}^n \left( D_t h_j(S^0), I_{S^0} - I_{S^0} \right) < 0, \quad \forall j \in \{1, \ldots, m\}.
\]
Then there exist \( \tau^0 \in \mathbb{R}^p \), \( \tau^0 > 0 \), \( \mu^0 \in \mathbb{R}^p \) and \( \lambda^0 \in \mathbb{R}_+^m \) such that \( (S^0, \tau^0, \mu^0, \lambda^0) \in D \) and the objective functions of \( (\text{VFP}_\mu) \) and \( (\text{GMWD}) \) have the same values at \( S^0 \) and \( (S^0, \tau^0, \mu^0, \lambda^0) \), respectively.

If problem \( (\text{VFP}_\mu) \) is \( (p_0, \rho_0) \)-pseudo quasi V-univex type I with \( p_0 + \rho_0^0 \geq 0 \) at all feasible solutions of \( (\text{GMWD}) \), according to the partition \( \{ J_0, J_1, \ldots, J_k \} \) with respect to \( \tau^0, \lambda^0 \) and conditions (4.1)–(4.3) of Theorem 4.1 are satisfied for all feasible solutions of \( (\text{GMWD}) \), then \( (S^0, \tau^0, \mu^0, \lambda^0) \in D \) is an efficient solution of \( (\text{GMWD}) \).

**Proof.** By Theorem 2.1, there exist \( \tau_i^0 \in \mathbb{R}^p \), \( \tau_i^0 > 0 \), and \( \lambda_i^0 \in \mathbb{R}_+^m \) such that, for any \( S \in S_0 \) we have
\[
\sum_{i=1}^p \tau_i^0 \sum_{t=1}^n \left( D_t f_i(S^0) - \mu_i^0 D_t g_i(S^0), I_{S_t} - I_{S^0_t} \right) + 
\]
\[ \sum_{j=1}^{m} \lambda_j^0 \sum_{i=1}^{n} \left( D_i h_j(S^0), I_{S^0}^i - I_{S^0}^0 \right) \geq 0 \] and \( \lambda_j^0 h_j(S^0) = 0, j \in \{1, \ldots, m\} \). Therefore \((S^0, \tau^0, \mu^0, \lambda^0) \in \mathcal{D}\), and obviously both problems (VFP) and (GMWD) have the same value of the objective function.

Suppose that \((S^0, \tau^0, \mu^0, \lambda^0)\) is not an efficient solution of (GMWD). Then there exists a point \((T^*, \tau^*, \mu^*, \lambda^*) \in \mathcal{D}\) such that \( f_i(S^0) - \mu_i^0 g_i(S^0) + (\lambda_j^0) T h_j(S^0) \leq f_i(T^*) - \mu_i^* g_i(T^*) + (\lambda_j^*) T h_j(T^*) \), with strict inequality for at least one \( i \), which contradicts the weak duality Theorem 4.1. Therefore, \((S^0, \tau^0, \mu^0, \lambda^0)\) is an efficient solution of (GMWD). \(\square\)

The following theorem can be proved following the line in the proof of the previous theorem.

**Theorem 4.4 (Strong duality).** Suppose that (j1) and (j2) of Theorem 4.3 are satisfied. Then there exist \( \tau^0 \in \mathbb{R}^p, \tau^0 > 0, \mu^0 \in \mathbb{R}^p \) and \( \lambda^0 \in \mathbb{R}_+^m \) such that \((S^0, \tau^0, \mu^0, \lambda^0) \in \mathcal{D}\) and the objective functions of (VFP) and (GMWD) have the same values at \( S^0 \) and \((S^0, \tau^0, \mu^0, \lambda^0)\), respectively.

If problem (VP\(_{\mu_0}\)) is \((\rho_0, \rho_0')\)-semi strictly V-univex type I with \( \rho_0 + \rho_0' \geq 0 \) at all feasible solutions of (GMWD), according to the partition \{\(J_0, J_1, \ldots, J_k\}\) with respect to \( \lambda^0 \), and conditions (4.11)–(4.13) of Theorem 4.2 are satisfied for all feasible solutions of (GMWD), then \((S^0, \tau^0, \mu^0, \lambda^0) \in \mathcal{D}\) is an efficient solution of (GMWD).

**Theorem 4.5 (Converse duality).** Suppose that:

(k1) \( (T^0, \tau^0, \mu^0, \lambda^0) \in \mathcal{D}\) with \( \tau^0 > 0 \);

(k2) \( T^0 \in \mathcal{S}_0 \);

(k3) the problem (VFP\(_{\mu}\)) is \((\rho, \rho')\)-V-univex type I at \( T^0 \), with \( \sum_{i=1}^{p} \frac{\alpha_i}{\alpha_i(S, T^0)} \)

\[ + \sum_{s=1}^{k} \frac{\rho_i'}{\rho_i(S, T^0)} \geq 0, \] according to the partition \{\(J_0, J_1, \ldots, J_k\}\) with respect to \( \lambda^0 \) and some positive functions \( \alpha_i, i \in \{1, \ldots, p\} \), and \( \beta_s, s \in \{1, \ldots, k\} \).

Further assume that the functions \( \varphi_0 \) and \( \varphi_1 \) have the properties

\[
\begin{align*}
    r < 0 & \Rightarrow \varphi_0(r) < 0 \\
    \varphi_0(0) & \leq 0 \\
    r_1 \leq r_2 & \Rightarrow \varphi_0(r_1) \leq \varphi_0(r_2),
\end{align*}
\]

(4.20)

\[
\begin{align*}
    r \geq 0 & \Rightarrow \varphi_1(r) \geq 0,
\end{align*}
\]

(4.21)

and

\[
\begin{align*}
    b_0(S, T^0) & > 0, \quad b_1(S, T^0) \geq 0, \quad \forall S \in S_0.
\end{align*}
\]

Then \( T^0 \) is an efficient solution of (VFP).
If, in addition, there exist positive real numbers \( n_i, m_i \) such that \( n_i < \alpha_i (S, T^0) < m_i \) for all \( S \in S_0 \) and \( i = 1, \ldots, p \), then \( T^0 \) is properly efficient for (VFP).

**Proof.** From hypothesis (k1), we have

\[
(4.23) \quad \sum_{j \in J_s} \lambda^0_j h_j (T^0) \geq 0, \quad s \in \{1, \ldots, k\}.
\]

By hypothesis (k3), using Definition 2.3, we have for any \( S \in S_0 \) and \( \forall i, 

\[
(4.24) \quad b_0 (S, T^0) \varphi_0 \[ \psi_i (S, \mu^0, \lambda^0_{j_0}) - \psi_i (T^0, \mu^0, \lambda^0_{j_0}) \] \geq \alpha_i (S, T^0) \langle D\psi_i (T^0, \mu^0, \lambda^0_{j_0}), I_S - I_{T^0} \rangle + \rho_i d^2 (S, T^0),
\]

and for \( \forall s, 

\[
(4.25) \quad -b_1 (S, T^0) \varphi_1 \left[ \sum_{j \in J_s} \lambda^0_j h_j (T^0) \right] \geq \beta_s (S, T^0) \sum_{j \in J_s} \lambda^0_j \langle Dh_j (T^0), I_S - I_{T^0} \rangle + \rho'_s d^2 (S, T^0).
\]

Since \( \alpha_i > 0, \beta_s > 0, \) for \( \forall i, s, \) and \( \tau^0 > 0, \lambda^0 \geq 0, \) it follows by (4.24), (4.25) and (k3) that

\[
(4.26) \quad \sum_{i=1}^{p} \frac{\tau^0_i}{\alpha_i (S, T^0)} b_0 (S, T^0) \varphi_0 \[ \psi_i (S, \mu^0, \lambda^0_{j_0}) - \psi_i (T^0, \mu^0, \lambda^0_{j_0}) \] - \sum_{s=1}^{k} \frac{b_1 (S, T^0)}{\beta_s (S, T^0)} \varphi_1 \left[ \sum_{j \in J_s} \lambda^0_j g_j (T^0) \right] \geq \sum_{i=1}^{p} \tau^0_i \langle D\psi_i (T^0, \mu^0, \lambda^0_{j_0}), I_S - I_{T^0} \rangle + \sum_{j=1, j \neq j_0}^{m} \lambda^0_j \langle Dh_j (T^0), I_S - I_{T^0} \rangle + \left( \sum_{i=1}^{p} \frac{\tau^0_i \rho_i}{\alpha_i (S, T^0)} + \sum_{s=1}^{k} \frac{\rho'_s}{\beta_s (S, T^0)} \right) d^2 (S, T^0) \geq 0.
\]

From (4.21), (4.23), and (4.26) it follows that for \( \forall S \in S_0, 

\[
(4.27) \quad \sum_{i=1}^{p} \frac{\tau^0_i}{\alpha_i (S, T^0)} b_0 (S, T^0) \varphi_0 \[ \psi_i (S, \mu^0, \lambda^0_{j_0}) - \psi_i (T^0, \mu^0, \lambda^0_{j_0}) \] \geq 0.
\]

Suppose that \( T^0 \) is not an efficient solution of (VFP). According to Lemma 3.1, \( T^0 \) is not an efficient solution of (VFP, \( \mu^0 \)) with \( \mu^0_i = \frac{f_i (S')}{g_i (S')}, \) \( i = 1, \ldots, p. \) Then there exists an \( S' \in S_0 \) such that \( f_i (S') - \mu^0_i g_i (S') \leq f_i (T^0) - \mu^0_i g_i (T^0) \) for
any $i = 1, \ldots, p$, with strict inequality for at least one $i$, which implies, by using (4.20) and (4.22), that

\begin{equation}
\sum_{i=1}^{p} \frac{\tau^0_i}{\alpha_i (S', T^0)} b_0 (S', T^0) \varphi_0 \left[ \psi_i (S', \mu^0, \lambda^0_0) - \psi_i (T^0, \mu^0, \lambda^0_0) \right] \leq \\
\leq \sum_{i=1}^{p} \frac{\tau^0_i}{\alpha_i (S', T^0)} b_0 (S', T^0) \varphi_0 \left[ f_i (S') - \psi_i (T^0, \mu^0, \lambda^0_0) \right] < 0.
\end{equation}

(4.28)

Obviously, (4.27) and (4.28) are in contradiction. Therefore, we get the conclusion of the theorem.

To establish proper efficiency of $T^0$ for (VFP), we define

$$M = \left(p - 1\right) \max \left\{ \frac{m_i \tau_j}{n_j \lambda_i} \mid i, j \in \{1, \ldots, p\}, i \neq j \right\}$$

and use (4.27) to get a contradiction.

**Theorem 4.6 (Converse duality).** Suppose that (k1) and (k2) of Theorem 4.5 are fulfilled and

(k3’) the problem (VFP) is $(\rho_0, \rho'_0)$-semi strictly pseudo $V$-univex type I at $T^0$, with $\rho_0 + \rho'_0 \geq 0$, according to the partition $\{J_0, J_1, \ldots, J_k\}$ with respect to $\tau^0$, $\lambda^0$, and some positive functions $\alpha_i, i \in \{1, \ldots, p\}, \beta_s, s \in \{1, \ldots, k\}$.

Further assume that the functions $\varphi_0$ and $\varphi_1$ have the properties

\begin{align*}
\varphi_0(r) &\geq 0 \Rightarrow r \geq 0, \\
r &\geq 0 \Rightarrow \varphi_1(r) \geq 0, \\
b_0 (S, T^0) &> 0, b_1 (S, T^0) > 0, \forall S \in S_0.
\end{align*}

(4.29) \hspace{1cm} (4.30) \hspace{1cm} (4.31)

Then $T^0$ is an efficient solution of (VFP).

If in addition there exist positive real numbers $n_i, m_i$ such that $n_i < \alpha_i (S, T^0) < m_i$ for all $S \in S_0$ and $i \in \{1, \ldots, p\}$, then $T^0$ is properly efficient for (VFP).

**Proof.** From hypothesis (k1) we have $\sum_{j \in J_s} \lambda^0_j h_j (T^0) \geq 0, s \in \{1, \ldots, k\}$.

Using (4.30) and the facts that the functions $\beta_s$ are positive and $b_1$ is nonnegative, we get

$$b_1 (S, T^0) \varphi_1 \left[ \sum_{s=1}^{k} \beta_s (S, T^0) \sum_{j \in J_s} \lambda^0_j h_j (T^0) \right] > 0, \forall S \in S_0.$$
From the negation of the relation (2.6) in the condition of hypothesis (k3'), we have
\[ \sum_{j=1, j \notin J_0}^{m} \lambda_j^0 \sum_{t=1}^{n} \left( D_t h_j (T^0), I_{S_t} - I_{T^0_t} \right) < -\rho_0' d^2 (S, T^0), \quad \forall S \in S_0. \]

Since \( \rho_0 + \rho_0' \geq 0 \), we get
\[ \sum_{j=1, j \notin J_0}^{m} \lambda_j^0 \sum_{t=1}^{n} \left( D_t h_j (T^0), I_{S_t} - I_{T^0_t} \right) < -\rho_0' d^2 (S, T^0), \quad \forall S \in S_0. \]

Using (4.32) and hypothesis (k1), we have
\[ \sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n} \left( D_t \psi_i (T^0), \mu_0^0, \lambda_j^0 \right) (I_{S_i} - I_{T^0_t}) > -\rho_0 d^2 (S, T^0), \quad \forall S \in S_0. \]

By Definition 2.5, we have for \( \forall S \in S_0 \),
\[ b_0 (S, T^0) \phi_0 \left[ \sum_{i=1}^{p} \tau_i^0 \alpha_i (S, T^0) (\psi_i (S, \mu_0^0, \lambda_j^0) - \psi_i (T^0, \mu_0^0, \lambda_j^0)) \right] \geq 0. \]

By (4.29) and (4.31) it follows
\[ \sum_{i=1}^{p} \tau_i^0 \alpha_i (S, T^0) (\psi_i (S, \mu_0^0, \lambda_j^0) - \psi_i (T^0, \mu_0^0, \lambda_j^0)) \geq 0, \quad \forall S \in S_0. \]

Comparing (4.33) with (4.10), the rest of the proof continues in the same way as for Theorem 4.1.

To prove that \( T^0 \) is properly efficient for (VFP) we can follow the argument given in the proof of Theorem 3.2 of Hanson et al. [2], and at the end we use (4.33) to get a contradiction.

**Theorem 4.7 (Converse duality).** Suppose that (k1) and (k2) of Theorem 4.5 are fulfilled and
\[ \text{(k3') the problem (VFP) is } (\rho_0, \rho_0')\text{-strictly pseudo quasi } V \text{-univex type I at } T^0, \text{ with } \rho_0 + \rho_0' \geq 0, \text{ according to the partition } \{ J_0, J_1, \ldots, J_k \} \text{ with respect to } \tau^0, \lambda^0 \text{ and some positive functions } \alpha_i, i \in \{ 1, \ldots, p \}, \beta_s, s \in \{ 1, \ldots, k \}. \]

Further assume that the functions \( \phi_0 \) and \( \phi_1 \) have the properties
\[ \begin{cases} r < 0 \Rightarrow \phi_0 (r) \leq 0, \\ r_1 \leq r_2 \Rightarrow \phi_0 (r_1) \leq \phi_0 (r_2), \\ r \geq 0 \Rightarrow \phi_1 (r) \geq 0, \\ b_0 (S, T^0) > 0, b_1 (S, T^0) \geq 0, \quad \forall S \in S_0. \end{cases} \]
Then \( T^0 \) is an efficient solution of \((VFP)\).

If, in addition, there exist positive real numbers \( n_i, m_i \) such that \( n_i < \alpha_i \left( S, T^0 \right) < m_i \) for all \( S \in \mathcal{S}_0 \) and \( i = 1, \ldots, p \), then \( T^0 \) is properly efficient for \((VFP)\).

**Proof.** From hypothesis \((k1)\) we have

\[
\sum_{j \in J_s} \lambda^0_j h_j \left( T^0 \right) \geq 0, \quad s \in \{1, \ldots, k\},
\]

which implies according to \((4.35)\) and the facts that the function \( \beta_s \) are positive, that

\[
b_1 \left( S, T^0 \right) \varphi_1 \left[ \sum_{s=1}^{k} \beta_s \left( S, T^0 \right) \sum_{j \in J_s} \lambda^0_j h_j \left( T^0 \right) \right] \geq 0, \quad \forall S \in \mathcal{S}_0.
\]

From the Definition 2.7 it follows

\[
(4.37) \quad \sum_{j=1, j \notin J_0}^m \lambda^0_j \sum_{i=1}^n \left\langle D_i h_j \left( T^0 \right), I_{S_i} - I_{T^0_i} \right\rangle \leq -\rho_0 d^2 \left( S, T^0 \right), \quad \forall S \in \mathcal{S}_0.
\]

From \((4.37)\), hypothesis \((k1)\), the fact that \( \rho_0 + \rho'_0 \geq 0 \) and relation \((2.9)\) we have for \( \forall S \in \mathcal{S}_0 \),

\[
(4.38) \quad b_0 \left( S, T^0 \right) \varphi_0 \left[ \sum_{i=1}^p \tau^0_i \alpha_i \left( S, T^0 \right) \left( \psi_i \left( S, \lambda^0_{j_0} \right) - \psi_i \left( T^0, \lambda^0_{j_0} \right) \right) \right] > 0.
\]

If \( T^0 \) is not an efficient solution of \((VFP)\), then there exists an \( S' \in \mathcal{S}_0 \) such that \( f_i \left( S' \right) - \mu^0_i g_i \left( S' \right) \leq f_i \left( T^0 \right) - \mu^0_i g_i \left( T^0 \right) \) for any \( i = 1, \ldots, p \), with strict inequality for at least one \( i \), which by \((4.34), (i2)\) and \( \alpha_i \) positive implies that

\[
(4.39) \quad \varphi_0 \left[ \sum_{i=1}^p \tau^0_i \alpha_i \left( S', T^0 \right) \left( \psi_i \left( S, \lambda^0_{j_0} \right) - \psi_i \left( T^0, \lambda^0_{j_0} \right) \right) \right] \leq 0.
\]

Since \((4.38)\) and \((4.39)\) contradict each other, it follows that \( T^0 \) is an efficient solution for \((VP)\). To establish the proper efficiency of \( T^0 \) for \((VP)\) we follow the same arguments as in the proof of Theorem 3.2 of Hanson et al. \cite{2}, and then we use \((4.38)\) to obtain a contradiction.

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