GENERALIZED \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-INVEX FUNCTIONS IN SEMIINFINITE FRACTIONAL PROGRAMMING. PART I: SUFFICIENT OPTIMALITY CONDITIONS

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In this paper, we formulate and discuss a fairly large number of sets of global sufficient optimality criteria using various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity assumptions for a semiinfinite fractional programming problem.

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1. INTRODUCTION

In this paper, we first propose further generalizations of the new classes of invex functions recently defined in [11], and then, using the new functions, we state and prove a fairly large number of sufficient optimality results for the following semiinfinite fractional programming problem:

\[\begin{align*}
(P) \quad \text{Minimize} \quad \varphi(x) = \frac{f(x)}{g(x)} \\
\text{subject to} \\
G_j(x, t) \leq 0 \quad \text{for all} \quad t \in T_j, \quad j \in q, \\
H_k(x, s) = 0 \quad \text{for all} \quad s \in S_k, \quad k \in r,
\end{align*}\]

where \(q\) and \(r\) are positive integers, \(X\) is a nonempty open convex subset of \(\mathbb{R}^n\) \((n\text{-dimensional Euclidean space})\), for each \(j \in q \equiv \{1, 2, \ldots, q\}\) and \(k \in r\), \(T_j\) and \(S_k\) are compact subsets of complete metric spaces, \(f\) and \(g\) are real-valued functions defined on \(X\), for each \(j \in q\), \(z \rightarrow G_j(z, t)\) is a real-valued function defined on \(X\) for all \(t \in T_j\), for each \(k \in r\), \(z \rightarrow H_k(z, s)\) is a real-valued function defined on \(X\) for all \(s \in S_k\), for each \(j \in q\) and \(k \in r\), \(t \rightarrow G_j(x, t)\) and \(s \rightarrow H_k(x, s)\) are continuous real-valued functions defined, respectively, on
$T_j$ and $S_k$ for all $x \in X$, and $g(x) > 0$ for all $x$ satisfying the constraints of $(P)$. Nonlinear programming problems like $(P)$ but with a finite number of constraints, that is, when the functions $G_j$ are independent of $t$, and the functions $H_k$ are independent of $s$, are known in the literature of mathematical programming as *fractional programming problems*. These problems have been the subject of vigorous investigations in the past four decades which have succeeded in producing a phenomenal profusion of publications dealing with various aspects of numerous classes of static and dynamic fractional optimization problems. One of the principal reasons for such a continual immense interest in this particular optimization paradigm appears to be its capability in providing realistically representative models for a number of significant classes of problems in the fields of operations research, management science, and economics. This feature is primarily due to the fact that in many areas, including resource allocation, transportation, production planning, sequencing, inventory control, financial planning, maintenance and replacement scheduling, and reliability assessment, ratios such as profit/capital, profit/revenue, return/cost, return/risk, cost/time, profit/time, etc., can serve as useful measures of system performance. Proper characterization and utilization of these measures often requires optimization of certain ratios which, in turn, gives rise to the formulation of fractional programming problems. In noneconomic situations, fractional programming problems have arisen in information theory, stochastic programming, numerical analysis, approximation theory, cluster analysis, graph theory, multifacility location theory, decomposition of large-scale mathematical programming problems, and goal programming, among others. For comprehensive surveys and extensive lists of references dealing with several aspects of fractional programming, including modeling properties, actual and potential areas of applications, optimality conditions, duality formulations, sensitivity and stability analysis, and computational algorithms, the reader is referred to [12, 18, 19, 52, 54, 55].

A mathematical programming problem with a finite number of variables and infinitely many constraints is called a *semiinfinite programming problem*. Problems of this type have been utilized for the modeling and analysis of a wide range of theoretical as well as concrete, real-world, practical problems. More specifically, semiinfinite programming concepts and techniques have found relevance and applications in approximation theory, statistics, game theory, engineering design (design of control systems, digital filters, electronic circuits, etc.), boundary value problems, defect minimization for operator equations, geometry, random graphs, graphs related to Newton flows, reliability testing, environmental protection planning, decision making under uncertainty, semidefi-
finite programming, geometric programming, disjunctive programming, optimal control problems, robotics, and continuum mechanics, among others. For a wealth of information pertaining to various aspects of semiinfinite programming, including areas of applications, optimality conditions, duality relations, and numerical algorithms, the reader is referred to [15, 23, 24, 27–31, 34–37, 41, 50, 56, 57]. Some relatively more recent applications of semiinfinite programming to data envelopment are discussed in [38], to anticipatory systems and gene-environment networks in [22, 58–61], to the analysis and implementation of Vasicek-type interest rate models in [16], to an interesting gemstone cutting problem in [62], to infinite kernel learning in [46, 47], and to basket option pricing in [20].

In stark contrast to the case of conventional fractional programming, semiinfinite programming problems with fractional objective functions have not received much attention in the related literature. Some small steps toward ameliorating this situation were taken in the recently published papers [66] and [67]. In [66], we establish a set of necessary optimality conditions and numerous sets of global sufficient optimality criteria under various generalized \((\mathcal{F}, \beta, \phi, \rho, \theta)\)-univexity assumptions. In [67], we use these optimality conditions to formulate and discuss several parametric and parameter-free duality models for \((P)\).

In the present study, we shall adopt a Dinkelbach-type approach [21] to formulate and discuss a fairly large number of sets of global sufficient optimality results for \((P)\) using some new classes of generalized invex functions, namely, (strictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invex functions, (strictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex functions, and (prestrictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex functions. The use of these optimality results in the construction and analysis of various parametric and parameter-free duality models for \((P)\) is discussed in [63–65].

The rest of this paper is organized as follows. In Section 2, we present a number of definitions and auxiliary results which will be needed in the sequel. In Section 3, we begin our discussion of sufficient optimality conditions where we clearly indicate how one can formulate and prove numerous sets of sufficiency criteria under a variety of generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity assumptions that are placed on the individual as well as certain combinations of the problem functions. Utilizing a partitioning scheme, in Section 4 we establish several sets of generalized sufficient optimality results each of which is in fact a family of such results whose members can easily be identified by appropriate choices of certain sets and functions. Finally, in Section 5 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem model considered in this paper.
Evidently, all the sufficient optimality results established in this paper can easily be modified and restated for the following classes of nonlinear programming problems, which are special cases of \((P)\):

\[(P1) \quad \text{Minimize} \quad f(x), \quad \text{where} \ \mathbb{F} \text{ (assumed to be nonempty)} \text{ is the feasible set of } (P), \text{ that is,} \]
\[
\mathbb{F} = \{ x \in X : G_j(x,t) \leq 0 \text{ for all } t \in T_j, \ j \in q, H_k(x,s) = 0 \text{ for all } s \in S_k, k \in r \};
\]

\[(P2) \quad \text{Minimize} \quad \frac{f(x)}{g(x)};
\]

\[(P3) \quad \text{Minimize} \quad f(x), \quad \text{where} \ \mathcal{G} = \{ x \in X : \tilde{G}_j(x) \leq 0, \ j \in q, \ \tilde{H}_k(x) = 0, \ k \in r \}.\]

Here \(X, f,\) and \(g\) are as defined in the description of \((P),\) and \(\tilde{G}_j, \ j \in q,\) and \(\tilde{H}_k, \ k \in r,\) are real-valued functions defined on \(X.\)

2. PRELIMINARIES

In this section we recall, for convenience of reference, the definitions of certain new classes of generalized invex functions which will be needed in the sequel. These functions were recently introduced by Antczak [11] which are certain extensions and variants of the invex functions originally introduced by Hanson [32]. For brief accounts of the evolution of these new classes of generalized convex functions as well as their relevance and applicability to optimality conditions and duality relations for various classes of mathematical programming problems, the reader is referred to [1–11, 40, 43, 53]. We begin by defining an invex function which has been instrumental in creating a vast array of interesting and important classes of generalized convex functions.

**Definition 2.1.** Let \(f\) be a differentiable real-valued function defined on \(\mathbb{R}^n.\) Then \(f\) is said to be \(\eta\text{-}\text{invex} \) (invex with respect to \(\eta\)) at \(y\) if there exists a function \(\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) such that for each \(x \in \mathbb{R}^n,\)

\[f(x) - f(y) \geq \langle \nabla f(y), \eta(x,y) \rangle,
\]

where \(\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)\) is the gradient of \(f\) at \(y,\) and \(\langle a, b \rangle\) denotes the inner product of the vectors \(a\) and \(b;\) \(f\) is said to be \(\eta\text{-}\text{invex on } \mathbb{R}^n\) if the above inequality holds for all \(x, y \in \mathbb{R}^n.\)
From this definition it is clear that every differentiable real-valued convex function is invex with \( \eta(x, y) = x - y \). This generalization of the concept of convexity was originally proposed by Hanson [32] who showed that for a nonlinear programming problem of the form

Minimize \( f(x) \) subject to \( g_i(x) \leq 0, \ i \in m, \ x \in \mathbb{R}^n \),

where the differentiable functions \( f, g_i : \mathbb{R}^n \to \mathbb{R}, \ i \in m \), are invex with respect to the same function \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \), the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for invariant convex) was coined by Craven [17] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define \( \eta \)-*pseudoinvex* and \( \eta \)-*quasiinvex* functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been generalized in several directions. For our present purposes, we shall need the recent extension of invexity, namely, \( B - (\tilde{p}, \tilde{r}) \)-invexity defined in [11]. Below, we reproduce the definitions of \( B - (\tilde{p}, \tilde{r}) \)-invex, \( B - (\tilde{p}, \tilde{r}) \)-pseudoinvex, and \( B - (\tilde{p}, \tilde{r}) \)-quasiinvex functions.

Let \( f \) be a differentiable real-valued function defined on \( X \).

**Definition 2.2** [11]. The function \( f \) is said to be \( B - (\tilde{p}, \tilde{r}) \)-invex with respect to \( \eta \) and \( b \) at \( x^* \in X \) if there exist a function \( \eta : X \times X \to \mathbb{R}^n \), a function \( b : X \times X \to \mathbb{R}_+ \equiv [0, \infty) \), and real numbers \( \tilde{r} \) and \( \tilde{p} \) such that for all \( x \in X \),

\[
\frac{1}{\tilde{r}} b(x, x^*) \left( e^{\tilde{r}[f(x) - f(x^*)]} - 1 \right) \geq \frac{1}{\tilde{p}} \langle \nabla f(x^*), e^{\tilde{p}\eta(x, x^*)} - 1 \rangle \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0,
\]

\[
\frac{1}{\tilde{r}} b(x, x^*) \left( e^{\tilde{r}[f(x) - f(x^*)]} - 1 \right) \geq \langle \nabla f(x^*), \eta(x, x^*) \rangle \text{ for } \tilde{p} = 0 \text{ and } \tilde{r} \neq 0,
\]

\[
b(x, x^*) [f(x) - f(x^*)] \geq \frac{1}{\tilde{p}} \langle \nabla f(x^*), e^{\tilde{p}\eta(x, x^*)} - 1 \rangle \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} = 0,
\]

\[
b(x, x^*) [f(x) - f(x^*)] \geq \langle \nabla f(x^*), \eta(x, x^*) \rangle \text{ for } \tilde{p} = 0 \text{ and } \tilde{r} = 0,
\]

where

\[
(e^{\tilde{p}\eta(x, x^*)} - 1) \equiv (e^{\tilde{p}\eta_1(x, x^*)} - 1, \ldots, e^{\tilde{p}\eta_n(x, x^*)} - 1).
\]

**Definition 2.3** [11]. The function \( f \) is said to be \( B - (\tilde{p}, \tilde{r}) \)-pseudoinvex with respect to \( \eta \) and \( b \) at \( x^* \in X \) if there exist a function \( \eta : X \times X \to \mathbb{R}^n \), a function \( b : X \times X \to \mathbb{R}_+ \), and real numbers \( \tilde{r} \) and \( \tilde{p} \) such that for all \( x \in X \),
\[
\frac{1}{\tilde{p}} \langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1 \rangle \geq 0 \quad \Rightarrow \quad \frac{1}{\tilde{r}} b(x, x^*) \left( e^{\tilde{r}[f(x) - f(x^*)]} - 1 \right) \geq 0
\]

for \( \tilde{p} \neq 0 \) and \( \tilde{r} \neq 0 \),

\[
\langle \nabla f(x^*), \eta(x, x^*) \rangle \geq 0 \Rightarrow \frac{1}{\tilde{p}} b(x, x^*) \left( e^{\tilde{r}[f(x) - f(x^*)]} - 1 \right) \geq 0 \quad \text{for} \quad \tilde{p} = 0 \quad \text{and} \quad \tilde{r} \neq 0,
\]

\[
\frac{1}{\tilde{p}} \langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1 \rangle \geq 0 \quad \Rightarrow \quad b(x, x^*)[f(x) - f(x^*)] \geq 0 \quad \text{for} \quad \tilde{p} \neq 0 \quad \text{and} \quad \tilde{r} = 0,
\]

\[
\langle \nabla f(x^*), \eta(x, x^*) \rangle \geq 0 \quad \Rightarrow \quad b(x, x^*)[f(x) - f(x^*)] \geq 0 \quad \text{for} \quad \tilde{p} = 0 \quad \text{and} \quad \tilde{r} = 0.
\]

**Definition 2.4** [11]. The function \( f \) is said to be \( B - (\tilde{p}, \tilde{r}) \)-quasiinvex with respect to \( \eta \) and \( b \) at \( x^* \in X \) if there exist a function \( \eta : X \times X \rightarrow \mathbb{R}^n \), a function \( b : X \times X \rightarrow \mathbb{R}_+ \), and real numbers \( \tilde{r} \) and \( \tilde{p} \) such that for all \( x \in X \),

\[
\frac{1}{\tilde{r}} b(x, x^*) \left( e^{\tilde{r}[f(x) - f(x^*)]} - 1 \right) \leq 0 \quad \Rightarrow \quad \frac{1}{\tilde{p}} \langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1 \rangle \leq 0
\]

for \( \tilde{p} \neq 0 \) and \( \tilde{r} \neq 0 \),

\[
\frac{1}{\tilde{r}} b(x, x^*) \left( e^{\tilde{r}[f(x) - f(x^*)]} - 1 \right) \leq 0 \quad \Rightarrow \quad \langle \nabla f(x^*), \eta(x, x^*) \rangle \leq 0 \quad \text{for} \quad \tilde{p} = 0 \quad \text{and} \quad \tilde{r} \neq 0,
\]

\[
b(x, x^*)[f(x) - f(x^*)] \leq 0 \quad \Rightarrow \quad \frac{1}{\tilde{p}} \langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1 \rangle \leq 0 \quad \text{for} \quad \tilde{p} \neq 0 \quad \text{and} \quad \tilde{r} = 0,
\]

\[
b(x, x^*)[f(x) - f(x^*)] \leq 0 \quad \Rightarrow \quad \langle \nabla f(x^*), \eta(x, x^*) \rangle \leq 0 \quad \text{for} \quad \tilde{p} = 0 \quad \text{and} \quad \tilde{r} = 0.
\]

In this paper, we shall utilize the following slightly modified and more general versions of the above classes of generalized invex functions.

**Definition 2.5.** The function \( f \) is said to be (strictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invex at \( x^* \in X \) if there exist functions \( \alpha : X \times X \rightarrow \mathbb{R} \), \( \beta : X \times X \rightarrow \mathbb{R} \), \( \gamma : X \times X \rightarrow \mathbb{R}_+ \), \( \eta : X \times X \rightarrow \mathbb{R}^n \), \( \rho : X \times X \rightarrow \mathbb{R} \), and \( \theta : X \times X \rightarrow \mathbb{R}^n \) such that for all \( x \in X (x \neq x^*) \),

\[
\frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left( e^{\alpha(x,x^*)[f(x) - f(x^*)]} - 1 \right) \left( > \right) \geq \frac{1}{\beta(x, x^*)} \langle \nabla f(x^*), e^{\beta(x,x^*)\eta(x,x^*)} - 1 \rangle
\]

\[
+ \rho(x, x^*)\|\theta(x, x^*)\|^2 \text{if} \ \alpha(x, x^*) \neq 0 \ \text{and} \ \beta(x, x^*) \neq 0 \quad \text{for all} \quad x \in X,
\]

\[
\frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left( e^{\alpha(x,x^*)[f(x) - f(x^*)]} - 1 \right) \left( > \right) \geq \langle \nabla f(x^*), \eta(x, x^*) \rangle
\]

\[
+ \rho(x, x^*)\|\theta(x, x^*)\|^2 \text{ if} \ \alpha(x, x^*) \neq 0 \ \text{and} \ \beta(x, x^*) = 0 \quad \text{for all} \quad x \in X,
\]
\( \gamma(x, x^*)[f(x) - f(x^*)](>) \geq \frac{1}{\beta(x, x^*)} \langle \nabla f(x^*), e^{\beta(x,x^*)\eta(x,x^*)} - 1 \rangle \)

\[ + \rho(x, x^*)\|\theta(x, x^*)\|^2 \text{ if } \alpha(x, x^*) = 0 \text{ and } \beta(x, x^*) \neq 0 \text{ for all } x \in X, \]

\( \gamma(x, x^*)[f(x) - f(x^*)](>) \geq \langle \nabla f(x^*), \eta(x, x^*) \rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \)

\[ \text{if } \alpha(x, x^*) = 0 \text{ and } \beta(x, x^*) = 0 \text{ for all } x \in X, \]

where \( \| \cdot \| \) is a norm on \( \mathbb{R}^n \).

The function \( f \) is said to be (strictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invex on \( X \) if it is (strictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invex at each \( x^* \in X \).

**Definition 2.6.** The function \( f \) is said to be (strictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex at \( x^* \in X \) if there exist functions \( \alpha : X \times X \to \mathbb{R}, \beta : X \times X \to \mathbb{R}, \gamma : X \times X \to \mathbb{R}_+, \eta : X \times X \to \mathbb{R}^n, \rho : X \times X \to \mathbb{R}, \) and \( \theta : X \times X \to \mathbb{R}^n \) such that for all \( x \in X (x \neq x^*) , \)

\[ \frac{1}{\beta(x, x^*)} \langle \nabla f(x^*), e^{\beta(x,x^*)\eta(x,x^*)} - 1 \rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \]

\[ \Rightarrow \frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left( e^{\alpha(x,x^*)[f(x)-f(x^*)]} - 1 \right) (>) \geq 0 \]

\[ \text{if } \alpha(x, x^*) \neq 0 \text{ and } \beta(x, x^*) \neq 0 \text{ for all } x \in X, \]

\[ \langle \nabla f(x^*), \eta(x, x^*) \rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \]

\[ \Rightarrow \frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left( e^{\alpha(x,x^*)[f(x)-f(x^*)]} - 1 \right) (>) \geq 0 \]

\[ \text{if } \alpha(x, x^*) \neq 0 \text{ and } \beta(x, x^*) = 0 \text{ for all } x \in X, \]

\[ \frac{1}{\beta(x, x^*)} \langle \nabla f(x^*), e^{\beta(x,x^*)\eta(x,x^*)} - 1 \rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \]

\[ \Rightarrow \gamma(x, x^*)[f(x)-f(x^*)](>) \geq 0 \text{ if } \alpha(x, x^*) = 0 \text{ and } \beta(x, x^*) \neq 0 \text{ for all } x \in X, \]

\[ \langle \nabla f(x^*), \eta(x, x^*) \rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \]

\[ \Rightarrow \gamma(x, x^*)[f(x)-f(x^*)](>) \geq \langle \nabla f(x^*), \eta(x, x^*) \rangle \]

\[ \text{if } \alpha(x, x^*) = 0 \text{ and } \beta(x, x^*) = 0 \text{ for all } x \in X. \]

The function \( f \) is said to be (strictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex on \( X \) if it is (strictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex at each \( x^* \in X \).

**Definition 2.7.** The function \( f \) is said to be (prestrictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex at \( x^* \in X \) if there exist functions \( \alpha : X \times X \to \mathbb{R}, \beta : X \times X \to \)
\[ \frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left( e^{\alpha(x, x^*)[f(x)-f(x^*)]} - 1 \right)(<) \leq 0 \]

\[ \Rightarrow \frac{1}{\beta(x, x^*)} \langle \nabla f(x^*), e^{\beta(x, x^*)\eta(x, x^*)} - 1 \rangle \leq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \]

if \( \alpha(x, x^*) \neq 0 \) and \( \beta(x, x^*) \neq 0 \) for all \( x \in X \),

\[ \frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left( e^{\alpha(x, x^*)[f(x)-f(x^*)]} - 1 \right)(<) \leq 0 \]

\[ \Rightarrow \langle \nabla f(x^*), \eta(x, x^*) \rangle \leq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \]

if \( \alpha(x, x^*) \neq 0 \) and \( \beta(x, x^*) = 0 \) for all \( x \in X \),

\[ \gamma(x, x^*)[f(x) - f(x^*)](<) \leq 0 \quad \Rightarrow \quad \langle \nabla f(x^*), \eta(x, x^*) \rangle \leq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \]

if \( \alpha(x, x^*) = 0 \) and \( \beta(x, x^*) = 0 \) for all \( x \in X \).

The function \( f \) is said to be (prestrictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex on \( X \) if it is (prestrictly) \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex at each \( x^* \in X \).

From the above definitions it is clear that if \( f \) is \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invex at \( x^* \), then it is both \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex and \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex at \( x^* \), if \( f \) is \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex at \( x^* \), then it is prestrictly \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex at \( x^* \), and if \( f \) is strictly \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex at \( x^* \), then it is \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex at \( x^* \).

In the proofs of the sufficient optimality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvexity (when \( \alpha(x, x^*) \neq 0 \) and \( \beta(x, x^*) \neq 0 \) for all \( x \in X \)) can be defined in the following equivalent way: the function \( f \) is said to be \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex at \( x^* \in X \) if there exist functions \( \alpha : X \times X \to \mathbb{R} \), \( \beta : X \times X \to \mathbb{R} \), \( \gamma : X \times X \to \mathbb{R}_+ \), \( \eta : X \times X \to \mathbb{R}^n \), \( \rho : X \times X \to \mathbb{R} \), and \( \theta : X \times X \to \mathbb{R}^n \) such that for all \( x \in X \),

\[ \frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left( e^{\alpha(x, x^*)[f(x)-f(x^*)]} - 1 \right) < 0 \Rightarrow \]
\[ \Rightarrow \frac{1}{\beta(x,x^*)} \left( \nabla f(x^*), e^{\beta(x,x^*)\eta(x,x^*)} - 1 \right) < -\rho(x,x^*) \|\theta(x,x^*)\|^2. \]

The concept of \( \eta \)-invexity has been extended in many ways, and various types of generalized \( \eta \)-invex functions have been utilized for establishing a variety of sufficient optimality criteria and duality relations for several classes of nonlinear programming problems. For more information about invex functions, the reader may consult \[13, 14, 25, 26, 33, 42, 44, 51\], and for recent surveys of these and other related functions, the reader is referred to \[39, 49\].

In formulating our sufficient optimality criteria, we have followed as our guide the form and features of the necessary optimality conditions for \((P)\). For a precise statement of this result, the reader is referred to \[63\].

3. SUFFICIENT OPTIMALITY CONDITIONS

In this section, we present a multitude of sufficiency results in which various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity assumptions are imposed on the individual as well as certain combinations of the problem functions.

With regard to the choice of the type of generalized invex functions, specified in Definitions 2.5–2.7, to be used in the statements and proofs of the our sufficient optimality results, we shall consistently use the cases in which the functions \(\alpha\) and \(\beta\) are nonzero for all \((x,y) \in X \times X\). All the sufficiency results established in this paper can be modified, restated, and proved for the other cases in a straightforward manner.

We begin our discussion of sufficiency criteria for \((P)\) with two simple sets of conditions. In the first set only \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity assumptions are imposed on the problem functions, whereas in the second set a Lagrangian-type function is assumed to satisfy a certain \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvexity requirement.

**Theorem 3.1.** Let \(x^* \in \mathbb{F}\), let \(\lambda^* = \varphi(x^*)\), let the functions \(f, g, z \rightarrow G_{j}(z,t)\), and \(z \rightarrow H_{k}(z,s)\) be differentiable at \(x^*\) for all \(t \in T_j\) and \(s \in S_k\), \(j \in q, k \in r\), and assume that there exist integers \(\nu_0\) and \(\nu\), with \(0 \leq \nu_0 \leq \nu \leq n+1\), such that there exist \(\nu_0\) indices \(j_m\), with \(1 \leq j_m \leq q\), together with \(\nu_0\) points \(t^m \in \hat{T}_{j_m}(x^*)\), \(m \in \nu_0\), \(\nu - \nu_0\) indices \(k_m\), with \(1 \leq k_m \leq r\), together with \(\nu - \nu_0\) points \(s^m \in S_{k_m}\), \(m \in \nu \setminus \nu_0\), and \(\nu\) real numbers \(v^*_m\), with \(v^*_m > 0\) for \(m \in \nu_0\), with the property that

\[(3.1) \quad \nabla f(x^*) - \lambda^* \nabla g(x^*) + \sum_{m=1}^{\nu_0} v^*_m \nabla G_{j_m}(x^*,t^m) + \sum_{m=\nu_0+1}^{\nu} v^*_m \nabla H_{k_m}(x^*,s^m) = 0.\]
Assume, furthermore, that either one of the following two sets of conditions holds:

(a) (i) the function \( z \to f(z) - \lambda^*g(z) \) is \((\alpha, \beta, \bar{\gamma}, \eta, \rho, \theta)^{-}\text{invex} \) at \( x^* \) and \( \bar{\gamma}(x, x^*) > 0 \) for all \( x \in \mathbb{F} \);
(ii) the function \( z \to G_{jm}(z, t^m) \) is \((\alpha, \beta, \bar{\gamma}_m, \eta, \hat{\rho}_m, \theta)^{-}\text{invex} \) at \( x^* \) for each \( m \in \nu_0 \);
(iii) the function \( z \to v^*_m H_{km}(z, s^m) \) is \((\alpha, \beta, \bar{\gamma}_m, \eta, \hat{\rho}_m, \theta)^{-}\text{invex} \) at \( x^* \) for each \( m \in \nu \setminus \nu_0 \);
(iv) \( \bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v^*_m \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} v^*_m \rho_m(x, x^*) \geq 0 \) for all \( x \in \mathbb{F} \);
(b) the Lagrangian-type function

\[
z \to L(z, v^*, \lambda^*, \bar{t}, \bar{s}) = f(z) - \lambda^*g(z) + \sum_{m=1}^{\nu_0} v^*_m G_{jm}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v^*_m H_{km}(z, s^m)
\]

is \((\alpha, \beta, \gamma, \eta, 0, \theta)^{-}\text{pseudoinvex} \) at \( x^* \) and \( \gamma(x, x^*) > 0 \) for all \( x \in \mathbb{F} \), where \( \bar{t} \equiv (t_1^1, t_2^1, \ldots, t_{\nu_0}^1) \) and \( \bar{s} \equiv (s_{\nu_0+1}, s_{\nu_0+2}, \ldots, s_{\nu}) \).

Then \( x^* \) is an optimal solution of \((P)\).

Proof. (a): Suppose to the contrary that \( x^* \) is not an optimal solution of \((P)\), and hence for some \( x \in \mathbb{F} \), \( \varphi(x) < \varphi(x^*) = \lambda^* \). This implies that

\[
f(x) - \lambda^*g(x) < 0.
\]

Since the exponential function \( z \to e^z \) is monotonically increasing, \( \alpha(x, x^*) \neq 0 \), and \( \bar{\gamma}(x, x^*) > 0 \) for all \( x \in \mathbb{F} \), it follows from \((3.2)\) (in both cases when \( \alpha(x, x^*) > 0 \) and \( \alpha(x, x^*) < 0 \)) that the following strict inequality holds:

\[
\frac{1}{\alpha(x, x^*)}\bar{\gamma}(x, x^*) \left( e^{\alpha(x, x^*)[f(x) - \lambda^*g(x)]} - 1 \right) < 0.
\]

In view of our \((\alpha, \beta, \gamma, \eta, \rho, \theta)^{-}\text{invexity} \) assumptions in (i)–(iii), we have

\[
\frac{1}{\alpha(x, x^*)}\bar{\gamma}(x, x^*) \left( e^{\alpha(x, x^*)[f(x) - \lambda^*g(x)]} - 1 \right) \geq \frac{1}{\beta(x, x^*)} \left\langle \nabla f(x^*) - \lambda^*\nabla g(x^*), e^{\beta(x, x^*)\eta(x, x^*)} - 1 \right\rangle + \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2;
\]

\[
\frac{1}{\alpha(x, x^*)}\bar{\gamma}_m(x, x^*) \left( e^{\alpha(x, x^*)[G_{jm}(x, t^m) - G_{jm}(x^*, t^m)]} - 1 \right) \geq \frac{1}{\beta(x, x^*)} \left\langle \nabla G_{jm}(x^*, t^m), e^{\beta(x, x^*)\eta(x, x^*)} - 1 \right\rangle + \hat{\rho}_m(x, x^*)\|\theta(x, x^*)\|^2, \quad m \in \nu_0,
\]
(3.6) \[
\frac{1}{\alpha(x, x^*)} \gamma m(x, x^*) \left( e^{\alpha(x, x^*)[u_m H_{km}(x, s^m) - v_m H_{km}(x, s^m)]} - 1 \right) \\
\geq \frac{1}{\beta(x, x^*)} \left( v_m^* \nabla H_{km}(x^*, s^m), e^{\beta(x, x^*)\eta(x, x^*)} - 1 \right) + \tilde{\rho}_m(x, x^*) \|\theta(x, x^*)\|^2, m \in \nu \setminus \nu_0.
\]

Multiplying (3.5) by \( v_m^* \), and then summing over \( m \in \nu_0 \), summing (3.6) over \( m \in \nu \setminus \nu_0 \), and finally adding (3.4) and the resulting inequalities, we get
\[
\frac{1}{\alpha(x, x^*)} \left\{ \gamma(x, x^*) \left( e^{\alpha(x, x^*)[\{f(x) - \lambda^* g(x) - [f(x^*) - \lambda^* g(x^*)]\}] - 1 \right) \right\} \\
+ \sum_{m=1}^{\nu_0} v_m^* \gamma m(x, x^*) \left( e^{\alpha(x, x^*)[G_{jm}(x, t^m) - G_{jm}(x^*, t^m)]} - 1 \right) \\
+ \sum_{m=\nu_0+1}^{\nu} \gamma m(x, x^*) \left( e^{\alpha(x, x^*)[u_m H_{km}(x, s^m) - v_m H_{km}(x^*, s^m)]} - 1 \right) \}
\geq \frac{1}{\beta(x, x^*)} \left( \nabla f(x^*) - \lambda^* \nabla g(x^*) + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{jm}(x^*, t^m) \right.
\left. + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{km}(x^*, s^m), e^{\beta(x, x^*)\eta(x, x^*)} - 1 \right) + \left[ \tilde{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \tilde{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} \tilde{\rho}_m(x, x^*) \right] \|\theta(x, x^*)\|^2.
\]

Now using (3.1) and (iv), and noticing that \( \varphi(x^*) = \lambda^* \), \( x^* \in \mathbb{F} \), \( G_{jm}(x^*, t^m) = 0 \) for all \( m \in \nu_0 \), and \( \gamma m(x, x^*) \geq 0 \) for each \( m \in \nu_0 \), we see that the above inequality reduces to
\[
\frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left( e^{\alpha(x, x^*)[f(x) - \lambda^* g(x)]} - 1 \right) \geq 0,
\]
which contradicts (3.3). Therefore, we conclude that \( x^* \) is an optimal solution of \( (P) \).

(b) : Let \( x \) be an arbitrary feasible solution of \( (P) \). From (3.1) we observe that
\[
\frac{1}{\beta(x, x^*)} \left( \nabla f(x^*) - \lambda^* \nabla g(x^*) + \sum_{m=1}^{\nu_0} v_m^* \nabla G_{jm}(x^*, t^m) \right.
\left. + \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla H_{km}(x^*, s^m), e^{\beta(x, x^*)\eta(x, x^*)} - 1 \right) = 0,
\]
which in view of our \( (\alpha, \beta, \gamma, \eta, 0, \theta) \)-pseudoinvexity assumption implies that
\[
\frac{1}{\alpha(x,x^*)} \gamma(x,x^*) \left( e^{\alpha(x,x^*)[L(x,v^*,\lambda^*,\bar{t},\bar{s})-L(x^*,v^*,\lambda^*,\bar{t},\bar{s})]} - 1 \right) \geq 0.
\]

We need to consider two cases: \( \alpha(x,x^*) > 0 \) and \( \alpha(x,x^*) < 0 \). If we assume that \( \alpha(x,x^*) > 0 \) and recall that \( \gamma(x,x^*) > 0 \), then the above inequality becomes
\[
e^{\alpha(x,x^*)[L(x,v^*,\lambda^*,\bar{t},\bar{s})-L(x^*,v^*,\lambda^*,\bar{t},\bar{s})]} \geq 1,
\]
which implies that
\[
L(x,v^*,\lambda^*,\bar{t},\bar{s}) \geq L(x^*,v^*,\lambda^*,\bar{t},\bar{s}).
\]

Because \( x^* \in \mathbb{F} \), \( t^m \in \hat{T}_{jm}(x^*) \), \( m \in \nu_0 \), and \( \lambda^* = \varphi(x^*) \), the right-hand side of the above inequality is equal to zero, and hence we have \( L(x,v^*,\lambda^*,\bar{t},\bar{s}) \geq 0 \). Inasmuch as \( x \in \mathbb{F} \), and \( v^*_m > 0 \), \( m \in \nu_0 \), this inequality simplifies to \( f(x) - \lambda^*g(x) \geq 0 \), and hence \( \varphi(x^*) = \lambda^* \leq \varphi(x) \). Since \( x \in \mathbb{F} \) was arbitrary, we conclude from this inequality that \( x^* \) is an optimal solution of \((P)\).

If we assume that \( \alpha(x,x^*) < 0 \), we arrive at the same conclusion. \( \square \)

**Theorem 3.2.** Let \( x^* \in \mathbb{F} \), let \( \lambda^* = \varphi(x^*) \), let the functions \( f, g, z \to G_j(z,t) \), and \( z \to H_k(z,s) \) be differentiable at \( x^* \) for all \( t \in T_j \) and \( s \in S_k, j \in q, k \in r \), and assume that there exist integers \( \nu_0 \) and \( \nu \), with \( 0 \leq \nu_0 \leq \nu \leq n+1 \), such that there exist \( \nu_0 \) indices \( j_m \), with \( 1 \leq j_m \leq q \), together with \( \nu_0 \) points \( t^m \in \hat{T}_{jm}(x^*) \), \( m \in \nu_0 \), \( \nu - \nu_0 \) indices \( k_m \), with \( 1 \leq k_m \leq r \), together with \( \nu - \nu_0 \) points \( s^m \in S_{k_m} \), \( m \in \nu \setminus \nu_0 \), and \( \nu \) real numbers \( v^*_m \), with \( v^*_m > 0 \) for each \( m \in \nu_0 \), such that (3.1) holds. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) \( z \to f(z) - \lambda^*g(z) \) is \( (\alpha, \beta, \gamma, \eta, \rho, \theta) \)-pseudoinvex at \( x^* \) and \( \bar{\gamma}(x,x^*) > 0 \) for all \( x \in \mathbb{F} \);

(ii) \( z \to G_{jm}(z,t^m) \) is \( (\alpha, \beta, \gamma_m, \eta, \rho_m, \theta) \)-quasiinvex at \( x^* \) for each \( m \in \nu_0 \);

(iii) \( z \to v^*_m H_{km}(z,s^m) \) is \( (\alpha, \beta, \gamma_m, \eta, \rho_m, \theta) \)-quasiinvex at \( x^* \) for each \( m \in \nu \setminus \nu_0 \);

(iv) \( \bar{\rho}(x,x^*) + \sum_{m=1}^{\nu_0} v^*_m \bar{\rho}_m(x,x^*) + \sum_{m=\nu_0+1}^{\nu} \bar{\rho}_m(x,x^*) \geq 0 \) for all \( x \in \mathbb{F} \);

(b) (i) \( z \to f(z) - \lambda^*g(z) \) is \( (\alpha, \beta, \gamma, \eta, \rho, \theta) \)-pseudoinvex at \( x^* \) and \( \bar{\gamma}(x,x^*) > 0 \) for all \( x \in \mathbb{F} \);

(ii) \( z \to \sum_{m=1}^{\nu_0} v^*_m G_{jm}(z,t^m) \) is \( (\alpha, \beta, \gamma, \eta, \rho, \theta) \)-quasiinvex at \( x^* \);

(iii) \( z \to v^*_m H_{km}(z,s^m) \) is \( (\alpha, \beta, \gamma_m, \eta, \rho_m, \theta) \)-quasiinvex at \( x^* \) for each \( m \in \nu \setminus \nu_0 \);

(iv) \( \bar{\rho}(x,x^*) + \bar{\rho}(x,x^*) + \sum_{m=\nu_0+1}^{\nu} \bar{\rho}_m(x,x^*) \geq 0 \) for all \( x \in \mathbb{F} \);
(c) (i) \( z \rightarrow f(z) - \lambda^* g(z) \) is \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex at \( x^* \) and \( \gamma(x, x^*) > 0 \) for all \( x \in \mathbb{F} \);

(ii) \( z \rightarrow G_{jm}(z, t^m) \) is \((\alpha, \beta, \hat{\gamma}_m, \eta, \hat{\rho}_m, \theta)\)-quasiinvex at \( x^* \) for each \( m \in \nu_0 \);

(iii) \( z \rightarrow \sum_{m=\nu_0+1}^{\nu_0+1} v_m^* H_{km}(z, s^m) \) is \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( x^* \);

(iv) \( \bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \rho_m(x, x^*) + \bar{\rho}(x, x^*) \geq 0 \) for all \( x \in \mathbb{F} \);

(d) (i) \( z \rightarrow f(z) - \lambda^* g(z) \) is \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex at \( x^* \) and \( \gamma(x, x^*) > 0 \) for all \( x \in \mathbb{F} \);

(ii) \( z \rightarrow \sum_{m=1}^{\nu_0} v_m^* G_{jm}(z, t^m) \) is \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( x^* \);

(iii) \( z \rightarrow \sum_{m=\nu_0+1}^{\nu_0+1} v_m^* H_{km}(z, s^m) \) is \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( x^* \);

(iv) \( \bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \bar{\rho}(x, x^*) \geq 0 \) for all \( x \in \mathbb{F} \);

(e) (i) \( z \rightarrow f(z) - \lambda^* g(z) \) is \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-pseudoinvex at \( x^* \) and \( \gamma(x, x^*) > 0 \) for all \( x \in \mathbb{F} \);

(ii) \( z \rightarrow \sum_{m=1}^{\nu_0} v_m^* G_{jm}(z, t^m) + \sum_{m=\nu_0+1}^{\nu_0+1} v_m^* H_{km}(z, s^m) \) is \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( x^* \);

(iii) \( \bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geq 0 \) for all \( x \in \mathbb{F} \).

Then \( x^* \) is an optimal solution of \((P)\).

Proof. Let \( x \) be an arbitrary feasible solution of \((P)\).

(a): Suppose to the contrary that \( x^* \) is not an optimal solution of \((P)\). As shown in the proof of part (a) of Theorem 3.1, this supposition leads to \((3.3)\).

Since \( x \in \mathbb{F} \) and \( t^m \in T_{jm}(x^*) \) for each \( m \in \nu_0 \), it is clear that \( G_{jm}(x, t^m) \leq 0 = G_{jm}(x^*, t^m) \), which implies that

\[
\frac{1}{\alpha(x, x^*)} \gamma_m(x, x^*) \left( e^{\alpha(x, x^*) [G_{jm}(x, t^m) - G_{jm}(x^*, t^m)]} - 1 \right) \leq 0
\]

because \( \alpha(x, x^*) \neq 0 \) and \( \gamma_m(x, x^*) \geq 0 \) for all \( x \in \mathbb{F} \) and \( m \in \nu_0 \). In view of (ii), this inequality implies that

\[
\frac{1}{\beta(x, x^*)} \left( \nabla G_{jm}(x^*, t^m), e^{\beta(x, x^*) \eta(x, x^*)} - 1 \right) \leq -\hat{\rho}_m(x, x^*) \| \theta(x, x^*) \|^2, \ m \in \nu_0.
\]

Multiplying this inequality by \( v_m^* \) and then summing over \( m \in \nu_0 \), we get

\[
(3.7) \quad \frac{1}{\beta(x, x^*)} \left< \sum_{m=1}^{\nu_0} v_m^* \nabla G_{jm}(x^*, t^m), e^{\beta(x, x^*) \eta(x, x^*)} - 1 \right> \leq -\sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) \| \theta(x, x^*) \|^2.
\]
In a similar manner, our assumptions in (iii) lead to the following inequality:

\[
\frac{1}{\beta(x,x^*)} \left( \sum_{m=\nu_0+1}^\nu v_m^* \nabla H_{km}(z^*, s^m), e^{\beta(x,x^*)} \eta(x,x^*) - 1 \right) \leq - \sum_{m=\nu_0+1}^\nu \rho_m(x,x^*) \| \theta(x,x^*) \|^2.
\]

Now combining (3.1), (3.7), and (3.8), and using (iv), we obtain

\[
\frac{1}{\beta(x,x^*)} \left( \nabla f(x^*) - \lambda^* \nabla g(x^*), e^{\beta(x,x^*)} \eta(x,x^*) - 1 \right) \geq - \rho(x,x^*) \| \theta(x,x^*) \|^2,
\]

which in view of (i) implies that

\[
\frac{1}{\alpha(x,x^* \bar{\gamma}(x,x^*) \left( e^{\alpha(x,x^*)} \{ f(x) - \lambda^* g(x) - [f(x^*) - \lambda^* g(x^*)] \} - 1 \right) \geq 0.
\]

Since \( \lambda^* = \varphi(x^*) \), this inequality contradicts (3.3), and hence we conclude that \( x^* \) is an optimal solution of (P).

(b)–(e) : The proofs are similar to that of part (a). □

**Theorem 3.3.** Let \( x^* \in \mathbb{F} \), let \( \lambda^* = \varphi(x^*) \), let the functions \( f \), \( g \), \( z \rightarrow G_j(z,t) \), and \( z \rightarrow H_k(z,s) \) be differentiable at \( x^* \) for all \( t \in T_j \) and \( s \in S_k \), \( j \in q \), \( k \in r \), and assume that there exist integers \( \nu_0 \) and \( \nu \), with \( 0 \leq \nu_0 \leq \nu \leq n+1 \), such that there exist \( \nu_0 \) indices \( j_m \), with \( 1 \leq j_m \leq q \), together with \( \nu_0 \) points \( t_m \in T_{\nu_0}(z^*) \), \( m \in \nu_0 \), \( \nu - \nu_0 \) indices \( k_m \), with \( 1 \leq k_m \leq r \), together with \( \nu - \nu_0 \) points \( s_m \in S_{\nu_0} \), \( m \in \nu \backslash \nu_0 \), and \( \nu \) real numbers \( v_m^* \), with \( v_m^* > 0 \) for \( m \in \nu_0 \), such that (3.1) holds. Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) 
(i) \( z \rightarrow f(z) - \lambda^* g(z) \) is prestrictly \((\alpha, \beta, \bar{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( x^* \) and \( \bar{\gamma}(x,x^*) > 0 \) for all \( x \in \mathbb{F} \);
(ii) \( z \rightarrow G_{j_m}(z,t^m) \) is \((\alpha, \beta, \bar{\gamma}_m, \eta, \hat{\rho}_m, \theta)\)-quasiinvex at \( x^* \) for each \( m \in \nu_0 \);
(iii) \( z \rightarrow v_m^* H_{km}(z, s^m) \) is \((\alpha, \beta, \bar{\gamma}_m, \eta, \hat{\rho}_m, \theta)\)-quasiinvex at \( x^* \) for each \( m \in \nu \backslash \nu_0 \);
(iv) \( \hat{\rho}(x,x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x,x^*) + \sum_{m=\nu_0+1}^\nu \hat{\rho}_m(x,x^*) > 0 \) for all \( x \in \mathbb{F} \);

(b) 
(i) \( z \rightarrow f(z) - \lambda^* g(z) \) is prestrictly \((\alpha, \beta, \bar{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( x^* \) and \( \bar{\gamma}(x,x^*) > 0 \) for all \( x \in \mathbb{F} \);
(ii) \( z \rightarrow \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(z,t^m) \) is \((\alpha, \beta, \bar{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( x^* \);
(iii) \( z \rightarrow v_m^* H_{km}(z, s^m) \) is \((\alpha, \beta, \bar{\gamma}_m, \eta, \hat{\rho}_m, \theta)\)-quasiinvex at \( x^* \) for each \( m \in \nu \backslash \nu_0 \);
Then we deduce that (3.8) remain valid for the present case. From (3.1), (3.7), (3.8), and (iv) we conclude that

\[ \bar{x} \in \mathbb{F}. \]

Proof. Let \( x \) be an arbitrary feasible solution of (P).

(a): Suppose to the contrary that \( x^* \) is not an optimal solution of (P). As observed in the proof of part (a) of Theorem 3.1, this supposition implies that (3.3) holds. Because of our assumptions specified in (ii) and (iii), (3.7) and (3.8) remain valid for the present case. From (3.1), (3.7), (3.8), and (iv) we deduce that

\[
\frac{1}{\alpha(x, x^*)} \left( \nabla f(x^*) - \lambda^* \nabla g(x^*) - \eta(x, x^*) \right) \geq \left( e^\beta(x, x^*) \eta(x, x^*) - 1 \right)
\]

which in view of (i) implies that

\[
\frac{1}{\alpha(x, x^*)} \bar{\gamma}(x, x^*) \left( e^{\alpha(x, x^*)} \{ f(x) - \lambda^* g(x) - [f(x^*) - \lambda^* g(x^*)] \} - 1 \right) \geq 0.
\]

Since \( f(x^*) - \lambda^* g(x^*) = 0 \), this inequality contradicts (3.3), and hence we conclude that \( x^* \) is an optimal solution of (P).

(b)–(e): The proofs are similar to that of part (a). □
Theorem 3.4. Let $x^* \in F$, let $\lambda^* = \varphi(x^*)$, let the functions $f, g, z \rightarrow G_j(z,t)$, and $z \rightarrow H_k(z,s)$ be differentiable at $x^*$ for all $t \in T_j$ and $s \in S_k$, $j \in Q$, $k \in R$, and assume that there exist integers $\nu_0$ and $\nu$, with $0 \leq \nu_0 \leq \nu \leq n+1$, such that there exist $\nu$ indices $j_m$, with $1 \leq j_m \leq q$, together with $\nu$ points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \nu_0$, $\nu - \nu_0$ indices $k_m$, with $1 \leq k_m \leq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \nu \setminus \nu_0$, and $\nu$ real numbers $v^*_m$ with $v^*_m > 0$ for $m \in \nu_0$, such that (3.1) holds. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:

(a) (i) $z \rightarrow f(z) - \lambda^* g(z)$ is prestrictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-quasiinvex at $x^*$ and $\tilde{\gamma}(x, x^*) > 0$ for all $x \in F$;

(ii) $z \rightarrow G_{j_m}(z, t^m)$ is strictly $(\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)$-pseudoinvex at $x^*$ for each $m \in \nu_0$;

(iii) $z \rightarrow v^*_m H_{k_m}(z, s^m)$ is $(\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)$-quasiinvex at $x^*$ for each $m \in \nu \setminus \nu_0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v^*_m \tilde{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^\nu \tilde{\rho}_m(x, x^*) \geq 0$ for all $x \in F$;

(b) (i) $z \rightarrow f(z) - \lambda^* g(z)$ is prestrictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-quasiinvex at $x^*$ and $\tilde{\gamma}(x, x^*) > 0$ for all $x \in F$;

(ii) $z \rightarrow \sum_{m=1}^{\nu_0} v^*_m G_{j_m}(z, t^m)$ is strictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-pseudoinvex at $x^*$;

(iii) $z \rightarrow v^*_m H_{k_m}(z, s^m)$ is $(\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)$-quasiinvex at $x^*$ for each $m \in \nu \setminus \nu_0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v^*_m \tilde{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^\nu \tilde{\rho}_m(x, x^*) \geq 0$ for all $x \in F$;

(c) (i) $z \rightarrow f(z) - \lambda^* g(z)$ is prestrictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-quasiinvex at $x^*$ and $\tilde{\gamma}(x, x^*) > 0$ for all $x \in F$;

(ii) $z \rightarrow G_{j_m}(z, t^m)$ is $(\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)$-quasiinvex at $x^*$ for each $m \in \nu_0$;

(iii) $z \rightarrow v^*_m H_{k_m}(z, s^m)$ is strictly $(\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)$-pseudoinvex at $x^*$ for each $m \in \nu \setminus \nu_0$;

(iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v^*_m \tilde{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^\nu \tilde{\rho}_m(x, x^*) \geq 0$ for all $x \in F$;

(d) (i) $z \rightarrow f(z) - \lambda^* g(z)$ is prestrictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-quasiinvex at $x^*$ and $\tilde{\gamma}(x, x^*) > 0$ for all $x \in F$;

(ii) $z \rightarrow G_{j_m}(z, t^m)$ is $(\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)$-quasiinvex at $x^*$ for each $m \in \nu_0$;

(iii) $z \rightarrow \sum_{m=\nu_0+1}^\nu v^*_m H_{k_m}(z, s^m)$ is strictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-pseudoinvex at $x^*$;

(iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v^*_m \tilde{\rho}_m(x, x^*) + \tilde{\rho}(x, x^*) \geq 0$ for all $x \in F$;
(e) \( z \to f(z) - \lambda^* g(z) \) is prestrictly \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-quasiinvex at \( x^* \) and \( \hat{\gamma}(x, x^*) > 0 \) for all \( x \in \mathbb{F} \);

(ii) \( z \to \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(z, t^m) \) is strictly \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-pseudoinvex at \( x^* \);

(iii) \( z \to \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(z, s^m) \) is \((\alpha, \beta, \gamma, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( x^* \);

(iv) \( \bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geq 0 \) for all \( x \in \mathbb{F} \);

(f) \( z \to f(z) - \lambda^* g(z) \) is prestrictly \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \theta)\)-quasiinvex at \( x^* \) and \( \bar{\gamma}(x, x^*) > 0 \) for all \( x \in \mathbb{F} \);

(ii) \( z \to \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(z, t^m) \) is \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \theta)\)-quasiinvex at \( x^* \);

(iii) \( z \to \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(z, s^m) \) is strictly \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \theta)\)-pseudoinvex at \( x^* \);

(iv) \( \bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geq 0 \) for all \( x \in \mathbb{F} \);

(g) \( z \to f(z) - \lambda^* g(z) \) is prestrictly \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \theta)\)-quasiinvex at \( x^* \) and \( \bar{\gamma}(x, x^*) > 0 \) for all \( x \in \mathbb{F} \);

(ii) \( z \to \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(z, s^m) \) is strictly \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \theta)\)-pseudoinvex at \( x^* \);

(iii) \( \bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geq 0 \) for all \( x \in \mathbb{F} \).

Then \( x^* \) is an optimal solution of \((P)\).

**Proof.** Let \( x \) be an arbitrary feasible solution of \((P)\).

(a): Suppose to the contrary that \( x^* \) is not an optimal solution of \((P)\). This supposition implies that \((3.3)\) holds. Since \( x \in \mathbb{F} \) and \( t^m \in \bar{T}_{j_m}(x^*) \) for each \( m \in \nu_0 \), we have \( G_{j_m}(x, t^m) \leq 0 = G_{j_m}(x^*, t^m) \), and hence

\[
\frac{1}{\alpha(x, x^*)} \hat{\gamma}_m(x, x^*) \left( e^{\alpha(x, x^*)}[G_{j_m}(x, t^m) - G_{j_m}(x^*, t^m)] - 1 \right) \leq 0,
\]

which in view of (ii) implies that

\[
\frac{1}{\beta(x, x^*)} \left< \nabla G_{j_m}(x^*, t^m), e^{\beta(x, x^*)\eta(x, x^*)} - 1 \right> \leq -\hat{\rho}_m \|\theta(x, x^*)\|^2.
\]

As \( v_m^* > 0 \) for each \( m \in \nu_0 \), the above inequalities yield

\[
\frac{1}{\beta(x, x^*)} \left< \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m), e^{\beta(x, x^*)\eta(x, x^*)} \right> \leq -\sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m \|\theta(x, x^*)\|^2.
\]

Now, combining this inequality with \((3.8)\) (which is valid for the present case
because of (iii)) and (3.1), and using (iv), we obtain

\[
\frac{1}{\beta(x,x^*)} \left( \nabla f(x^*) - \lambda^* \nabla g(x^*) \right), e^{\beta(x,x^*)} \eta(x,x^*) - 1 \right) \geq \\
\left[ \sum_{m=1}^{\nu_0} v^*_m \hat{\rho}_m(x,x^*) + \sum_{m=\nu_0+1}^{\nu} \check{\rho}_m(x,x^*) \right] \|\theta(x,x^*)\|^2 > -\check{\rho}(x,x^*) \|\theta(x,x^*)\|^2,
\]

which by virtue of (i) implies that

\[
\frac{1}{\alpha(x,x^*)} \gamma(x,x^*) \left( e^{\alpha(x,x^*)}[f(x) - \lambda^* g(x)] - 1 \right) \geq 0.
\]

This inequality contradicts (3.3), and hence we conclude that \(x^*\) is an optimal solution of \((P)\).

(b)–(g): The proofs are similar to that of part (a). \(\square\)

In the remainder of this section, we briefly discuss certain modifications of Theorems 3.1–3.4 obtained by replacing (3.1) with an inequality. We begin by stating the following variant of Theorem 3.1; its proof is almost identical to that of Theorem 3.1 and hence omitted.

**Theorem 3.5.** Let \(x^* \in \mathbb{F}\), let \(\lambda^* = \varphi(x^*)\), let the functions \(f, g, z \to G_j(z,t)\), and \(z \to H_k(z,s)\) be differentiable at \(x^*\) for all \(t \in T_j\) and \(s \in S_k\), \(j \in q, k \in r\), and assume that there exist integers \(\nu_0\) and \(\nu\), with \(0 \leq \nu_0 \leq \nu \leq n+1\), such that there exist \(\nu_0\) indices \(j_m\), with \(1 \leq j_m \leq q\), together with \(\nu_0\) points \(t^m \in \hat{T}_{j_m}(x^*)\), \(m \in \nu_0\), \(\nu - \nu_0\) indices \(k_m\), with \(1 \leq k_m \leq r\), together with \(\nu - \nu_0\) points \(s^m \in S_{k_m}\), \(m \in \nu \setminus \nu_0\), and \(\nu\) real numbers \(v^*_m\) with \(v^*_m > 0\) for \(m \in \nu_0\), such that the following inequality holds:

\[
(3.9) \frac{1}{\beta(x,x^*)} \left\langle \nabla f(x^*) - \lambda^* \nabla g(x^*) + \sum_{m=1}^{\nu_0} v^*_m \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0+1}^{\nu} v^*_m \nabla H_{k_m}(x^*, s^m), e^{\beta(x,x^*)} \eta(x,x^*) - 1 \right\rangle \geq 0 \text{ for all } x \in \mathbb{F},
\]

where \(\beta : X \times X \to \mathbb{R}\) and \(\eta : X \times X \to \mathbb{R}^n\) are given functions. Furthermore, assume that either one of the two sets of conditions specified in Theorem 3.1 is satisfied. Then \(x^*\) is an optimal solution of \((P)\).

Although the proofs of Theorems 3.1 and 3.5 are essentially the same, their contents are somewhat different. This can easily be seen by comparing (3.1) with (3.9). We observe that any quintuple \((x^*, v^*, \lambda^*, \check{\tau}, \check{s})\) that satisfies (3.1) also satisfies (3.9), but the converse is not necessarily true. Moreover, (3.1) is a system of \(n\) equations, whereas (3.9) is a single inequality. Evidently,
from a computational point of view, (3.1) is preferable to (3.9) because of the dependence of the latter on the feasible set of \((P)\).

In a similar manner, one can easily state the modified versions of Theorems 3.2–3.4. However, we shall not state them explicitly.

4. GENERALIZED SUFFICIENCY CRITERIA

In this section, we discuss several families of sufficient optimality results under various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity hypotheses imposed on certain combinations of the problem functions. This is accomplished by employing a certain partitioning scheme which was originally proposed in [45] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let \(\nu_0\) and \(\nu\) be integers, with \(1 \leq \nu_0 \leq \nu\), and let \(\{J_0, J_1, \ldots, J_M\}\) and \(\{K_0, K_1, \ldots, K_M\}\) be partitions of the sets \(\nu_0\) and \(\nu \setminus \nu_0\), respectively; thus, \(J_i \subseteq \nu_0\) for each \(i \in M \cup \{0\}\), \(J_i \cap J_j = \emptyset\) for each \(i, j \in M \cup \{0\}\) with \(i \neq j\), and \(\bigcup_{i=0}^{M} J_i = \nu_0\). Obviously, similar properties hold for \(\{K_0, K_1, \ldots, K_M\}\). Moreover, if \(m_1\) and \(m_2\) are the numbers of the partitioning sets of \(\nu_0\) and \(\nu \setminus \nu_0\), respectively, then \(M = \max\{m_1, m_2\}\) and \(J_i = \emptyset\) or \(K_i = \emptyset\) for \(i > \min\{m_1, m_2\}\).

In addition, we use the real-valued functions \(z \to \Phi(z, v, \lambda, \bar{t}, \bar{s})\) and \(z \to \Lambda_\tau(z, v, \bar{t}, \bar{s})\) defined, for fixed \(v, \lambda, \bar{t} \equiv (t^1, t^2, \ldots, t^{\nu_0})\), and \(\bar{s} \equiv (s^{\nu_0+1}, s^{\nu_0+2}, \ldots, s^{\nu})\), on \(X\) as follows:

\[
\Phi(z, v, \lambda, \bar{t}, \bar{s}) = f(z) - \lambda g(z) + \sum_{m \in J_0} v_m G_{jm}(z, t^m) + \sum_{m \in K_0} v_m H_{km}(z, s^m),
\]

\[
\Lambda_\tau(z, v, \bar{t}, \bar{s}) = \sum_{m \in J_\tau} v_m G_{jm}(z, t^m) + \sum_{m \in K_\tau} v_m H_{km}(z, s^m), \quad \tau \in M.
\]

Making use of the sets and functions defined above, we can now formulate our collection of generalized sufficiency results for \((P)\) as follows.

**Theorem 4.1.** Let \(x^* \in F\), let \(\lambda^* = \varphi(x^*)\), let the functions \(f, g, z \to G_j(z, t)\), and \(z \to H_k(z, s)\) be differentiable at \(x^*\) for all \(t \in T_j\) and \(s \in S_k\), \(j \in q, k \in r\), and assume that there exist integers \(\nu_0\) and \(\nu\), with \(0 \leq \nu_0 \leq \nu \leq n+1\), such that there exist \(\nu_0\) indices \(j_m\), with \(1 \leq j_m \leq q\), together with \(\nu_0\) points \(t^m \in \hat{T}_{jm}(x^*)\), \(m \in \nu_0\), \(\nu - \nu_0\) indices \(k_m\), with \(1 \leq k_m \leq r\), together with \(\nu - \nu_0\) points \(s^m \in S_{k_m}\), \(m \in \nu \setminus \nu_0\), and \(\nu\) real numbers \(v_m^*\), with \(v_m^* > 0\) for \(m \in \nu_0\), such that (3.1) holds. Assume, furthermore, that any one of the following four sets of hypotheses is satisfied:

(a) (i) \( z \rightarrow \Phi(z, v^*, \lambda^*, \bar{t}, \bar{s}) \) is prestrictly \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiinvex at \( x^* \) and \( \tilde{\gamma}(x, x^*) > 0 \) for all \( x \in \mathcal{F} \);

(ii) for each \( \tau \in \mathcal{M} \), \( z \rightarrow \Lambda_\tau(z, v^*, \bar{t}, \bar{s}) \) is strictly \((\alpha, \beta, \tilde{\gamma}_\tau, \eta, \tilde{\rho}_\tau, \theta)\)-pseudoinvex at \( x^* \);

(iii) \( \tilde{\rho}(x, x^*) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, x^*) \geq 0 \) for all \( x \in \mathcal{F} \);

(b) (i) \( z \rightarrow \Phi(z, v^*, \lambda^*, \bar{t}, \bar{s}) \) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-pseudoinvex at \( x^* \) and \( \tilde{\gamma}(x, x^*) > 0 \) for all \( x \in \mathcal{F} \);

(ii) for each \( \tau \in \mathcal{M} \), \( z \rightarrow \Lambda_\tau(z, v^*, \bar{t}, \bar{s}) \) is \((\alpha, \beta, \tilde{\gamma}_\tau, \eta, \tilde{\rho}_\tau, \theta)\)-quasiinvex at \( x^* \);

(iii) \( \tilde{\rho}(x, x^*) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, x^*) \geq 0 \) for all \( x \in \mathcal{F} \);

(c) (i) \( z \rightarrow \Phi(z, v^*, \lambda^*, \bar{t}, \bar{s}) \) is prestrictly \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiinvex at \( x^* \) and \( \tilde{\gamma}(x, x^*) > 0 \) for all \( x \in \mathcal{F} \);

(ii) for each \( \tau \in \mathcal{M} \), \( z \rightarrow \Lambda_\tau(z, v^*, \bar{t}, \bar{s}) \) is \((\alpha, \beta, \tilde{\gamma}_\tau, \eta, \tilde{\rho}_\tau, \theta)\)-quasiinvex at \( x^* \);

(iii) \( \tilde{\rho}(x, x^*) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, x^*) > 0 \) for all \( x \in \mathcal{F} \);

(d) (i) \( z \rightarrow \Phi(z, v^*, \lambda^*, \bar{t}, \bar{s}) \) is prestrictly \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiinvex at \( x^* \) and \( \tilde{\gamma}(x, x^*) > 0 \) for all \( x \in \mathcal{F} \);

(ii) for each \( \tau \in \mathcal{M}_1 \), \( z \rightarrow \Lambda_\tau(z, v^*, \bar{t}, \bar{s}) \) is \((\alpha, \beta, \tilde{\gamma}_\tau, \eta, \tilde{\rho}_\tau, \theta)\)-quasiinvex at \( x^* \), and for each \( \tau \in \mathcal{M}_2 \neq \emptyset \), \( z \rightarrow \Lambda_\tau(z, v^*, \bar{t}, \bar{s}) \) is strictly \((\alpha, \beta, \tilde{\gamma}_\tau, \eta, \tilde{\rho}_\tau, \theta)\)-pseudoinvex at \( x^* \), where \( \{\mathcal{M}_1, \mathcal{M}_2\} \) is a partition of \( \mathcal{M} \);

(iii) \( \tilde{\rho}(x, x^*) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, x^*) \geq 0 \) for all \( x \in \mathcal{F} \).

Then \( x^* \) is an optimal solution of \((P)\).

*Proof.* Let \( x \) be an arbitrary feasible solution of \((P)\).

(a): It is clear that \((3.1)\) can be expressed as follows:

\[
\nabla f(x^*) - \lambda^* \nabla g(x^*) + \sum_{m \in J_0} v^*_m \nabla G_{j_m}(x^*, t^m) + \sum_{m \in K_0} v^*_m \nabla H_{k_m}(x^*, s^m) \\
+ \sum_{\tau=1}^{M} \left[ \sum_{m \in J_\tau} v^*_m \nabla G_{j_m}(x^*, t^m) + \sum_{m \in K_\tau} v^*_m \nabla H_{k_m}(x^*, s^m) \right] = 0.
\]

Since \( x, x^* \in \mathcal{F}, v^* \geq 0 \), and \( t^m \in \hat{T}_{j_m}(x^*), m \in \nu_0 \), it follows that for each \( \tau \in \mathcal{M} \),

\[
\Lambda_\tau(x, v^*, \bar{t}, \bar{s}) = \sum_{m \in J_\tau} v^*_m G_{j_m}(x, t^m) + \sum_{m \in K_\tau} v^*_m H_{k_m}(x, s^m) \\
\leq 0
\]
and hence

\[ \frac{1}{\alpha(x, x^*)} \tilde{\gamma}(x, x^*) \left( e^{\alpha(x, x^*)} \left[ \Lambda_\tau(x, v^*, \bar{t}, \bar{s}) - \Lambda_\tau(x, v^*, \bar{t}, \bar{s}) \right] - 1 \right) \leq 0, \]

which because of (ii) implies that

\[ \frac{1}{\beta(x, x^*)} \left( \sum_{m \in J_\tau} v_m^* \nabla G_{jm}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* \nabla H_{km}(x^*, s^m) \right) \left[ e^{\beta(x, x^*)} \eta(x, x^*) - 1 \right] < -\tilde{\rho}_{\tau}(x, x^*) \| \theta(x, x^*) \|^2. \]

Summing over \( \tau \), we obtain

\[ (4.2) \quad \frac{1}{\beta(x, x^*)} \left( \sum_{\tau=1}^M \left[ \sum_{m \in J_\tau} v_m^* \nabla G_{jm}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* \nabla H_{km}(x^*, s^m) \right] \right) \left[ e^{\beta(x, x^*)} \eta(x, x^*) - 1 \right] < -\sum_{\tau=1}^M \tilde{\rho}_{\tau}(x, x^*) \| \theta(x, x^*) \|^2. \]

Combining (4.1) and (4.2), and using (iii) we get

\[ \frac{1}{\beta(x, x^*)} \left( \nabla f(x^*) - \lambda^* \nabla g(x^*) + \sum_{m \in J_0} v_m^* \nabla G_{jm}(x^*, t^m) + \sum_{m \in K_0} v_m^* \nabla H_{km}(x^*, s^m) \right) \left[ e^{\beta(x, x^*)} \eta(x, x^*) - 1 \right] > \sum_{\tau=1}^M \tilde{\rho}_{\tau}(x, x^*) \| \theta(x, x^*) \|^2 \]

\[ \geq -\rho(x, x^*) \| \theta(x, x^*) \|^2, \]

which by virtue of (i) implies that

\[ \frac{1}{\alpha(x, x^*)} \tilde{\gamma}(x, x^*) \left( e^{\alpha(x, x^*)} [\Phi(x, v^*, \lambda^*, \bar{t}, \bar{s}) - \Phi(x, v^*, \lambda^*, \bar{t}, \bar{s})] - 1 \right) \geq 0. \]

Inasmuch as \( \tilde{\gamma}(x, x^*) > 0 \), this inequality implies that

\[ \Phi(x, v^*, \lambda^*, \bar{t}, \bar{s}) \geq \Phi(x^*, v^*, \lambda^*, \bar{t}, \bar{s}) = 0, \]

where the equality follows from the fact that \( \lambda^* = \varphi(x^*), \) \( t^m \in \hat{T}_{jm}(x^*), \) and \( x^* \in \mathbb{F}. \) Because \( x \in \mathbb{F} \) and \( v_m^* > 0 \) for each \( m \in \nu_0, \) this inequality further reduces to \( f(x) - \lambda^* g(x) \geq 0, \) and hence \( \varphi(x^*) = \lambda^* \leq \varphi(x). \) Therefore, we conclude that \( x^* \) is an optimal solution of \( (P). \)

(b): Proceeding as in the proof of part (a), we see that (ii) leads to the following inequality:
\[ \frac{1}{\beta(x, x^\ast)} \left( \sum_{\tau=1}^{M} \left[ \sum_{m \in J_{\tau}} v_m^* \nabla G_{jm}(x^*, t^m) + \sum_{m \in K_{\tau}} v_m^* \nabla H_{km}(x^*, s^m) \right] e^{\beta(x, x^\ast)} \eta(x, x^\ast) - 1 \right) \leq - \sum_{\tau=1}^{M} \tilde{\rho}_{\tau}(x, x^\ast) \| \theta(x, x^\ast) \|_2^2. \]

Now combining this inequality with (4.1) and using (iii), we obtain

\[ \frac{1}{\beta(x, x^\ast)} \left( \nabla f(x^\ast) - \lambda^* \nabla g(x^\ast) + \sum_{m \in J_0} v_m^* \nabla G_{jm}(x^*, t^m) + \sum_{m \in K_0} v_m^* \nabla H_{km}(x^*, s^m), e^{\beta(x, x^\ast)} \eta(x, x^\ast) - 1 \right) \geq \sum_{\tau=1}^{M} \tilde{\rho}_{\tau}(x, x^\ast) \| \theta(x, x^\ast) \|_2^2 \geq - \bar{\rho}(x, x^\ast) \| x - x^\ast \|^2, \]

which by virtue of (i) implies that

\[ \frac{1}{\alpha(x, x^\ast)} \tilde{\gamma}(x, x^\ast) \left( e^{\alpha(x, x^\ast)} \Phi(x, v^\ast, \lambda^*, \bar{t}, \bar{s}) - \Phi(x^\ast, v^\ast, \lambda^*, \bar{t}, \bar{s}) \right) - 1 \right) \geq 0. \]

The rest of the proof is identical to that of part (a).

(c) and (d): The proofs are similar to those of parts (a) and (b). □

Each one of the eight sets of conditions given in Theorem 4.1 and its modified version obtained by replacing (3.1) with (3.9), can be viewed as a family of sufficient optimality conditions whose members can easily be identified by appropriate choices of the partitioning sets \( J_\mu \) and \( K_\mu, \mu \in M \cup \{0\} \). Evidently, the special cases and variants of these eight sets of conditions collectively provide a multitude of sufficient optimality results for \((P)\).
Minimize \( x \in \mathcal{F} \) \( \max_{y \in \mathcal{Y}} \frac{f(x, y)}{g(x, y)} \),

and

Minimize \( x \in \mathcal{F} \) \( \left( \frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right) \).

We shall investigate these classes of semiinfinite programming problems in subsequent papers.

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