In this paper, we discuss a fairly large number of first-order duality results under various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity assumptions for a semiinfinite fractional programming problem.

AMS 2010 Subject Classification: 49N15, 90C30, 90C32, 90C34, 90C46.

Key words: Semiinfinite programming, fractional programming, generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invex functions, infinitely many equality and inequality constraints, first-order dual problems, duality theorems.

1. INTRODUCTION

In this paper, we state and prove a fairly large number of parametric and parameter-free duality results under various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity assumptions for the following semiinfinite fractional programming problem:

\[
(P) \quad \text{Minimize } \varphi(x) = \frac{f(x)}{g(x)}
\]

subject to

\[
G_j(x, t) \leq 0 \quad \text{for all } t \in T_j, \quad j \in q,
\]

\[
H_k(x, s) = 0 \quad \text{for all } s \in S_k, \quad k \in r,
\]

\[
x \in X,
\]

where \(q\) and \(r\) are positive integers, \(X\) is a nonempty open convex subset of \(\mathbb{R}^n\) (\(n\)-dimensional Euclidean space), for each \(j \in q \equiv \{1, 2, \ldots, q\}\) and \(k \in r\), \(T_j\) and \(S_k\) are compact subsets of complete metric spaces, \(f\) and \(g\) are real-valued functions defined on \(X\), for each \(j \in q\), \(z \rightarrow G_j(z, t)\) is a real-valued function defined on \(X\) for all \(t \in T_j\), for each \(k \in r\), \(z \rightarrow H_k(z, s)\) is a real-valued function defined on \(X\) for all \(s \in S_k\), for each \(j \in q\) and \(k \in r\), \(t \rightarrow G_j(x, t)\) and \(s \rightarrow H_k(x, s)\) are continuous real-valued functions defined, respectively, on \(T_j\) and \(S_k\) for all \(x \in X\), and \(g(x) > 0\) for all \(x\) satisfying the constraints of \((P)\).
The present study is essentially a continuation of the investigation initiated in the companion paper [2] where some information about fractional programming is presented, the current status of semiinfinite programming is briefly discussed and numerous key references are cited, an overview of the concept of \((\alpha, \beta, \gamma, \eta, \rho, \theta)-\text{invexity}\) and some of its extensions is given, and a fairly large number of sets of global sufficient optimality results under various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)-\text{invexity}\) assumptions are established. For the necessary background material and preliminaries, the reader is referred to [2]. Here we shall make use of the sufficient optimality results developed in [2] and a set of necessary optimality conditions, which will be recalled in the next section, and construct several first-order duality models for \((P)\) and prove appropriate duality theorems. Second- and higher-order counterparts of these results are discussed in [3, 4].

Although presently various duality results exist in the related literature for several classes of static and dynamic fractional optimization problems with a finite number of constraints, so far no such results are available for any kind of **semiinfinite** fractional programming problems involving generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)-\text{invexity}\) functions. To the best of our knowledge, all the duality results established in this paper are new in the area of **semiinfinite programming**.

The rest of the paper is organized as follows. In Section 2, we recall an auxiliary result which will be needed in the sequel. We begin our discussion of duality for \((P)\) in Section 3 where we formulate two parametric duality models with somewhat restricted constraint structures, and prove weak, strong, and strict converse duality theorems under appropriate generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)-\text{invexity}\) assumptions. In Section 4, we present another pair of parametric duality models with relatively more general constraint structures that allow for a greater variety of generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)-\text{invexity}\) hypotheses under which duality can be established. We continue our investigation of parametric duality for \((P)\) in Section 5 where we consider two generalized duality models and prove several sets of duality results. These duality models are, in fact, two families of dual problems for \((P)\) whose members can easily be identified by appropriate choices of certain sets and functions. The nonparametric counterparts of these generalized duality models are taken up in Section 6, and a number of duality results are established. Finally, in Section 7, we summarize our main results and also point out some additional research opportunities arising from certain modifications of the principal problem model \((P)\).

### 2. PRELIMINARIES

For proving our duality theorems, we shall need a few basic definitions and auxiliary results. For the definitions of the new classes of generalized
(α, β, γ, η, ρ, θ)-inve functions that will be utilized in the statements and proofs of our duality theorems, the reader is referred to [2], and the two auxiliary results that will be needed in the sequel for proving strong and strict converse duality theorems are stated below.

**Theorem 2.1.** [5]. Let $x^* \in F$ and $\lambda^* = f(x^*)/g(x^*)$, let $f$ and $g$ be continuously differentiable at $x^*$, for each $j \in q$, let the function $z \rightarrow G_j(z,t)$ be continuously differentiable at $x^*$ for all $t \in T_j$, and for each $k \in r$, let the function $z \rightarrow H_k(z,s)$ be continuously differentiable at $x^*$ for all $s \in S_k$. If $x^*$ is an optimal solution of (P), if the generalized Guignard constraint qualification holds at $x^*$, and if the set cone$\{\nabla G_j(x^*,t) : t \in \hat{T}_j(x^*), j \in q\} + \text{span}\{\nabla H_k(x^*,s) : s \in S_k, k \in r\}$ is closed, then there exist integers $\nu_0^*$ and $\nu^*$, with $0 \leq \nu_0^* \leq \nu^* \leq n + 1$, such that there exist $\nu_0^*$ indices $j_m$ with $1 \leq j_m \leq q$, together with $\nu_0^*$ points $t^m \in \hat{T}_j(x^*)$, $m \in \nu_0^*$, $\nu^* - \nu_0^*$ indices $k_m$, with $1 \leq k_m \leq r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_k$ for $m \in \nu^* \setminus \nu_0^*$, and $\nu^*$ real numbers $v_m^*$, with $v_m^* > 0$ for $m \in \nu_0^*$, with the property that

$$
\nabla f(x^*) - \lambda^* \nabla g(x^*) + \sum_{m=1}^{\nu_0^*} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_{k_m}(x^*, s^m) = 0,
$$

where $\hat{T}_j(x^*) = \{t \in T_j : G_j(x^*, t) = 0\}$ and $\nu^* \setminus \nu_0^*$ is the complement of the set $\nu_0^*$ relative to $\nu^*$.

Eliminating $\lambda^*$ and redefining the Lagrange multipliers $v_m^*, m \in \nu^*$, we obtain the following parameter-free version of Theorem 2.1.

**Theorem 2.2.** Let $x^* \in F$, let $f$ and $g$ be continuously differentiable at $x^*$, for each $j \in q$, let the function $z \rightarrow G_j(z,t)$ be continuously differentiable at $x^*$ for all $t \in T_j$, and for each $k \in r$, let the function $z \rightarrow H_k(z,s)$ be continuously differentiable at $x^*$ for all $s \in S_k$. If $x^*$ is an optimal solution of (P), if the generalized Guignard constraint qualification holds at $x^*$, and if the set cone$\{\nabla G_j(x^*,t) : t \in \hat{T}_j(x^*), j \in q\} + \text{span}\{\nabla H_k(x^*,s) : s \in S_k, k \in r\}$ is closed, then there exist integers $\nu_0^*$ and $\nu^*$, with $0 \leq \nu_0^* \leq \nu^* \leq n + 1$, such that there exist $\nu_0^*$ indices $j_m$, with $1 \leq j_m \leq q$, together with $\nu_0^*$ points $t^m \in \hat{T}_j(x^*)$, $m \in \nu_0^*$, $\nu^* - \nu_0^*$ indices $k_m$, with $1 \leq k_m \leq r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_k$ for $m \in \nu^* \setminus \nu_0^*$, and $\nu^*$ real numbers $v_m^*$, with $v_m^* > 0$ for $m \in \nu_0^*$, with the property that

$$
g(x^*) \nabla f(x^*) - f(x^*) \nabla g(x^*) + \sum_{m=1}^{\nu_0^*} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_{k_m}(x^*, s^m) = 0.
$$

We shall call $x$ a normal feasible solution of (P) if $x$ satisfies all the constraints of (P), if the generalized Guignard constraint qualification holds.
at $x$, and if the set $\text{cone}\{\nabla G_j(x, t) : t \in \hat{T}_j(x), j \in q\} + \text{span}\{\nabla H_k(x, s) : s \in S_k, k \in r\}$ is closed.

The forms and features of these results will serve as our guide in this paper for constructing several parametric and nonparametric duality models for $(P)$ and proving appropriate duality theorems.

In the remainder of this paper, we assume that the functions $f, g, G_j(\cdot, t), t \in T_j, j \in q$, and $H_k(\cdot, s), s \in S_k, k \in r$, are continuously differentiable on the open set $X$.

With regard to the choice of the type of generalized invex functions, specified in Definitions 2.5–2.7 in [2], to be used in the statements and proofs of our duality theorems, we shall consistently use the cases in which the functions $\alpha$ and $\beta$ are nonzero for all $(x, y) \in X \times X$. All the duality results established in this paper can be modified, restated, and proved for the other cases in a straightforward manner.

3. DUALITY MODEL I

In this section, we consider two dual problems with relatively simple constraint structures and prove weak, strong, and strict converse duality theorems under appropriate $(\alpha, \beta, \gamma, \eta, 0, \theta)$-invexity conditions. More general duality models and results for $(P)$ will be discussed in the subsequent sections.

Let

$$\mathbb{H} = \left\{(y, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu_0}, \bar{t}, \bar{s}) : y \in X; \lambda \in \mathbb{R}; 0 \leq \nu_0 \leq \nu \leq n + 1; v \in \mathbb{R}^\nu, v_i > 0, 1 \leq i \leq \nu_0; J_{\nu_0} = (j_1, j_2, \ldots, j_{\nu_0}), 1 \leq j_i \leq q; K_{\nu_0} = (k_{\nu_0 + 1}, \ldots, k_\nu), 1 \leq k_i \leq r; \bar{t} = (t^1, t^2, \ldots, t^{\nu_0}), t^i \in T_{j_i}; \bar{s} = (s^{\nu_0 + 1}, \ldots, s^\nu), s^i \in S_{k_i}\right\}.$$ 

Consider the following two problems:

$$\text{(DI)} \quad \sup_{(y, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu_0}, \bar{t}, \bar{s}) \in \mathbb{H}} \lambda \ subject \ to \ \text{(3.1)} \ \nabla f(y) - \lambda \nabla g(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m) = 0,$$

$$\text{(3.2)} \quad f(y) - \lambda g(y) + \sum_{m=1}^{\nu_0} v_m G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y, s^m) \geq 0;$$

$$\text{(\tilde{DI})} \quad \sup_{(y, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu_0}, \bar{t}, \bar{s}) \in \mathbb{H}} \lambda \ subject \ to \ \text{(3.2)} \ and$$
\[ (3.3) \quad \frac{1}{\beta(x,y)} \left\langle \nabla f(y) - \lambda \nabla g(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{jm}(y, t^m) \right. \\
\left. + \sum_{m=\nu_0+1}^\nu v_m \nabla H_{km}(y, s^m), e^{\beta(x,y)\eta(x,y)} - 1 \right \rangle \geq 0 \text{ for all } x \in \mathbb{F}, \]

where \( \beta \) is a function from \( X \times X \) to \( \mathbb{R} \) and \( \eta \) is a function from \( X \times X \) to \( \mathbb{R}^n \).

The structures of the two problems designated above as \((DI)\) and \((\tilde{D}I)\) are based directly on the form and contents of the necessary optimality conditions of Theorem 2.1. This is, of course, the standard method for constructing Wolfe-type dual problems. Comparing \((DI)\) and \((\tilde{D}I)\), we see that \((\tilde{D}I)\) is relatively more general than \((DI)\) in the sense that any feasible solution of \((DI)\) is also feasible for \((\tilde{D}I)\), but the converse is not necessarily true. Furthermore, we observe that (3.1) is a system of \( n \) equations, whereas (3.3) is a single inequality. Clearly, from a computational point of view, \((DI)\) is preferable to \((\tilde{D}I)\) because of the dependence of (3.3) on the feasible set of \((P)\). Despite these apparent differences, it turns out that the statements and proofs of all the duality theorems for \((P) - (DI)\) and \((P) - (\tilde{D}I)\) are almost identical and, therefore, we shall consider only the pair \((P) - (DI)\).

The next two theorems show that \((DI)\) is a dual problem for \((P)\).

**Theorem 3.1 (Weak Duality).** Let \( x \) and \((y, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu_0}, \bar{t}, \bar{s})\) be arbitrary feasible solutions of \((P)\) and \((DI)\), respectively, and assume that the Lagrangian-type function

\[ z \to L(z, v, \lambda, \bar{t}, \bar{s}) = f(z) - \lambda g(z) + \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m) + \sum_{m=\nu_0+1}^\nu v_m H_{km}(z, s^m) \]

is \((\alpha, \beta, \gamma, \eta, 0, \theta)\)-pseudoinvex at \( y \) and \( \gamma(x, y) > 0 \), where \( \bar{t} \equiv (t^1, t^2, \ldots, t^\nu) \) and \( \bar{s} \equiv (s^{\nu_0+1}, s^{\nu_0+2}, \ldots, s^\nu) \). Then \( \varphi(x) \geq \lambda \).

**Proof.** From (3.1) we observe that

\[ \frac{1}{\beta(x,y)} \left\langle \nabla f(y) - \lambda \nabla g(y) + \sum_{m=1}^{\nu_0} \nabla v_m G_{jm}(y, t^m) \right. \\
\left. + \sum_{m=\nu_0+1}^\nu \nabla v_m H_{km}(y, s^m), e^{\beta(x,y)\eta(x,y)} - 1 \right \rangle = 0, \]

which in view of our \((\alpha, \beta, \gamma, \eta, 0, \theta)\)-pseudoinvexity assumption implies that

\[ \frac{1}{\alpha(x,y)} \gamma(x, y) \left( e^{\alpha(x,y)[L(x,v,\lambda,\bar{t},\bar{s})-L(y,v,\lambda,\bar{t},\bar{s})]} - 1 \right) \geq 0. \]
We need to consider two cases: \( \alpha(x, y) > 0 \) and \( \alpha(x, y) < 0 \). If we assume that \( \alpha(x, y) > 0 \) and recall that \( \gamma(x, y) > 0 \), then the above inequality becomes

\[
e^{\alpha(x,y)[L(x,v,\lambda,t,s)−L(y,v,\lambda,t,s)]} \geq 1,
\]

which implies that

\[
L(x, v, \lambda, t, s) \geq L(y, v, \lambda, t, s).
\]

In view of the dual feasibility of \((y, v, \lambda, \nu_0, J_{\nu_0}, K_{\nu_0\setminus\nu_0}, \bar{t}, \bar{s})\) and (3.2), the right-hand side of the above inequality is greater than or equal to zero, and hence we have \( L(x, v, \lambda, t, s) \geq 0 \). Inasmuch as \( x \in \mathbb{R} \), and \( v_m > 0, m \in \nu_0 \), this inequality simplifies to \( f(x) − \lambda g(x) \geq 0 \). Hence, \( \varphi(x) = f(x)/g(x) \geq \lambda \).

If we assume that \( \alpha(x, y) < 0 \), we arrive at the same conclusion. \( \square \)

**Theorem 3.2 (Strong Duality).** Let \( x^* \) be a normal optimal solution of \((P)\) and assume that for each feasible solution \((y, v, \lambda, \nu_0, J_{\nu_0}, K_{\nu_0\setminus\nu_0}, \bar{t}, \bar{s})\) of \((DI)\), the conditions specified in Theorem 3.1 are satisfied. Then there exist \( v^*, \lambda^*, \nu^*, J_{\nu^*_0}, K_{\nu^*\setminus\nu^*_0}, \bar{t}^*, \) and \( \bar{s}^* \) such that \( (x^*, v^*, \lambda^*, \nu^*, \nu^*_0, J_{\nu^*_0}, K_{\nu^*\setminus\nu^*_0}, \bar{t}^*, \bar{s}^*) \) is an optimal solution of \((DI)\) and \( \varphi(x^*) = \lambda^* \).

**Proof.** Since \( x^* \) is a normal optimal solution of \((P)\), by Theorem 2.1, there exist \( v^*, \lambda^*, \nu^*, J_{\nu^*_0}, K_{\nu^*\setminus\nu^*_0}, \bar{t}^*, \) and \( \bar{s}^* \) such that \( w^* \equiv (x^*, v^*, \lambda^*, \nu^*, \nu^*_0, J_{\nu^*_0}, K_{\nu^*\setminus\nu^*_0}, \bar{t}^*, \bar{s}^*) \) is a feasible solution of \((DI)\) and \( \varphi(x^*) = \lambda^* \). If \( w^* \) were not optimal, then there would exist a feasible solution \((\tilde{x}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{v}_0, J_{\tilde{v}_0}, K_{\tilde{v}_0\setminus\tilde{v}_0}, \tilde{t}, \tilde{s})\) of \((DI)\) such that \( \varphi(x^*) = \lambda^* < \tilde{\lambda} \), which contradicts Theorem 3.1. Therefore, we conclude that \( w^* \) is an optimal solution of \((DI)\). \( \square \)

We also have the following converse duality result for \((P)\) and \((DI)\).

**Theorem 3.3 (Strict Converse Duality).** Let \( x^* \) be a normal optimal solution of \((P)\), let \((\tilde{x}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{v}_0, J_{\tilde{v}_0}, K_{\tilde{v}_0\setminus\tilde{v}_0}, \tilde{t}, \tilde{s})\) be an optimal solution of \((DI)\), and assume that the function \( z \rightarrow L(z, \tilde{v}, \tilde{\lambda}, \tilde{t}, \tilde{s}) \) is strictly \((\alpha, \beta, \gamma, \eta, 0, \theta)\)-pseudoinvex at \( \tilde{x} \) and \( \gamma(x^*, \tilde{x}) > 0 \). Then \( \tilde{x} = x^* \), that is, \( \tilde{x} \) is an optimal solution of \((P)\) and \( \varphi(x^*) = \lambda \).

**Proof.** (a): Suppose to the contrary that \( \tilde{x} \neq x^* \). By Theorem 2.1, there exist \( v^*, \lambda^*, \nu^*, \nu^*_0, J_{\nu^*_0}, K_{\nu^*\setminus\nu^*_0}, \bar{t}^*, \) and \( \bar{s}^* \) such that \( (x^*, v^*, \lambda^*, \nu^*, \nu^*_0, J_{\nu^*_0}, K_{\nu^*\setminus\nu^*_0}, \bar{t}^*, \bar{s}^*) \) is a feasible solution of \((DI)\) and \( \varphi(x^*) = \lambda^* \). Now proceeding as in the proof of Theorem 3.1 (with \( x \) replaced by \( x^* \) and \((y, v, \lambda, \nu_0, J_{\nu_0}, K_{\nu_0\setminus\nu_0}, \bar{t}, \bar{s})\) by \((\tilde{x}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{v}_0, J_{\tilde{v}_0}, K_{\tilde{v}_0\setminus\tilde{v}_0}, \tilde{t}, \tilde{s})\)) and using the condition set forth above, we arrive at the strict inequality \( f(x^*) − \tilde{\lambda} g(x^*) > 0 \), and so \( \varphi(x^*) > \tilde{\lambda} \), which contradicts the fact that \( \varphi(x^*) = \lambda^* \leq \tilde{\lambda} \). Therefore, we conclude that \( \tilde{x} = x^* \) and \( \varphi(x^*) = \tilde{\lambda} \). \( \square \)
4. DUALITY MODEL II

In this section, we consider two duality models with special constraint structures that allow for a greater variety of generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity conditions under which duality can be established.

Consider the following two problems:

\[ (DII) \quad \sup_{(y, v, \lambda, \nu, \nu_0, J_0, K_{\nu \setminus \nu_0, \tilde{t}, \tilde{s}})} \lambda \]

subject to

\[
\nabla f(y) - \lambda \nabla g(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m) = 0, \\
\]

\[
f(y) - \lambda g(y) \geq 0, \\
G_{j_m}(y, t^m) \geq 0, \quad m \in \nu_0, \\
v_m H_{k_m}(y, s^m) \geq 0, \quad m \in \nu \setminus \nu_0; \\
\]

\[
(\tilde{DII}) \quad \sup_{(y, v, \lambda, \nu, \nu_0, J_0, K_{\nu \setminus \nu_0, \tilde{t}, \tilde{s}})} \lambda \]

subject to (3.3) and (4.2)–(4.4).

The remarks and observations made earlier about the relationships between \((DII)\) and \((\tilde{DII})\) are, of course, also valid for \((DII)\) and \((\tilde{DII})\).

The next two theorems show that \((DII)\) is a dual problem for \((P)\).

**Theorem 4.1 (Weak Duality).** Let \(x\) and \(w \equiv (y, v, \lambda, \nu_0, J_0, K_{\nu \setminus \nu_0, \tilde{t}, \tilde{s}})\) be arbitrary feasible solutions of \((P)\) and \((DII)\), respectively, and assume that any one of the following five sets of hypotheses is satisfied:

(a) (i) \(z \to f(z) - \lambda g(z)\) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-pseudoinvex at \(y\) and \(\tilde{\gamma}(x, y) > 0\);

(ii) for each \(m \in \nu_0\), \(z \to G_{j_m}(z, t^m)\) is \((\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)\)-quasiinvex at \(y\);

(iii) for each \(m \in \nu \setminus \nu_0\), \(z \to v_m H_{k_m}(z, s^m)\) is \((\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)\)-quasiinvex at \(y\);

(iv) \(\rho(x, y) + \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) + \sum_{m=\nu_0+1}^{\nu} \hat{\rho}_m(x, y) \geq 0\);

(b) (i) \(z \to f(z) - \lambda g(z)\) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-pseudoinvex at \(y\) and \(\tilde{\gamma}(x, y) > 0\);

(ii) \(z \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(z, t^m)\) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiinvex at \(y\);

(iii) for each \(m \in \nu \setminus \nu_0\), \(z \to v_m H_{k_m}(z, s^m)\) is \((\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)\)-quasiinvex at \(y\);

(iv) \(\tilde{\rho}(x, y) + \hat{\rho}(x, y) + \sum_{m=\nu_0+1}^{\nu} \hat{\rho}_m(x, y) \geq 0\);

(c) (i) \(z \to f(z) - \lambda g(z)\) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-pseudoinvex at \(y\) and \(\tilde{\gamma}(x, y) > 0\);

(ii) for each \(m \in \nu_0\), \(z \to G_{j_m}(z, t^m)\) is \((\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)\)-quasiinvex at \(y\);
(iii) \( z \to \sum_{m=\nu_0+1}^{\nu} v_m H_{km}(z, s^m) \) is \((\alpha, \beta, \check{\gamma}, \eta, \check{\rho}, \theta)\)-quasiconvex at \( y \);
(iv) \( \check{\rho}(x, y) + \sum_{m=1}^{\nu_0} v_m \check{\rho}_m(x, y) + \check{\rho}(x, y) \geq 0; \)
(d) \( z \to f(z) - \lambda g(z) \) is \((\alpha, \beta, \check{\gamma}, \eta, \check{\rho}, \theta)\)-pseudoconvex at \( y \) and \( \check{\gamma}(x, y) > 0; \)
(ii) \( z \to \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m) \) is \((\alpha, \beta, \check{\gamma}, \eta, \check{\rho}, \theta)\)-quasiconvex at \( y; \)
(iii) \( z \to \sum_{m=\nu_0+1}^{\nu} v_m H_{km}(z, s^m) \) is \((\alpha, \beta, \check{\gamma}, \eta, \check{\rho}, \theta)\)-quasiconvex at \( y; \)
(iv) \( \check{\rho}(x, y) + \hat{\rho}(x, y) + \check{\rho}(x, y) \geq 0; \)
(e) \( z \to f(z) - \lambda g(z) \) is \((\alpha, \beta, \check{\gamma}, \eta, \check{\rho}, \theta)\)-pseudoconvex at \( y \) and \( \check{\gamma}(x, y) > 0; \)
(ii) \( z \to \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{km}(z, s^m) \) is \((\alpha, \beta, \check{\gamma}, \eta, \check{\rho}, \theta)\)-quasiconvex at \( y; \)
(iii) \( \check{\rho}(x, y) + \hat{\rho}(x, y) \geq 0. \)

Then \( \varphi(x) \geq \lambda. \)

Proof. (a): From the primal feasibility of \( x \) and (4.3) we see that \( G_{jm}(x, t^m) \leq 0 \leq G_{jm}(y, t^m) \) for each \( m \in \nu_0 \), and hence
\[
\frac{1}{\alpha(x, y)} \check{\gamma}_m(x, y) \left( e^{\alpha(x,y)}[G_{jm}(x,t^m) - G_{jm}(y,t^m)] - 1 \right) \leq 0,
\]
which in view of (ii) implies that
\[
\frac{1}{\beta(x, y)} \left\langle \nabla G_{jm}(y, t^m), e^{\beta(x,y)\eta(x,y)} - 1 \right\rangle \leq -\hat{\rho}_m(x, y)\|\theta(x, y)\|^2.
\]
As \( v_m > 0 \) for each \( m \in \nu_0 \), the above inequalities yield
(4.5)
\[
\frac{1}{\beta(x, y)} \left\langle \sum_{m=1}^{\nu_0} v_m \nabla G_{jm}(y, t^m), e^{\beta(x,y)\eta(x,y)} - 1 \right\rangle \leq -\sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y)\|\theta(x, y)\|^2.
\]
Similarly, from the primal feasibility of \( x \), (4.4), and (iii) we deduce that
(4.6)
\[
\frac{1}{\beta(x, y)} \left\langle \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{km}(y, s^m), e^{\beta(x,y)\eta(x,y)} - 1 \right\rangle \leq -\sum_{m=\nu_0+1}^{\nu} \check{\rho}_m(x, y)\|\theta(x, y)\|^2.
\]
Combining (4.1) with (4.5) and (4.6), and using (iv), we obtain
\[
\frac{1}{\beta(x, y)} \left\langle \nabla f(y) - \lambda \nabla g(y), e^{\beta(x,y)\eta(x,y)} - 1 \right\rangle \geq \left[ \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) + \sum_{m=\nu_0+1}^{\nu} \check{\rho}_m(x, y) \right]\|\theta(x, y)\|^2 \geq -\check{\rho}(x, y)\|\theta(x, y)\|^2,
\]
which in view of (i) implies that
\[
\frac{1}{\alpha(x, y)} \check{\gamma}(x, y) \left( e^{\alpha(x,y)}\{f(x) - \lambda g(x) - [f(y) - \lambda g(y)]\} - 1 \right) \geq 0.
\]
Since $\tilde{\gamma}(x, y) > 0$, this inequality implies that
\[ f(x) - \lambda g(x) \geq f(y) - \lambda g(y), \]
and therefore using (4.2) we have the desired inequality $\varphi(x) \geq \lambda$.

(b)–(e): The proofs are similar to that of part (a). □

**Theorem 4.2 (Strong Duality).** Let $x^*$ be a normal optimal solution of (P) and assume that any one of the five sets of conditions set forth in Theorem 4.1 is satisfied for all feasible solutions of (DII). Then there exist $v^*, \lambda^*, \nu^*, J_{\nu^*\setminus\nu_0^*}, \tilde{t}^*$, and $\tilde{s}^*$ such that $(x^*, \nu^*, \lambda^*, \nu^*, J_{\nu^*\setminus\nu_0^*}, \tilde{t}^*, \tilde{s}^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$.

**Proof.** The proof is similar to that of Theorem 3.2. □

**Theorem 4.3 (Strict Converse Duality).** Let $x^*$ be a normal optimal solution of (P), let $\tilde{w} = (\tilde{x}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}_0, J_{\tilde{\nu}_0\setminus\tilde{\nu}_0}, \tilde{t}, \tilde{s})$ be an optimal solution of (DII), and assume that any one of the five sets of conditions specified in Theorem 4.1 is satisfied and that the function $z \rightarrow f(z) - \tilde{\lambda} g(z)$ is strictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-pseudoinvex at $\tilde{x}$. Then $\tilde{x} = x^*$, that is, $\tilde{x}$ is an optimal solution of (P), and $\varphi(x^*) = \tilde{\lambda}$.

**Proof.** (a): Suppose to the contrary that $\tilde{x} \neq x^*$. By Theorem 2.1, there exist $v^*, \lambda^*, \nu^*, J_{\nu^*\setminus\nu_0^*}, \tilde{t}^*$, and $\tilde{s}^*$ such that $(x^*, \nu^*, \lambda^*, \nu^*, J_{\nu^*\setminus\nu_0^*}, \tilde{t}^*, \tilde{s}^*)$ is a feasible solution of (DII) and $\varphi(x^*) = \lambda^*$. Now proceeding as in the proof of Theorem 4.1 (with $x$ replaced by $x^*$ and $w$ by $\tilde{w}$) and using our strict $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-pseudoinvexity assumption, we arrive at the strict inequality $f(x^*) - \tilde{\lambda} g(x^*) > 0$ and so $\varphi(x^*) > \tilde{\lambda}$, which contradicts the fact that $\varphi(x^*) = \lambda^* \leq \tilde{\lambda}$. Therefore, we conclude that $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.

(b)–(e): The proofs are similar to that of part (a). □

**Theorem 4.4 (Weak Duality).** Let $x$ and $(y, v, \lambda, \nu, J_{\nu_0}, K_{\nu_0\setminus\nu_0^*}, \tilde{t}, \tilde{s})$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:

(a) (i) $z \rightarrow f(z) - \lambda g(z)$ is prestrictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-quasiinvex at $y$ and $\tilde{\gamma}(x, y) > 0$;

(ii) for each $m \in \nu_0$, $z \rightarrow G_{j_m}(z, t^m)$ is $(\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)$-quasiinvex at $y$;

(iii) for each $m \in \nu \setminus \nu_0$, $z \rightarrow v_m H_{k_m}(z, s^m)$ is $(\alpha, \beta, \tilde{\gamma}_m, \eta, \tilde{\rho}_m, \theta)$-quasiinvex at $y$;

(iv) $\tilde{\rho}(x, y) + \sum_{m=1}^{\nu_0} v_m \tilde{\rho}_m(x, y) + \sum_{m=\nu_0+1}^{\nu} \tilde{\rho}_m(x, y) > 0$;

(b) (i) $z \rightarrow f(z) - \lambda g(z)$ is prestrictly $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-quasiinvex at $y$ and $\tilde{\gamma}(x, y) > 0$;

(ii) $z \rightarrow \sum_{m=1}^{\nu_0} v_m G_{j_m}(z, t^m)$ is $(\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)$-quasiinvex at $y$;
The desired inequality

\[ \phi \lambda g \]

Since \( \bar{x} \), which in view of (i) implies that

\[ z \to f(z) - \lambda g(z) \] is prestrictly \( (\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \bar{\theta}) \)-quasiinvex at \( y \) and \( \bar{\gamma}(x, y) > 0 \);

\[ \rho(x, y) + \hat{\rho}(x, y) + \sum_{m=1}^{\nu_0} \hat{\rho}_m(x, y) > 0; \]

(c)

\[ z \to f(z) - \lambda g(z) \] is prestrictly \( (\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \bar{\theta}) \)-quasiinvex at \( y \) and \( \bar{\gamma}(x, y) > 0 \);

\[ z \to \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m) \] is \( (\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \bar{\theta}) \)-quasiinvex at \( y \);

\[ z \to \sum_{m=1}^{\nu_0} v_m H_{km}(z, s^m) \] is \( (\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \bar{\theta}) \)-quasiinvex at \( y \);

\[ \rho(x, y) + \hat{\rho}(x, y) + \hat{\rho}(x, y) > 0; \]

(d)

\[ z \to f(z) - \lambda g(z) \] is prestrictly \( (\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \bar{\theta}) \)-quasiinvex at \( y \) and \( \bar{\gamma}(x, y) > 0 \);

\[ z \to \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{km}(z, s^m) \] is \( (\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \bar{\theta}) \)-quasiinvex at \( y \);

\[ \bar{\rho}(x, y) + \hat{\rho}(x, y) > 0; \]

(e)

Then \( \varphi(x) \geq \lambda \).

**Proof.** (a) : Because of our assumptions specified in (ii) and (iii), (4.5) and (4.6) remain valid for the present case. From (4.1), (4.5), (4.6), and (iv) we deduce that

\[
\frac{1}{\beta(x, y)} \langle \nabla f(y) - \lambda \nabla g(y), e^{\beta(x,y)\eta(x,y)} - 1 \rangle \\
\geq \left[ \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) + \sum_{m=\nu_0+1}^{\nu} \hat{\rho}_m(x, y) \right] \| \theta(x, y) \|^2 > -\bar{\rho}(x, y) \| \theta(x, y) \|^2,
\]

which in view of (i) implies that

\[
\frac{1}{\alpha(x, y)} \bar{\gamma}(x, y) \left( e^{\alpha(x,y)\{f(x)-\lambda g(x)-|f(y)-\lambda g(y)|\}} - 1 \right) \geq 0.
\]

Since \( \bar{\gamma}(x, y) > 0 \), it follows from this inequality that \( f(x) - \lambda g(x) \geq f(y) - \lambda g(y) \geq 0 \), where the second inequality follows from (4.2). Therefore, we have the desired inequality \( \varphi(x) \geq \lambda \).

(b)–(e): The proofs are similar to that of part (a). \( \square \)

**Theorem 4.5 (Strong Duality).** Let \( x^* \) be a normal optimal solution of (P) and assume that any one of the five sets of conditions set forth in
Theorem 4.4 is satisfied for all feasible solutions of (DII). Then there exist $v^*, \lambda^*, \nu^*, J^*_0, K_{\nu^*\nu^*_0}, \bar{t}^*$, and $\bar{s}^*$ such that $(x^*, v^*, \lambda^*, \nu^*, J^*_0, K_{\nu^*\nu^*_0}, \bar{t}^*, \bar{s}^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$.

Proof. The proof is similar to that of Theorem 3.2. □

**Theorem 4.6 (Strict Converse Duality).** Let $x^*$ be a normal optimal solution of (P), let $\bar{w} \equiv (\bar{x}, \bar{v}, \bar{\lambda}, \bar{\nu}_0, J_{\bar{v}_0}, K_{\bar{v}_0\bar{v}_0}, \bar{t}, \bar{s})$ be an optimal solution of (DII), and assume that any one of the five sets of conditions set forth in Theorem 4.4 is satisfied for the feasible solution $\bar{z}$ of (DII), and assume that any one of the five sets of conditions set forth in Theorem 4.4 is satisfied for all feasible solutions of (P) and (DII), respectively, and assume that any one of the five sets of conditions set forth in Theorem 4.4 is satisfied for the feasible solution $\bar{w}$ of (DII) and that the function $z \to f(z) - \bar{\lambda}g(z)$ is $\alpha, \beta, \gamma, \eta, \rho, \theta$-quasiinvex at $\bar{x}$. Then $\bar{x} = x^*$ and $\varphi(x^*) = \lambda^*$.

Proof. The proof is similar to that of Theorem 4.3. □

**Theorem 4.7 (Weak Duality).** Let $x$ and $(y, v, \lambda, \nu, J_0, K_{\nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following seven sets of hypotheses is satisfied:

(a) (i) $z \to f(z) - \lambda g(z)$ is prestrictly $\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta$-quasiinvex at $y$ and $\bar{\gamma}(x, y) > 0$;

(ii) for each $m \in \nu_0$, $z \to G_{jm}(z, t^m)$ is strictly $\alpha, \beta, \hat{\gamma}_m, \eta, \hat{\rho}_m, \theta$-pseudoinvex at $y$;

(iii) for each $m \in \nu \setminus \nu_0$, $z \to v_m H_{km}(z, s^m)$ is $\alpha, \beta, \bar{\gamma}_m, \eta, \bar{\rho}_m, \theta$-quasiinvex at $y$;

(iv) $\bar{\rho}(x, y) + \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) + \sum_{m=\nu_0+1}^\nu \hat{\rho}_m(x, y) \geq 0$;

(b) (i) $z \to f(z) - \lambda g(z)$ is prestrictly $\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta$-quasiinvex at $y$ and $\bar{\gamma}(x, y) > 0$;

(ii) $z \to \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m)$ is strictly $\alpha, \beta, \hat{\gamma}_m, \eta, \hat{\rho}_m, \theta$-pseudoinvex at $y$;

(iii) for each $m \in \nu \setminus \nu_0$, $z \to v_m H_{km}(z, s^m)$ is $\alpha, \beta, \bar{\gamma}_m, \eta, \bar{\rho}_m, \theta$-quasiinvex at $y$;

(iv) $\bar{\rho}(x, y) + \hat{\rho}(x, y) + \sum_{m=\nu_0+1}^\nu \hat{\rho}_m(x, y) \geq 0$;

(c) (i) $z \to f(z) - \lambda g(z)$ is prestrictly $\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta$-quasiinvex at $y$ and $\bar{\gamma}(x, y) > 0$;

(ii) for each $m \in \nu_0$, $z \to G_{jm}(z, t^m)$ is $\alpha, \beta, \hat{\gamma}_m, \eta, \hat{\rho}_m, \theta$-quasiinvex at $y$;

(iii) for each $m \in \nu \setminus \nu_0$, $z \to v_m H_{km}(z, s^m)$ is strictly $\alpha, \beta, \bar{\gamma}_m, \eta, \bar{\rho}_m, \theta$-pseudoinvex at $y$;

(iv) $\bar{\rho}(x, y) + \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) + \sum_{m=\nu_0+1}^\nu \hat{\rho}_m(x, y) \geq 0$;

(d) (i) $z \to f(z) - \lambda g(z)$ is prestrictly $\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta$-quasiinvex at $y$ and $\bar{\gamma}(x, y) > 0$;
(ii) for each \( m \in \nu_0 \), \( z \rightarrow G_{jm}(z, t^m) \) is \((\alpha, \beta, \hat{\gamma}_m, \eta, \hat{\rho}_m, \theta)\)-quasiinvex at \( y \);

(iii) \( z \rightarrow \sum_{m=\nu_0+1}^{\nu} v_m H_{km}(z, s^m) \) is strictly \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-pseudoinvex at \( y \);

(iv) \( \bar{\rho}(x, y) + \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) + \bar{\rho}(x, y) \geq 0 \);

(e) (i) \( z \rightarrow f(z) - \lambda g(z) \) is prestrictly \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( y \) and \( \hat{\gamma}(x, y) > 0 \);

(ii) \( z \rightarrow \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m) \) is strictly \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-pseudoinvex at \( y \);

(iii) \( z \rightarrow \sum_{m=\nu_0+1}^{\nu} v_m H_{km}(z, s^m) \) is \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( y \);

(iv) \( \bar{\rho}(x, y) + \hat{\rho}(x, y) + \bar{\rho}(x, y) \geq 0 \);

(f) (i) \( z \rightarrow f(z) - \lambda g(z) \) is prestrictly \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( y \) and \( \hat{\gamma}(x, y) > 0 \);

(ii) \( z \rightarrow \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m) \) is \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( y \);

(iii) \( z \rightarrow \sum_{m=\nu_0+1}^{\nu} v_m H_{km}(z, s^m) \) is strictly \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-pseudoinvex at \( y \);

(iv) \( \bar{\rho}(x, y) + \hat{\rho}(x, y) + \bar{\rho}(x, y) \geq 0 \);

(g) (i) \( z \rightarrow f(z) - \lambda g(z) \) is prestrictly \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-quasiinvex at \( y \) and \( \hat{\gamma}(x, y) > 0 \);

(ii) \( z \rightarrow \sum_{m=1}^{\nu_0} v_m G_{jm}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{km}(z, s^m) \) is strictly \((\alpha, \beta, \hat{\gamma}, \eta, \hat{\rho}, \theta)\)-pseudoinvex at \( y \);

(iii) \( \bar{\rho}(x, y) + \hat{\rho}(x, y) \geq 0 \).

Then \( \varphi(x) \geq \lambda \).

Proof. (a): Suppose to the contrary that \( \varphi(x) < \lambda \). This means that \( f(x) - \lambda g(x) < 0 \), and hence

\[
\frac{1}{\alpha(x, y)} \hat{\gamma}(x, y) \left( e^{\alpha(x, y)} \{ f(x) - \lambda g(x) - [f(y) - \lambda g(y)] \} - 1 \right) < 0,
\]

which in view of (i) implies

\[
\frac{1}{\beta(x, y)} \langle \nabla f(y) - \lambda \nabla g(y), e^{\beta(x, y)} \eta(x, y) - 1 \rangle \leq -\bar{\rho}(x, y) \| \theta(x, y) \|^2.
\]

From the primal feasibility of \( x \) and (4.3) we see that \( G_{jm}(x, t^m) \leq 0 \leq G_{jm}(y, t^m) \) for each \( m \in \nu_0 \), and hence

\[
\frac{1}{\alpha(x, y)} \hat{\gamma}_m(x, y) \left( e^{\alpha(x, y)} \{ G_{jm}(x, t^m) - G_{jm}(y, t^m) \} - 1 \right) \leq 0,
\]

which in view of (ii) implies that

\[
\frac{1}{\beta(x, y)} \langle \nabla G_{jm}(y, t^m), e^{\beta(x, y)} \eta(x, y) - 1 \rangle < -\hat{\rho}_m(x, y) \| \theta(x, y) \|^2.
\]
As $v_m > 0$ for each $m \in \nu_0$, the above inequalities yield (4.8)
\[
\frac{1}{\beta(x, y)} \left\langle \sum_{m=1}^{\nu_0} v_m \nabla G_{jm}(y, t^m), e^{\beta(x,y)}\eta(x,y) - 1 \right\rangle < -\sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) \|\theta(x, y)\|^2.
\]
Now combining (4.1), (4.6) (which is valid for the present case because of our assumptions set forth in (iii)), (4.8), and (iv), we get
\[
\frac{1}{\beta(x, y)} \left\langle \nabla f(y) - \lambda \nabla g(y), e^{\beta(x,y)}\eta(x,y) - 1 \right\rangle > -\bar{\rho}(x, y) \|\theta(x, y)\|^2,
\]
which contradicts (4.7). Therefore, we conclude that $\varphi(x) \geq \lambda$.

(b)–(g): The proofs are similar to that of part (a). □

**Theorem 4.8 (Strong Duality).** Let $x^*$ be a normal optimal solution of $(P)$ and assume that any one of the seven sets of conditions set forth in Theorem 4.7 is satisfied for all feasible solutions of (DII). Then there exist $v^*, \lambda^*, \nu^*, J_{\nu^*}, K_{\nu^*\backslash\nu_0^*}, \bar{\nu}^*$, and $\bar{s}^*$ such that $(x^*, v^*, \lambda^*, \nu^*, J_{\nu^*}, K_{\nu^*\backslash\nu_0^*}, \bar{\nu}^*, \bar{s}^*)$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$.

**Proof.** The proof is similar to that of Theorem 3.2. □

**Theorem 4.9 (Strict Converse Duality).** Let $x^*$ be a normal optimal solution of $(P)$, let $\tilde{w} \equiv (\tilde{x}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}_0\backslash\tilde{\nu}_0}, \tilde{t}, \tilde{s})$ be an optimal solution of (DII), and assume that any one of the seven sets of conditions set forth in Theorem 4.7 is satisfied for the feasible solution $\tilde{w}$ of (DII), and the function $z \to f(z) - \tilde{\lambda} g(z)$ is $(\alpha, \beta, \gamma, \eta, \bar{\rho}, \theta)$-quasiinvex at $\tilde{x}$. Then $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.

**Proof.** (a): By Theorem 2.1, there exist $v^*, \lambda^*, \nu^*, J_{\nu^*}, K_{\nu^*\backslash\nu_0^*}, \bar{\nu}^*$, and $\bar{s}^*$ such that $(x^*, v^*, \lambda^*, \nu^*, J_{\nu^*}, K_{\nu^*\backslash\nu_0^*}, \bar{\nu}^*, \bar{s}^*)$ is a feasible solution of (DII) and $\varphi(x^*) = \lambda^*$. Suppose to the contrary that $\tilde{x} \neq x^*$. From the primal feasibility of $x^*$ and (4.3) we see that $G_{jm}(x^*, t^m) \leq 0 \leq G_{jm}(\tilde{x}, \tilde{t}^m)$ for each $m \in \nu_0$, and hence
\[
\frac{1}{\alpha(x^*, \tilde{x})} \hat{\gamma}_m(x^*, \tilde{x}) \left( e^{\alpha(x^*, \tilde{x})} [G_{jm}(x^*, t^m) - G_{jm}(\tilde{x}, \tilde{t}^m)] - 1 \right) \leq 0,
\]
which in view of (ii) implies that
\[
\frac{1}{\beta(x^*, \tilde{x})} \left\langle \nabla G_{jm}(\tilde{x}, \tilde{t}^m), e^{\beta(x^*, \tilde{x})}\eta(x^*, \tilde{x}) - 1 \right\rangle < -\hat{\rho}_m(x^*, \tilde{x}) \|\theta(x^*, \tilde{x})\|^2.
\]
As $\tilde{v}_m > 0$ for each $m \in \tilde{\nu}_0$, the above inequalities yield (4.9)
\[
\frac{1}{\beta(x^*, \tilde{x})} \left\langle \sum_{m=1}^{\tilde{\nu}_0} \tilde{v}_m \nabla G_{jm}(\tilde{x}, \tilde{t}^m), e^{\beta(x^*, \tilde{x})}\eta(x^*, \tilde{x}) - 1 \right\rangle < -\sum_{m=1}^{\tilde{\nu}_0} \tilde{v}_m \hat{\rho}_m(x^*, \tilde{x}) \|\theta(x^*, \tilde{x})\|^2.
\]
Now, combining (4.1), (4.6) (which is valid for the present case because of our assumptions set forth in (iii)), (4.9), and (iv), we get

\[ \frac{1}{\beta(x^*, \bar{x})} \langle \nabla f(\bar{x}) - \bar{\lambda} \nabla g(\bar{x}), e^{\beta(x^*, \bar{x})} \eta(x^*, \bar{x}) - 1 \rangle > -\bar{\rho}(x^*, \bar{x}) \| \theta(x^*, \bar{x}) \|^2, \]

which by virtue of the \((\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta)\)-quasiinvexity of the function \(z \rightarrow f(z) - \bar{\lambda} g(z)\) at \(\bar{x}\) implies that

\[ \frac{1}{\alpha(x^*, \bar{x})} \bar{\gamma}(x^*, \bar{x}) \left( e^{\alpha(x^*, \bar{x})} \{ f(x^*) - \bar{\lambda} g(x^*) - [f(\bar{x}) - \bar{\lambda} g(\bar{x})] \} - 1 \right) > 0. \]

Since \(\bar{\gamma}(x^*, \bar{x}) > 0\) and \(f(\bar{x}) - \bar{\lambda} g(\bar{x}) \geq 0\), this inequality implies that \(f(x^*) - \bar{\lambda} g(x^*) > 0\), and so \(\varphi(x^*) > \bar{\lambda}\), which contradicts the fact that \(\varphi(x^*) = \lambda^* \leq \bar{\lambda}\). Therefore, we conclude that \(\bar{x} = x^*\) and \(\varphi(x^*) = \bar{\lambda}\).

(b)–(g): The proofs are similar to that of part (a). □

5. DUALITY MODEL III

In this section, we discuss several families of duality results under various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity hypotheses imposed on certain combinations of the problem functions. This is accomplished by employing a certain partitioning scheme which was originally proposed in [1] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let \(\nu_0\) and \(\nu\) be integers, with \(1 \leq \nu_0 \leq \nu \leq n+1\), and let \(\{J_0, J_1, \ldots, J_M\}\) and \(\{K_0, K_1, \ldots, K_M\}\) be partitions of the sets \(\nu_0\) and \(\nu \setminus \nu_0\), respectively; thus, \(J_i \subseteq \nu_0\) for each \(i \in \nu \setminus \{0\}\), \(J_i \cap J_j = \emptyset\) for each \(i, j \in \nu \setminus \{0\}\) with \(i \neq j\), and \(\bigcup_{i=0}^{M} J_i = \nu_0\). Obviously, similar properties hold for \(\{K_0, K_1, \ldots, K_M\}\). Moreover, if \(m_1\) and \(m_2\) are the numbers of the partitioning sets of \(\nu_0\) and \(\nu \setminus \nu_0\), respectively, then \(M = \max\{m_1, m_2\}\) and \(J_i = \emptyset\) or \(K_i = \emptyset\) for \(i > \min\{m_1, m_2\}\).

In addition, we use the real-valued functions \(z \rightarrow \Phi(z, \nu, \lambda^*, \bar{\nu}, \bar{s})\), and \(z \rightarrow \Lambda_{\tau}(z, \nu, \bar{\nu}, \bar{s})\) defined, for fixed \(\nu, \lambda, \nu_0, J_{\nu_0}, K_{\nu \setminus \nu_0}, \bar{\nu}, \) and \(\bar{s}\), on \(X\) as follows:

\[ \Phi(z, \nu, \lambda^*, \bar{\nu}, \bar{s}) = f(z) - \lambda g(z) + \sum_{m \in J_0} v_m G_{j_m}(z, t^m) + \sum_{m \in K_0} v_m H_{k_m}(z, s^m), \]

\[ \Lambda_{\tau}(z, \nu, \bar{\nu}, \bar{s}) = \sum_{m \in J_{\tau}} v_m G_{j_m}(z, t^m) + \sum_{m \in K_{\tau}} v_m H_{k_m}(z, s^m), \quad \tau \in \nu \setminus \{0\}. \]

Making use of the sets and functions defined above, we can state our general duality models as follows:
subject to

\[(\tilde{DIII}) \quad \sup_{(y, v, \lambda, \nu, 0, J_{\nu}, K_{\nu \setminus 0}, \tilde{t}, \tilde{s}) \in \mathbb{H}} \lambda \]

subject to (3.3), (5.2), and (5.3).

The remarks and observations made earlier about the relationships between (DI) and (\tilde{D}) are, of course, also valid for (DIII) and (\tilde{DIII}).

The next two theorems show that (DIII) is a dual problem for (P).

**Theorem 5.1 (Weak Duality).** Let \(x\) and \(w = (y, v, \lambda, \nu, 0, J_{\nu}, K_{\nu \setminus 0}, \tilde{t}, \tilde{s})\) be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following four sets of hypotheses is satisfied:

(a) (i) \(z \to \Phi(z, v, \lambda, \tilde{t}, \tilde{s})\) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-pseudoinvex at \(y\) and \(\tilde{\gamma}(x, y) > 0\);
(ii) for each \(\tau \in M\), \(z \to \Lambda_{\tau}(z, v, \tilde{t}, \tilde{s})\) is \((\alpha, \beta, \tilde{\gamma}_{\tau}, \eta, \tilde{\rho}_{\tau}, \theta)\)-quasiconvex at \(y\);
(iii) \(\bar{\rho}(x, y) + \sum_{\tau=1}^{M} \bar{\rho}_{\tau}(x, y) \geq 0\);

(b) (i) \(z \to \Phi(z, v, \lambda, \tilde{t}, \tilde{s})\) is prestrictly \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiconvex at \(y\) and \(\tilde{\gamma}(x, y) > 0\);
(ii) for each \(\tau \in M\), \(z \to \Lambda_{\tau}(z, v, \tilde{t}, \tilde{s})\) is \((\alpha, \beta, \tilde{\gamma}_{\tau}, \eta, \tilde{\rho}_{\tau}, \theta)\)-quasiconvex at \(y\);
(iii) \(\bar{\rho}(x, y) + \sum_{\tau=1}^{M} \bar{\rho}_{\tau}(x, y) > 0\);

(c) (i) \(z \to \Phi(z, v, \lambda, \tilde{t}, \tilde{s})\) is prestrictly \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiconvex at \(y\) and \(\tilde{\gamma}(x, y) > 0\);
(ii) for each \(\tau \in M\), \(z \to \Lambda_{\tau}(z, v, \tilde{t}, \tilde{s})\) is strictly \((\alpha, \beta, \tilde{\gamma}_{\tau}, \eta, \tilde{\rho}_{\tau}, \theta)\)-pseudoinvex at \(y\);
(iii) \(\bar{\rho}(x, y) + \sum_{\tau=1}^{M} \bar{\rho}_{\tau}(x, y) \geq 0\);

(d) (i) \(z \to \Phi(z, v, \lambda, \tilde{t}, \tilde{s})\) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiconvex at \(y\) and \(\tilde{\gamma}(x, y) > 0\);
(ii) for each \(\tau \in M_1\), \(z \to \Lambda_{\tau}(z, v, \tilde{t}, \tilde{s})\) is \((\alpha, \beta, \tilde{\gamma}_{\tau}, \eta, \tilde{\rho}_{\tau}, \theta)\)-quasiconvex at \(y\), and for each \(\tau \in M_2 \neq \emptyset\), \(z \to \Lambda_{\tau}(z, v, \tilde{t}, \tilde{s})\) is strictly \((\alpha, \beta, \tilde{\gamma}_{\tau}, \eta, \tilde{\rho}_{\tau}, \theta)\)-pseudoinvex at \(y\), where \(\{M_1, M_2\}\) is a partition of \(M\);
(iii) $\tilde{\rho}(x, y) + \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, y) \geq 0$.

Then $\varphi(x) \geq \lambda$.

Proof. (a): It is clear that (5.1) can be expressed as follows:

$$
\nabla f(y) - \lambda \nabla g(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y, s^m) + \sum_{m \in J_\tau} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{km}(y, s^m) = 0.
$$

Since for each $\tau \in M$,

$$
\Lambda_\tau(x, v, \bar{t}, \bar{s}) = \sum_{m \in J_\tau} v_m G_{jm}(x, t^m) + \sum_{m \in K_\tau} v_m H_{km}(x, s^m)
\leq 0 \text{ (by the primal feasibility of } x \text{ and positivity of } v_m, m \in \nu_0) \\
\leq \sum_{m \in J_\tau} v_m G_{jm}(y, t^m) + \sum_{m \in K_\tau} v_m H_{km}(y, s^m) \text{ (by (5.3))}
= \Lambda_\tau(y, v, \bar{t}, \bar{s}),
$$

it follows that

$$
\frac{1}{\alpha(x, y)} \tilde{\gamma}_\tau(x, y) (e^{\alpha(x, y)|\Lambda_\tau(x, v, \bar{t}, \bar{s}) - \Lambda_\tau(y, v, \bar{t}, \bar{s})|} - 1) \leq 0,
$$

which in view of (ii) implies that

$$
\frac{1}{\beta(x, y)} \left( \sum_{m \in J_\tau} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{km}(y, s^m), e^{\beta(x, y)\eta(x, y)} - 1 \right) \\
\leq - \tilde{\rho}_\tau(x, y) \|\theta(x, y)\|^2.
$$

Summing over $\tau$, we obtain

$$
\frac{1}{\beta(x, y)} \left( \sum_{\tau=1}^{M} \sum_{m \in J_\tau} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{km}(y, s^m), e^{\beta(x, y)\eta(x, y)} - 1 \right) \\
\leq - \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, y) \|\theta(x, y)\|^2.
$$

Combining (5.4) and (5.5), and using (iii) we get

$$
\frac{1}{\beta(x, y)} \left( \nabla f(y) - \lambda \nabla g(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y, s^m), e^{\beta(x, y)\eta(x, y)} - 1 \right) \geq \sum_{\tau=1}^{M} \tilde{\rho}_\tau(x, y) \|\theta(x, y)\|^2 \\
\geq - \tilde{\rho}(x, y) \|\theta(x, y)\|^2,
$$
which by virtue of (i) implies that
\[
\frac{1}{\alpha(x,y)} \tilde{\gamma}(x,y) \left( e^{\alpha(x,y)[\Phi(x,v,\lambda,\bar{t},\bar{s}) - \Phi(y,v,\lambda,\bar{t},\bar{s})]} - 1 \right) \geq 0,
\]
Since \( \tilde{\gamma}(x,y) > 0 \), we can deduce from this inequality that \( \Phi(x,v,\lambda,\bar{t},\bar{s}) \geq \Phi(y,v,\lambda,\bar{t},\bar{s}) \geq 0 \), where the second inequality follows from (5.2). Therefore, we have
\[
0 \leq \Phi(x,v,\lambda,\bar{t},\bar{s}) = f(x) - \lambda g(x) + \sum_{m \in J_0} v_m G_{jm}(x,t^m) + \sum_{m \in K_0} v_m H_{km}(x,s^m)
\leq f(x) - \lambda g(x) \quad \text{(by the primal feasibility of } x).\]
Hence, \( \varphi(x) \geq \lambda \).

(b): The proof is similar to that of part (a).

(c) : Suppose to the contrary that \( \varphi(x) < \lambda \). This gives \( f(x) - \lambda g(x) < 0 \). Using this inequality and keeping in mind that \( v_m > 0 \) for each \( m \in \nu_0 \), we have
\[
\Phi(x,v,\lambda,\bar{t},\bar{s})
\leq f(x) - \lambda g(x) \quad \text{(by the primal feasibility of } x)
\leq 0
\leq \Phi(y,v,\lambda,\bar{t},\bar{s}) \quad \text{(by (5.2))},
\]
and hence
\[
\frac{1}{\alpha(x,y)} \tilde{\gamma}(x,y) \left( e^{\alpha(x,y)[\Phi(x,v,\lambda,\bar{t},\bar{s}) - \Phi(y,v,\lambda,\bar{t},\bar{s})]} - 1 \right) < 0,
\]
which in view of (i) implies that
\[
\frac{1}{\beta(x,y)} \left\langle \nabla f(y) - \lambda \nabla g(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y,t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y,s^m), e^{\beta(x,y)\eta(x,y)} - 1 \right\rangle \leq -\tilde{\rho}(x,y)\|\theta(x,y)\|^2.
\]
As shown in the proof of part (a), for each \( \tau \in M, \) we have \( \Lambda_\tau(x,v,\bar{t},\bar{s}) \leq \Lambda_\tau(y,v,\bar{t},\bar{s}) \), and hence
\[
\frac{1}{\alpha(x,y)} \tilde{\gamma}_\tau(x,y) \left( e^{\alpha(x,y)[\Lambda_\tau(x,v,\bar{t},\bar{s}) - \Lambda_\tau(y,v,\bar{t},\bar{s})]} - 1 \right) \leq 0,
\]
which in view of (ii) implies that
\[
\frac{1}{\beta(x,y)} \left\langle \sum_{m \in J_\tau} v_m \nabla G_{jm}(y,t^m) + \sum_{m \in K_\tau} v_m \nabla H_{km}(y,s^m), e^{\beta(x,y)\eta(x,y)} - 1 \right\rangle < -\tilde{\rho}_\tau(x,y)\|\theta(x,y)\|^2.
\]
Summing over \( \tau \), we obtain
\[
\frac{1}{\beta(x,y)} \left( \sum_{m \in J_0} v_m \nabla G_{jm}(y,t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y,s^m) \right), \varepsilon^{\beta(x,y)\eta(x,y)} - 1 \right) \\
< - \sum_{\tau=1}^M \tilde{\rho}_\tau(x,y) \| \theta(x,y) \|^2.
\]
Combining this inequality with (5.4) and using (iii), we get
\[
\frac{1}{\beta(x,y)} \left( \nabla f(y) - \lambda \nabla g(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y,t^m) \\
+ \sum_{m \in K_0} v_m \nabla H_{km}(y,s^m), \varepsilon^{\beta(x,y)\eta(x,y)} - 1 \right) > \sum_{\tau=1}^M \tilde{\rho}_\tau(x,y) \| \theta(x,y) \|^2 \\
\geq -\tilde{\rho}(x,y) \| \theta(x,y) \|^2,
\]
which contradicts (5.6). Therefore, we conclude that \( \varphi(x) \geq \lambda \).

(d): The proof is similar to that of part (c). \( \square \)

**THEOREM 5.2 (Strong Duality).** Let \( x^* \) be a normal optimal solution of (P) and assume that any one of the four sets of conditions set forth in Theorem 5.1 is satisfied for all feasible solutions of (DIII). Then there exist \( v^*, \lambda^*, \nu^*, \nu_*^0, J_{\nu^*0}, K_{\nu^*0}, t^*, s^* \) such that \((x^*, v^*, \lambda^*, \nu^*, \nu_*^0, J_{\nu^*0}, K_{\nu^*0}, t^*, s^*)\) is an optimal solution of (DIII) and \( \varphi(x^*) = \lambda^* \).

**Proof.** The proof is similar to that of Theorem 3.2. \( \square \)

**THEOREM 5.3 (Strict Converse Duality).** Let \( x^* \) be a normal optimal solution of (P), let \( \tilde{w} \equiv (\tilde{x}, \tilde{v}, \tilde{\lambda}, \tilde{\nu}, \tilde{\nu}_0, J_{\tilde{\nu}_0}, K_{\tilde{\nu}_0}, \tilde{t}, \tilde{s}) \) be an optimal solution of (DIII), and assume that any one of the following four sets of conditions holds:

(a) The assumptions specified in part (a) of Theorem 5.1 are satisfied for the feasible solution \( \tilde{w} \) of (DIII), and the function \( z \to \Phi(z, \tilde{v}, \tilde{\lambda}, \tilde{t}, \tilde{s}) \) is strictly \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-pseudoinvex at \( \tilde{x} \).

(b) The assumptions specified in part (b) of Theorem 5.1 are satisfied for the feasible solution \( \tilde{w} \) of (DIII), and the function \( z \to \Phi(z, \tilde{v}, \tilde{\lambda}, \tilde{t}, \tilde{s}) \) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiiinvex at \( \tilde{x} \).

(c) The assumptions specified in part (c) of Theorem 5.1 are satisfied for the feasible solution \( \tilde{w} \) of (DIII), and the function \( z \to \Phi(z, \tilde{v}, \tilde{\lambda}, \tilde{t}, \tilde{s}) \) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiiinvex at \( \tilde{x} \).

(d) The assumptions specified in part (d) of Theorem 5.1 are satisfied for the feasible solution \( \tilde{w} \) of (DIII), and the function \( z \to \Phi(z, \tilde{v}, \tilde{\lambda}, \tilde{t}, \tilde{s}) \) is \((\alpha, \beta, \tilde{\gamma}, \eta, \tilde{\rho}, \theta)\)-quasiiinvex at \( \tilde{x} \).

Then \( \tilde{x} = x^* \) and \( \varphi(x^*) = \tilde{\lambda} \).
Proof. (a) : Since $x^*$ is a normal optimal solution of ($P$), by Theorem 2.1, there exist $v^*, \lambda^*, \nu^*, \nu^*_0, J^*_{\nu^*_0}, K_{\nu^*_0} \setminus \nu^*_0, \bar{t}^*$, and $ar{s}^*$ such that $(x^*, \nu^*, \lambda^*, \nu^*, J^*_0, K_{\nu^*_0} \setminus \nu^*_0, \bar{t}^*, \bar{s}^*)$ is a feasible solution of ($DIII$) and $\varphi(x^*) = \lambda^*$. Suppose to the contrary that $\bar{x} \neq x^*$. Now proceeding as in the proof of part (a) of Theorem 5.1 (with $x$ replaced by $x^*$ and $w$ by $\bar{w}$), we arrive at the inequality

$$\frac{1}{\beta(x^*, \bar{x})} \left( \nabla f(\bar{x}) - \bar{\lambda} \nabla g(\bar{x}) + \sum_{m \in J_0} \bar{v}_m \nabla G_{jm}(\bar{x}, \bar{t}^m) + \sum_{m \in K_0} \bar{v}_m \nabla H_{km}(\bar{x}, \bar{s}^m) \right) e^{\beta(x^*, \bar{x})} \eta(x^*, \bar{x}) - 1 \geq -\bar{\rho}(x^*, \bar{x}) \|\theta(x^*, \bar{x})\|^2,$$

which by virtue of our strict $HA(\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta)$-pseudoinvexity assumption implies that

$$\frac{1}{\alpha(x^*, \bar{x})} \bar{\gamma}(x^*, \bar{x}) \left( e^{\alpha(x^*, \bar{x})}[\Phi(x^*, \bar{v}, \bar{\lambda}, \bar{t}, \bar{s}) - \Phi(\bar{x}, \bar{v}, \bar{\lambda}, \bar{t}, \bar{s})] - 1 \right) > 0.$$

Since $\bar{\gamma}(x^*, \bar{x}) > 0$, this inequality implies that

$$\Phi(x^*, \bar{v}, \bar{\lambda}, \bar{t}, \bar{s}) > \Phi(\bar{x}, \bar{v}, \bar{\lambda}, \bar{t}, \bar{s}) \geq 0,$$

where the second inequality follows from the dual feasibility of $\bar{w}$ and (5.2). In view of the primal feasibility of $x^*$ and dual feasibility of $\bar{w}$, the above inequality reduces to $f(x^*) - \bar{\lambda}g(x^*) > 0$, and so we get $\varphi(x^*) > \bar{\lambda}$, which contradicts the fact that $\varphi(x^*) = \lambda^* \leq \bar{\lambda}$. Therefore, we conclude that $\bar{x} = x^*$ and $\varphi(x^*) = \bar{\lambda}$.

(b)–(d): The proofs are similar to that of part (a). \(\Box\)

As pointed out earlier, the duality models ($DIII$) and ($\bar{DIII}$) can be viewed as two families of dual problems for ($P$) whose members can easily be identified by appropriate choices of the partitioning sets $J_\mu$ and $K_\mu$, $\mu \in M \cup \{0\}$. The special cases and variants of ($DIII$) and ($\bar{DIII}$) provide a multitude of dual problems for various classes of semiinfinite as well as conventional nonlinear programming problems.

The remainder of this paper will be devoted to a brief discussion of some nonparametric duality models for ($P$). It is of course possible to formulate the nonparametric analogues of the duality models ($DI$), ($\bar{DI}$), ($DII$) and ($\bar{DII}$), and state and prove the counterparts of Theorems 3.1–3.3 and Theorems 4.1–4.9. However, for the sake of avoiding excessive repetition, we shall not treat these nonparametric duality models separately. Instead, in the next section we shall consider two general nonparametric duality models which subsume the semiinfinite versions of several well-known nonparametric dual problems that have been studied previously in the literature of fractional programming.
6. Duality Model IV

In this section, we formulate two generalized nonparametric duality models for \((P)\) and prove appropriate duality theorems under various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity conditions.

In addition to the notation used in Section 5, we also need the function \(z \to \Pi(z, y, v, \bar{t}, \bar{s})\) defined, for fixed \(y, v, \bar{t}, \bar{s}\), on \(X\) by

\[
\Pi(z, y, v, \bar{t}, \bar{s}) = g(y) \left[ f(z) + \sum_{m \in J_0} v_m G_{jm}(z, t^m) + \sum_{m \in K_0} v_m H_{km}(z, s^m) \right]
- \left[ f(y) + \Lambda_0(y, v, \bar{t}, \bar{s}) \right] g(z).
\]

Let

\[
\mathbb{K} = \left\{ (y, v, \nu, \nu_0, J_{\nu_0}, K_{\nu_0}, \bar{t}, \bar{s}) : y \in X; 0 \leq \nu_0 \leq \nu \leq n+1; v \in \mathbb{R}^\nu, v_i > 0, \right. \\
\left. \text{for } 1 \leq i \leq \nu; J_{\nu_0} = (j_1, j_2, \ldots, j_{\nu_0}), 1 \leq j_i \leq q; K_{\nu_0} = (k_{\nu_0+1}, \ldots, k_{\nu}), 1 \leq k_i \leq r; \bar{t} = (t^1, t^2, \ldots, t^{\nu_0}), t^i \in T_{j_i}; \bar{s} = (s^{\nu_0+1}, \ldots, s^\nu), s^i \in S_{k_i} \right\}.
\]

Consider the following two nonparametric duality models for \((P)\):

\[
(DIV) \quad \sup_{(y, v, \nu, \nu_0, J_{\nu_0}, K_{\nu_0}, \bar{t}, \bar{s}) \in \mathbb{K}} \frac{f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})}{g(y)}
\]
subject to

\[
(6.1) \quad g(y) \left[ \nabla f(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y, s^m) \right] - [f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})] \nabla g(y) + \sum_{m \in \nu_0 J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in (\nu \backslash \nu_0) \backslash K_0} v_m \nabla H_{km}(y, s^m) = 0,
\]

\[
(6.2) \quad \sum_{m \in J_{\tau}} v_m G_{jm}(y, t^m) + \sum_{m \in K_{\tau}} v_m H_{km}(y, s^m) \geq 0, \quad \tau \in M;
\]

\[
(\bar{D}IV) \quad \sup_{(y, v, \nu, \nu_0, J_{\nu_0}, K_{\nu_0}, \bar{t}, \bar{s}) \in \mathbb{K}} \frac{f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})}{g(y)}
\]
subject to \((6.2)\) and

\[
\frac{1}{\beta(x, y)} \left( g(y) \left[ \nabla f(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y, s^m) \right] - [f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})] \nabla g(y) + \sum_{m \in \nu_0 J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in (\nu \backslash \nu_0) \backslash K_0} v_m \nabla H_{km}(y, s^m),
\]

\[
e^{\beta(x, y)\eta(x, y)} - 1 \right) \geq 0 \text{ for all } x \in \mathbb{F},
\]

where \(\beta : X \times X \to \mathbb{R}\) and \(\eta : X \times X \to \mathbb{R}^n\) are given functions.
The remarks and observations made earlier about the relationships between \((DI)\) and \((\bar{D}I)\) are, of course, also valid for \((DIV)\) and \((\bar{D}IV)\).

The next two theorems show that \((DIV)\) is a dual problem for \((P)\).

**THEOREM 6.1 (Weak Duality).** Let \(x\) and \(w \equiv (y, v, \nu, \nu_0, J_{\nu_0}, K_{\nu_k}, \bar{t}, \bar{s})\) be arbitrary feasible solutions of \((P)\) and \((DIV)\), respectively, and assume that

\[
f(y) + \Lambda_0(y, v, \bar{t}, \bar{s}) \geq 0, \quad g(y) > 0, \quad \text{and any one of the following four sets of hypotheses is satisfied:}
\]

(a) \(z \rightarrow \Pi(z, y, v, \bar{t}, \bar{s})\) is \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \bar{\theta})\)-pseudoinvex at \(y\) and \(\bar{\gamma}(x, y) > 0\);

(ii) for each \(\tau \in M\), \(z \rightarrow \Lambda_\tau(z, v, \bar{t}, \bar{s})\) is \((\alpha, \beta, \bar{\gamma}_\tau, \eta, \bar{\rho}_\tau, \theta)\)-quasiinvex at \(y\);

(iii) \(\bar{\rho}(x, y) + \sum_{\tau=1}^{m} \bar{\rho}_\tau(x, y) \geq 0\);

(b) \(z \rightarrow \Pi(z, y, v, \bar{t}, \bar{s})\) is prestrictly \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \bar{\theta})\)-quasiinvex at \(y\) and \(\bar{\gamma}(x, y) > 0\);

(ii) for each \(\tau \in M\), \(z \rightarrow \Lambda_\tau(z, v, \bar{t}, \bar{s})\) is \((\alpha, \beta, \bar{\gamma}_\tau, \eta, \bar{\rho}_\tau, \theta)\)-quasiinvex at \(y\);

(iii) \(\bar{\rho}(x, y) + \sum_{\tau=1}^{m} \bar{\rho}_\tau(x, y) > 0\);

(c) \(z \rightarrow \Pi(z, y, v, \bar{t}, \bar{s})\) is prestrictly \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \bar{\theta})\)-quasiinvex at \(y\) and \(\bar{\gamma}(x, y) > 0\);

(ii) for each \(\tau \in M\), \(z \rightarrow \Lambda_\tau(z, v, \bar{t}, \bar{s})\) is strictly \((\alpha, \beta, \bar{\gamma}_\tau, \eta, \bar{\rho}_\tau, \theta)\)-pseudoinvex at \(y\);

(iii) \(\bar{\rho}(x, y) + \sum_{\tau=1}^{m} \bar{\rho}_\tau(x, y) \geq 0\);

(d) \(z \rightarrow \Pi(z, y, v, \bar{t}, \bar{s})\) is prestrictly \((\alpha, \beta, \gamma, \eta, \bar{\rho}, \bar{\theta})\)-quasiinvex at \(y\) and \(\bar{\gamma}(x, y) > 0\);

(ii) for each \(\tau \in M_1\), \(z \rightarrow \Lambda_\tau(z, v, \bar{t}, \bar{s})\) is \((\alpha, \beta, \bar{\gamma}_\tau, \eta, \bar{\rho}_\tau, \theta)\)-quasiinvex at \(y\), and for each \(\tau \in M_2 \neq \emptyset\), \(z \rightarrow \Lambda_\tau(z, v, \bar{t}, \bar{s})\) is strictly \((\alpha, \beta, \bar{\gamma}_\tau, \eta, \bar{\rho}_\tau, \theta)\)-pseudoinvex at \(y\), where \(\{M_1, M_2\}\) is a partition of \(M\);

(iii) \(\bar{\rho}(x, y) + \sum_{\tau=1}^{m} \bar{\rho}_\tau(x, y) \geq 0\).

Then \(\varphi(x) \geq \psi(w)\), where \(\psi\) is the objective function of \((DIV)\).

**Proof.** (a): It is clear that (6.1) can be expressed as follows:

\[
(6.3) \quad g(y) \left[ \nabla f(y) + \sum_{m \in J_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{k_m}(y, s^m) \right]
\]

\[- \left[ f(y) + \Lambda_0(y, v, \bar{t}, \bar{s}) \right] \nabla g(y) + \sum_{\tau=1}^{M} \left[ \sum_{m \in J_\tau} v_m \nabla G_{j_m}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{k_m}(y, s^m) \right] = 0.
\]

Since for each \(\tau \in M\),
\[ \Lambda \tau(x, v, \bar{t}, s) = \sum_{m \in J \tau} v_m G_{jm}(x, t^m) + \sum_{m \in K \tau} v_m H_{km}(x, s^m) \leq 0 \quad \text{(by the primal feasibility of } x \text{ and positivity of } v_m, \, m \in \nu_0) \]
\[ \leq \sum_{m \in J \tau} v_m G_{jm}(y, t^m) + \sum_{m \in K \tau} v_m H_{km}(y, s^m) \quad \text{(by (6.2))} \]
\[ = \Lambda \tau(y, v, \bar{t}, s), \]
it follows that
\[ \frac{1}{\alpha(x, y)} \tilde{\gamma}(x, y) \left( e^{\alpha(x, y)[\Lambda \tau(x, v, \bar{t}, s) - \Lambda \tau(y, v, \bar{t}, s)]} - 1 \right) \leq 0, \]
which in view of (ii) implies that
\[ \frac{1}{\beta(x, y)} \left( \sum_{m \in J \tau} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K \tau} v_m \nabla H_{km}(y, s^m) \right) \left( e^{\beta(x, y)\eta(x, y)} - 1 \right) \leq -\tilde{\rho}(x, y) \| \theta(x, y) \|^2. \]
Summing over \( \tau \), we obtain
\[ (6.4) \quad \frac{1}{\beta(x, y)} \left( \sum_{\tau=1}^{M} \left[ \sum_{m \in J \tau} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K \tau} v_m \nabla H_{km}(y, s^m) \right] \right) \left( e^{\beta(x, y)\eta(x, y)} - 1 \right) \leq -\sum_{\tau=1}^{M} \tilde{\rho}(x, y) \| \theta(x, y) \|^2. \]
Combining (6.3) and (6.4), and using (iii) we get
\[ \frac{1}{\beta(x, y)} \left( g(y) \left[ \nabla f(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y, s^m) \right] \right. \]
\[ - \left. [f(y) + \Lambda_0(y, v, \bar{t}, s)] \nabla g(y), e^{\beta(x, y)\eta(x, y)} - 1 \right) \geq \sum_{\tau=1}^{M} \tilde{\rho}(x, y) \| \theta(x, y) \|^2 \]
\[ \geq -\overline{\rho}(x, y) \| \theta(x, y) \|^2, \]
which by virtue of (i) implies that
\[ \frac{1}{\alpha(x, y)} \tilde{\gamma}(x, y) \left( e^{\alpha(x, y)[\Pi(x, y, v, \bar{t}, s) - \Pi(y, y, v, \bar{t}, s)]} - 1 \right) \geq 0. \]
Since \( \tilde{\gamma}(x, y) > 0 \), we can deduce from this inequality that \( \Pi(x, y, v, \bar{t}, s) \geq \Pi(y, y, v, \bar{t}, s) = 0 \), where the equality follows from the definitions of \( \Pi \) and \( \Lambda_0 \). Therefore, we have
\[ 0 \leq \Pi(x, y, v, \bar{t}, s) = g(y) \left[ f(x) + \sum_{m \in J_0} v_m G_{jm}(x, t^m) + \sum_{m \in K_0} v_m H_{km}(x, s^m) \right] - [f(y) + \Lambda_0(y, v, \bar{t}, s)] g(x) \]
\[ \leq g(y) f(x) - [f(y) + \Lambda_0(y, v, \bar{t}, s)] g(x) \quad \text{(by the primal feasibility of } x). \]
Hence, we get the desired inequality

\[ \varphi(x) = \frac{f(x)}{g(x)} \geq \frac{f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})}{g(y)} = \psi(w). \]

(b): The proof is similar to that of part (a).

(c): Suppose to the contrary that \( \varphi(x) < \psi(w) \). This means that \( g(y) f(x) - [f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})] g(x) < 0 \). Using this inequality and keeping in mind that \( v_m > 0 \) for each \( m \in \nu_0 \), we have

\[
\Pi(x, y, v, \bar{t}, \bar{s}) = g(y) \left[ f(x) + \sum_{m \in J_0} v_m G_{jm}(x, t^m) + \sum_{m \in K_0} v_m H_{km}(x, s^m) \right] \\
- [f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})] g(x) \\
\leq g(y) f(x) - [f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})] g(x) \text{ (by the primal feasibility of } x) \\
< 0 \\
= \Pi(y, y, u, v, \bar{t}, \bar{s}) \text{ (by the definitions of } \Pi \text{ and } \Lambda_0),
\]

and hence

\[
\frac{1}{\alpha(x, y)} \tilde{\gamma}(x, y) \left( e^{\alpha(x, y)[\Pi(x, y, v, \bar{t}, \bar{s}) - \Pi(y, y, v, \bar{t}, \bar{s})]} - 1 \right) < 0,
\]

which in view of (i) implies that

\[
(6.5) \quad \frac{1}{\beta(x, y)} \left< g(y) \left[ \nabla f(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y, s^m) \right] \\
- [f(y) + \Lambda_0(y, v, \bar{t}, \bar{s})] \nabla g(y), e^{\beta(x, y)\eta(x, y)} - 1 \right> \leq -\tilde{\rho}(x, y) \|\theta(x, y)\|^2.
\]

As shown in the proof of part (a), for each \( \tau \in \mathcal{M} \), we have \( \Lambda_\tau(x, v, \bar{t}, \bar{s}) \leq \Lambda_\tau(y, v, \bar{t}, \bar{s}) \), and hence

\[
\frac{1}{\alpha(x, y)} \tilde{\gamma}_\tau(x, y) \left( e^{\alpha(x, y)[\Lambda_\tau(x, v, \bar{t}, \bar{s}) - \Lambda_\tau(y, v, \bar{t}, \bar{s})]} - 1 \right) \leq 0,
\]

which in view of (ii) implies that

\[
\frac{1}{\beta(x, y)} \left< \sum_{m \in J_\tau} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{km}(y, s^m), e^{\beta(x, y)\eta(x, y)} - 1 \right> < -\tilde{\rho}_\tau(x, y) \|\theta(x, y)\|^2.
\]

Summing over \( \tau \), we obtain

\[
\frac{1}{\beta(x, y)} \left< \sum_{\tau=1}^M \left[ \sum_{m \in J_\tau} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_\tau} v_m \nabla H_{km}(y, s^m) \right], e^{\beta(x, y)\eta(x, y)} - 1 \right> < -\sum_{\tau=1}^M \tilde{\rho}_\tau(x, y) \|\theta(x, y)\|^2.
\]

Combining this inequality with (6.3) and using (iii), we get
\[
\frac{1}{\beta(x,y)} \left[ g(y) \left( \nabla f(y) + \sum_{m \in J_0} v_m \nabla G_{jm}(y, t^m) + \sum_{m \in K_0} v_m \nabla H_{km}(y, s^m) \right) \right] \\
- \left[ f(y) + \Lambda_0(y, v, \bar{t}, \bar{s}) \right] \nabla g(y), e^{\beta(x,y)\eta(x,y)} - 1 > \sum_{\tau=1}^{M} \bar{\rho}_\tau(x,y) \|\theta(x,y)\|^2 \\
\geq -\bar{\rho}(x,y) \|\theta(x,y)\|^2,
\]

which contradicts (6.5). Therefore, we conclude that \( \varphi(x) \geq \psi(w) \).

(d): The proof is similar to that of part (c). \( \square \)

**Theorem 6.2 (Strong Duality).** Let \( x^* \) be a normal optimal solution of \( (P) \) and assume that any one of the four sets of conditions set forth in Theorem 6.1 is satisfied for all feasible solutions of \( (\text{DIV}) \). Then there exist \( v^*, \nu^*, \nu_0^*, J_{\nu^*} \setminus \nu_0^*, \bar{t}^*, \) and \( \bar{s}^* \) such that \( w^* \equiv (x^*, v^*, \nu^*, J_{\nu^*} \setminus \nu_0^*, K_{\nu^*} \setminus \nu_0^*, \bar{t}^*, \bar{s}^*) \) is an optimal solution of \( (\text{DIV}) \) and \( \varphi(x^*) = \psi(w^*) \).

**Proof.** (a): Since \( x^* \) is a normal optimal solution of \( (P) \), by Theorem 2.2, there exist \( \bar{v}, \nu^*, \nu_0^*, J_{\nu^*} \setminus \nu_0^*, \bar{t}^*, \) and \( \bar{s}^* \) such that
\[
g(x^*) \nabla f(x^*) - f(x^*) \nabla g(x^*) + \sum_{m=1}^{\nu_0^*} \bar{v}_m \nabla G_{jm}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} \bar{v}_m \nabla H_{km}(x^*, s^m) = 0.
\]

From this equation and the fact that \( x^* \in \mathbb{F} \) and \( t^m \in \hat{T}_{jm}(x^*) \) for each \( m \in \nu_0^* \), it is easily seen that
\[
\frac{1}{\beta(x,y)} \left[ g(y) \left( \nabla f(x^*) + \sum_{m \in J_0} v_m^* \nabla G_{jm}(x^*, t^m) + \sum_{m \in K_0} v_m^* \nabla H_{km}(x^*, s^m) \right) \right] \\
- \left[ f(x^*) + \Lambda_0(x^*, v^*, \bar{t}^*, \bar{s}^*) \right] \nabla g(y) = \sum_{m \in \nu_0^* \setminus J_0} v_m^* \nabla G_{jm}(x^*, t^m) + \sum_{m \in \nu_0^* \setminus K_0} v_m^* \nabla H_{km}(x^*, s^m) = 0,
\]

(6.6)
\[
\sum_{m \in J_0} v_m^* G_{jm}(x^*, t^m) + \sum_{m \in K_0} v_m^* H_{km}(x^*, s^m) = 0, \quad \tau \in M \cup \{0\},
\]

(6.7)
\[
\varphi(x^*) = \frac{f(x^*) + \Lambda_0(x^*, v^*, \bar{t}^*, \bar{s}^*)}{g(x^*)},
\]

(6.8)

where \( v_m^* = \bar{v}_m/g(x^*) \) for each \( m \in J_0 \cup K_0 \), and \( v_m^* = \bar{v}_m \) for each \( m \in \nu_0 \setminus (J_0 \cup K_0) \). From (6.6) and (6.7) it is clear that \( w^* \) is a feasible solution of \( (\text{DIV}) \). From (6.8) we see that \( \varphi(x^*) = \psi(w^*) \) and hence the optimality of \( w^* \) for \( (\text{DIV}) \) follows from Theorem 6.1.

(b)–(d): The proofs are similar to that of part (a). \( \square \)
Theorem 6.3 (Strict Converse Duality). Let $x^*$ be a normal optimal solution of $(P)$, let $\bar{w} = (\bar{x}, \bar{v}, \bar{\nu}, \nu_0, J_{\nu_0}, K_{\nu_0}, \tilde{t}, \tilde{s})$ be an optimal solution of (DIV), and assume that any one of the following four sets of conditions holds:

(a) The assumptions specified in part (a) of Theorem 6.1 are satisfied for the feasible solution $\bar{w}$ of (DIV), and $z \rightarrow \Pi(z, \bar{x}, \bar{v}, \bar{t}, \bar{s})$ is strictly $(\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta)$-pseudoinvex at $\bar{x}$;

(b) The assumptions specified in part (b) of Theorem 6.1 are satisfied for the feasible solution $\bar{w}$ of (DIV), and $z \rightarrow \Pi(z, \bar{x}, \bar{v}, \bar{t}, \bar{s})$ is $(\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta)$-quasiinvex at $\bar{x}$;

(c) The assumptions specified in part (c) of Theorem 6.1 are satisfied for the feasible solution $\bar{w}$ of (DIV), and $z \rightarrow \Pi(z, \bar{x}, \bar{v}, \bar{t}, \bar{s})$ is $(\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta)$-quasiinvex at $\bar{x}$;

(d) The assumptions specified in part (d) of Theorem 6.1 are satisfied for the feasible solution $\bar{w}$ of (DIV), and $z \rightarrow \Pi(z, \bar{x}, \bar{v}, \bar{t}, \bar{s})$ is $(\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta)$-quasiinvex at $\bar{x}$.

Then $\bar{x} = x^*$ and $\varphi(x^*) = \psi(\bar{w})$.

Proof. (a): Suppose to the contrary that $\bar{x} \neq x^*$. Since $x^*$ is a normal optimal solution of $(P)$, by Theorem 6.2, there exist $v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu_0^*}, t^*$, and $s^*$ such that $w^* \equiv (x^*, v^*, \nu^*, \nu_0^*, J_{\nu_0^*}, K_{\nu_0^*}, t^*, s^*)$ is a feasible solution of (DIV) and $\varphi(x^*) = \psi(w^*)$. Now, proceeding as in the proof of part (a) of Theorem 6.1 (with $x$ replaced by $x^*$ and $w$ by $\bar{w}$), we arrive at the inequality

$$\frac{1}{\beta(x^*, \bar{x})} \left[ g(\bar{x}) \left[ \nabla f_i(\bar{x}) + \sum_{m \in J_0} \bar{v}_m \nabla G_{jm}(\bar{x}, \bar{t}^m) + \sum_{m \in K_0} \bar{v}_m \nabla H_{km}(\bar{x}, \bar{s}^m) \right] - [f(\bar{x}) + \Lambda_0(\bar{x}, \bar{v}, \bar{t}, \bar{s})] \nabla g(\bar{x}), e^{\beta(x^*, \bar{x})} \eta(x^*, \bar{x}) - 1 \right] \geq -\bar{\rho}(x^*, \bar{x}) \left\| \theta(x^*, \bar{x}) \right\|^2,$$

which by virtue of our strict $(\alpha, \beta, \bar{\gamma}, \eta, \bar{\rho}, \theta)$-pseudoinvexity assumption implies that

$$\frac{1}{\alpha(x^*, \bar{x})} \bar{\gamma}(x^*, \bar{x}) \left( e^{\Pi(x^*, \bar{x}, \bar{v}, \bar{t}, \bar{s})} - \Pi(\bar{x}, \bar{v}, \bar{t}, \bar{s}) - 1 \right) > 0.$$

Since $\bar{\gamma}(x^*, \bar{x}) > 0$, this inequality implies that

$$\Pi(x^*, \bar{x}, \bar{v}, \bar{t}, \bar{s}) > \Pi(\bar{x}, \bar{v}, \bar{t}, \bar{s}) = 0.$$

In view of the primal feasibility of $x^*$, this inequality reduces to

$$g(\bar{x}) f(x^*) - [f(\bar{x}) + \Lambda_0(\bar{x}, \bar{v}, \bar{t}, \bar{s})] g(x^*) > 0,$$

and so we get

$$\varphi(x^*) = \frac{f(x^*)}{g(x^*)} > \frac{f(\bar{x}) + \Lambda_0(\bar{x}, \bar{v}, \bar{t}, \bar{s})}{g(\bar{x})} = \psi(\bar{w}),$$
which contradicts the fact that \( \varphi(x^*) = \psi(w^*) \leq \psi(\tilde{w}) \). Hence \( \tilde{x} = x^* \) and \( \varphi(x^*) = \psi(\tilde{w}) \).

(b)–(d): The proofs are similar to that of part (a). □

As pointed out earlier, the duality models \((DIV)\) and \((\tilde{DIV})\) can be viewed as two families of nonparametric dual problems for \((P)\) whose members can easily be identified by appropriate choices of the partitioning sets \(J_\mu\) and \(K_\mu, \mu \in M \cup \{0\}\). In particular, if we let \(J_0 = \nu_0\) and \(K_0 = \nu' \setminus \nu_0\), then we see that \((DIV)\) and \((\tilde{DIV})\) reduce to the following dual problems for \((P)\):

\[
(DIV_a) \quad \sup_{(y,v,\nu,\nu_0,J_\nu,\tilde{v},\tilde{s}) \in K} \frac{f(y) + \Lambda(y, v, \tilde{t}, \tilde{s})}{g(y)}
\]

subject to

\[
g(y) \left[ \nabla f(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t_m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s_m) \right] - [f(y) + \Lambda(y, v, \tilde{t}, \tilde{s})] \nabla g(y) = 0,
\]

where

\[
\Lambda(y, v, \tilde{t}, \tilde{s}) = \sum_{m=1}^{\nu_0} v_m G_{j_m}(y, t_m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(y, s_m);
\]

\[
(\tilde{DIV}_a) \quad \sup_{(y,v,\nu,\nu_0,J_\nu,\tilde{v},\tilde{s}) \in K} \frac{f(y) + \Lambda(y, v, \tilde{t}, \tilde{s})}{g(y)}
\]

subject to

\[
\frac{1}{\beta(x,y)} \left[ \nabla f(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t_m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s_m) \right] - [f(y) + \Lambda(y, v, \tilde{t}, \tilde{s})] \nabla g(y), e^{\beta(x,y)\eta(x,y)} - 1 \right] \geq 0 \quad \text{for all } x \in F,
\]

where \(\beta\) is a function from \(X \times X\) to \(\mathbb{R}\) and \(\eta\) is a function from \(X \times X\) to \(\mathbb{R}^n\).

Similarly, if we choose \(J_0 = \emptyset\) and \(K_0 = \emptyset\), then we obtain the following dual problems for \((P)\):

\[
(DIV_b) \quad \sup_{(y,v,\nu,\nu_0,J_\nu,\tilde{v},\tilde{s}) \in K} \frac{f(y)}{g(y)}
\]

subject to

\[
g(y) \nabla f(y) - f(y) \nabla g(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t_m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s_m) = 0;
\]

\[
(6.9) \quad \sum_{m \in J_\tau} v_m G_{j_m}(y, t_m) + \sum_{m \in K_\tau} v_m H_{k_m}(y, s_m) \geq 0, \quad \tau \in M;
\]
(DIV\(_b\)) \[\sup_{(y,v,\nu,\nu_0,J_{\nu_0},K_{\nu_0}\setminus\nu,\bar{t},\bar{s})\in K} \frac{f(y)}{g(y)}\]

subject to (6.9) and

\[\left\langle g(y)\nabla f(y) - f(y)\nabla g(y) + \sum_{m=1}^{\nu_0} v_m \nabla G_j(y,t^m)\right.\]
\[\left.+ \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_k(y,s^m), e^\beta(x,y)\eta(x,y) - 1\right\rangle \geq 0 \quad \text{for all } x \in F,\]

where \(\beta\) is a function from \(X \times X\) to \(\mathbb{R}\) and \(\eta\) is a function from \(X \times X\) to \(\mathbb{R}^n\).

In this manner, one can easily identify many other special cases of (DIV) and (DIV\(_b\)).

The dual problems (DIV\(_a\)), (DIV\(_a\)), (DIV\(_b\)), and (DIV\(_b\)) are the semiinfinite versions of some popular dual problems that have been investigated previously in the area of fractional programming. Evidently, one can easily specialize and restate Theorems 6.1–6.3 for these special cases of (DIV) and (DIV\(_b\)).

7. CONCLUDING REMARKS

In this study, we have constructed six first-order parametric and two first-order nonparametric duality models and established a fairly large number of duality results under various generalized \((\alpha, \beta, \gamma, \eta, \rho, \theta)\)-invexity hypotheses for a semiinfinite fractional programming problem. It appears that all these results are new in the area of semiinfinite programming. Furthermore, the style and techniques employed in this paper can be utilized to establish similar duality results for some other classes of related optimization problems. For example, it seems reasonable to expect that a similar approach can be used to investigate the optimality and duality aspects of the following classes of semiinfinite continuous minmax and multiobjective fractional programming problems:

\[
\begin{align*}
\text{Minimize} & \quad \max_{y \in Y} \frac{f(x,y)}{g(x,y)}, \\
\text{Minimize} & \quad \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right).
\end{align*}
\]

We shall investigate these classes of semiinfinite programming problems in subsequent papers.
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