We discuss Darboux-Staude type of thread configurations for the ellipsoid similar to Chasles-Graves type of thread configurations for the ellipse. These threads are formed by rectilinear segments, geodesic and line of curvature segments on the considered ellipsoid and with tangents tangent to the given ellipsoid and a fixed confocal hyperboloid with one sheet and preserve constant length when the vertices of the configuration move on confocal ellipsoids.

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1. INTRODUCTION

In trying to familiarize ourselves with classical results about confocal quadrics, Staude’s 1882 thread construction of confocal ellipsoids appears as an important result. In Hilbert-Cohn-Vossen ([10], §4) this construction appears as a generalization of thread construction of confocal ellipses, the two foci (singular sets of singular ellipses and hyperbolas of the confocal family and by means of which one jumps from one type of confocal conics to the other) being replaced with orthogonal focal curves (an ellipse and a hyperbola in orthogonal planes and with vertices of the one being the foci of the other) and thus, singular sets of singular ellipsoids (hyperboloids) of the confocal family and by means of which one jumps from one type of confocal quadrics to another.

Note however, that Staude’s original construction as it appears in Salmon ([11], §421 a,b) allows the initial fixed ellipsoid and hyperboloid with one sheet of the confocal family to be nonsingular, the singular case being obtained as a limit (also, in the last sections §397–§421 of Salmon [11], other type of thread configurations for quadrics are discussed; for example, Chasles’s result about a thread fixed at two points on a quadric, stretched with a pen and the pen thus moving on a line of curvature of a quadric confocal to the given one).

According to Salmon ([11], §421 a,b) this non-singular thread construction of confocal ellipsoids can be interpreted as the generalization of Graves’s non-singular thread construction of confocal ellipses using a thread passed around a given ellipse.
According to the Editor Reginald A.P. Rogers’ preface to Salmon [11] (referring to Staude’s thread construction of confocal quadrics) “In the Golden Age of Euclidean geometry, analogues of these types were of great interest to men like Jacobi, MacCullagh, Chasles and M. Roberts, but Staude’s constructions have virtually brought the subject to a conclusion. Staude’s treatment is also an excellent illustration of the elementary and visible meaning of elliptic and hyper-elliptic integrals.”

Unfortunately, Staude’s original paper [12] on addition of hyper-elliptic integrals is beyond our grasp, mainly due to our ignorance of German; however, Darboux’s generalization from 1870 of Chasles’s result from moving polygons circumscribed to an ellipse, with vertices situated on confocal ellipses (and thus, of constant perimeter) to moving polygons circumscribed to a quadric, with vertices situated on other quadrics confocal to the given one (and thus, of constant perimeter) seems to be discussed at length there, so it may be the case that Staude himself has other results about thread configurations for quadrics (see also footnote in Darboux ([4], Vol 2, §466)); in fact, it is the similarities of the arguments from Salmon ([11], §421 a,b) with those of Darboux ([4], Vol 2, Livre IV, Ch. XIV) that drew our attention to this project.

In Coolidge ([3], Ch. XIII) Staude’s thread construction of confocal quadrics is generalized to non-Euclidean geometries (space forms).

For Chasles’s result, the basic result is a theorem due to Graves (an identity involving an elliptic integral which at the geometric level boils down to the excess between the sum of the lengths of tangents to the given ellipse from a point on an ellipse confocal to the given one and the subtended arc of the given ellipse being independent of the point on the confocal ellipse); conversely, Chasles’s result implies Graves’s.

For Darboux’s generalization of Chasles’s result to dimension three the differential equation of a line tangent to two confocal quadrics and of its linear element in elliptic coordinates plays a fundamental role. On one hand, the linear element is a perfect square, which allows separation and separate accounting of the elliptic variables and on the other hand straight lines, having tangents tangent to two confocal quadrics (Chasles-Jacobi), have linear element amenable to these type of computations; moreover, reflections in confocal quadrics (which appear at the vertices of thread configurations) are accounted just by changing the sign of the variation of the corresponding elliptic coordinate; thus, threads formed by rectilinear segments tangent to the given ellipsoid and hyperboloid with one sheet will preserve constant length when the vertices will move on ellipsoids confocal to the given one. However, Darboux’s result is not of a general nature; it requires certain rationality conditions similar to the rationality conditions required by closed geodesics on ellipsoids.
Note that the same theorem of Chasles-Jacobi allows the computations of lines in elliptic coordinates to be extended to geodesic segments with tangents tangent to the same two confocal quadrics; their part involving the linear element is also trivially extended to segments of intersections of the two given quadrics (lines of curvature). While line of curvature segments are not locally length minimizing under the condition of being situated on one side of a quadric, they are situated on the same side of two quadrics and thus, are allowed in thread configurations as boundary requirements; if one (or both) quadric(s) becomes singular, then the line of curvature segments become geodesic segments and we have a genuine variational problem.

Staude’s thread construction of the ellipsoid as it appears in Salmon ([11], §421 a,b) has only one vertex and two (either possibly void) line of curvature segments; thus, in this vein Chasles’s and Graves’s result are in a relation analogous to that of Darboux’s and Staude’s.

We can extend threads to allow them to be formed by rectilinear segments, geodesic segments and line of curvature segments with common tangent at the points of change from one type of segment to the other (thus, the threads will be analytic on pieces and (excepting vertices) with continuous derivative); keeping in mind variations of elliptic coordinates (certain of their extreme values are alternatively attained) their length will be constant.

For threads without line of curvature segments a rationality condition remains; if we allow line of curvature segments, then the rationality condition disappears.

We also derive the algebraic structure of vertex configuration via the Ivory affine transformation between confocal quadrics.

Using this local result and Darboux’s constant perimeter property, concerning moving polygons circumscribed to a given set of $n$ and inscribed in arbitrarily many $n$-dimensional confocal quadrics, we derive Darboux’s result as a variational principle, valid in a more general form (complex setting) and which covers all (totally real) quadrics in the complex Euclidean space.

All results except Proposition 3.3, §5 and §6 are classical and due as indicated; §5 and §6 are generalizations of the classical results.

In §5 we generalize Staude’s thread construction of the ellipsoid to arbitrary many vertices on the ellipsoid and in §6 we provide a generalization of Darboux’s result to arbitrary confocal quadrics and using only algebraic computations instead of computations in elliptic coordinates.

A simple internet search with keywords reveals most of the current literature in this area.

There is some recent work on Chasles-Darboux type results, related to closed billiard trajectories, confocal quadrics and hyper-elliptic integrals (see
Dragović-Radnović [6, 7] for a synthetic approach, Flatto [9], Tabachnikov [13] and their references) and to closed geodesics on the ellipsoid (classically studied by Jacobi, Weierstrass, etc. and recently by Knorrer, Moser, etc., see Abenda-Fedorov [1], Fedorov [8] and their references). For example, in Darboux’s result when the vertices are situated all on the same ellipsoid the polygon in question can be viewed as a closed geodesic on a degenerate 3-dimensional ellipsoid (the double cover of the interior of the 2-dimensional ellipsoid in question) and closed geodesics on ellipsoids require certain rationality conditions; these rationality conditions remain valid in the degenerate case and are precisely those found by Darboux.

2. CONFOCAL QUADRICS IN CANONICAL FORM

Consider the complexified Euclidean space

$$(\mathbb{C}^{n+1}, \langle \ldots \rangle), \quad \langle x, y \rangle := x^T y, \quad |x|^2 := x^T x, \quad x, y \in \mathbb{C}^{n+1}$$

with standard basis $\{e_j\}_{j=1,\ldots,n+1}, \quad e_j^T e_k = \delta_{jk}$.

Isotropic (null) vectors are those vectors $v$ of length 0 ($|v|^2 = 0$); since most vectors are not isotropic we shall call a vector simply vector and we shall only emphasize isotropic when the vector is assumed to be isotropic. The same denomination will apply in other settings: for example, we call quadric a non-degenerate quadric (a quadric projectively equivalent to the complex unit sphere).

**Definition 2.1.** A quadric $x \subset \mathbb{C}^{n+1}$ is given by the quadratic equation

$$Q(x) := \begin{bmatrix} x^T & A & B \\ 1 & B^T & C \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = x^T(Ax + 2B) + C = 0,$$

$$A = A^T \in M_{n+1}(\mathbb{C}), \quad B \in \mathbb{C}^{n+1}, \quad C \in \mathbb{C}, \quad \begin{vmatrix} A & B \\ B^T & C \end{vmatrix} \neq 0.$$

There are many definitions of totally real (sub)spaces of $\mathbb{C}^{n+1}$, some even involving a hermitian inner product, but all definitions coincide: an $(n + 1)$-totally real subspace of $\mathbb{C}^{n+1}$ is of the form $(R, t)(\mathbb{R}^k \times (i\mathbb{R})^{n+1-k}), \quad k = 0, \ldots, n+1$, where $(R, t) \in O_{n+1}(\mathbb{C}) \ltimes \mathbb{C}^{n+1}$. Now, a totally real quadric is simply an $n$-dimensional quadric in an $(n + 1)$-totally real subspace of $\mathbb{C}^{n+1}$.

A metric classification of all (totally real) quadrics in $\mathbb{C}^{n+1}$ requires the notion of symmetric Jordan canonical form of a symmetric complex matrix. The symmetric Jordan blocks are:

$$J_1 := 0 = 0_{1,1} \in M_1(\mathbb{C}), \quad J_2 := f_1 f_1^T \in M_2(\mathbb{C}), \quad J_3 := f_1 e_3^T + e_3 f_1^T \in M_3(\mathbb{C}),$$

$$J_4 := f_1 f_2^T + f_2 f_1^T + e_2 f_1^T \in M_4(\mathbb{C}), \quad J_5 := f_1 f_2^T + f_2 e_5^T + e_5 f_2^T + e_2 f_1^T \in M_5(\mathbb{C}),$$
\[ J_6 := f_1 \overline{f}_2^T + f_2 \overline{f}_3^T + f_3 \overline{f}_2^T + \overline{f}_3 f_2^T + \overline{f}_2 f_1^T \in M_6(\mathbb{C}), \]

eetc., where \( f_j := \frac{e_{2j-1} + ie_{2j}}{\sqrt{2}} \) are the standard isotropic vectors (at least the blocks \( J_2, J_3 \) were known to the classical geometers). Any symmetric complex matrix can be brought via conjugation with a complex rotation to the symmetric Jordan canonical form, that is a matrix block decomposition with blocks of the form \( a_j I_p + J_p \); totally real quadrics are obtained for eigenvalues \( a_j \) of the quadratic part \( A \) defining the quadric being real or coming in complex conjugate pairs \( a_j, \overline{a}_j \) with subjacent symmetric Jordan blocks of same dimension \( p \). Just as the usual Jordan block \( \sum_{j=1}^{p} e_j e_j^T \) is nilpotent with \( e_{p+1} \) cyclic vector of order \( p \), \( J_p \) is nilpotent with \( \overline{f}_1 \) cyclic vector of order \( p \), so we can take square roots of symmetric Jordan matrices without isotropic kernels \( (\sqrt{a} I_p + J_p) \), \( a \in \mathbb{C}^*, \sqrt{a} := \sqrt{re^{i\theta}} \) for \( a = re^{2i\theta}, 0 < r, -\pi < 2\theta \leq \pi \), two matrices with same symmetric Jordan decomposition type (that is \( J_p \) is replaced with a polynomial in \( J_p \)) commute, etc.

**Definition 2.2.** The confocal family \( \{x_z\}_{z \in \mathbb{C}} \) of a quadric \( x_0 \subset \mathbb{C}^{n+1} \) in canonical form (depending on as few constants as possible) is given in the projective space \( \mathbb{CP}^{n+1} \) by the equation

\[
Q_z(x_z) := \begin{bmatrix} x_z \\ 1 \end{bmatrix}^T \left( \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} - z \begin{bmatrix} I_{n+1} & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} x_z \\ 1 \end{bmatrix} = 0,
\]

where

- \( A = A^T \in \text{GL}_{n+1}(\mathbb{C}) \) symmetric Jordan, \( B = 0 \in \mathbb{C}^{n+1}, C = -1 \), for quadrics with center;
- \( A = A^T \in \text{M}_{n+1}(\mathbb{C}) \) symmetric Jordan, \( \ker(A) = \mathbb{C}e_{n+1}, B = -e_{n+1}, C = 0 \), for quadrics without center and
- \( A = A^T \in \text{M}_{n+1}(\mathbb{C}) \) symmetric Jordan, \( \ker(A) = \mathbb{C}f_1, B = -\overline{f}_1, C = 0 \), for isotropic quadrics without center.

The totally real confocal family of a totally real quadric is obtained for \( z \in \mathbb{R} \).

From the definition one can see that the family of quadrics confocal to \( x_0 \) is the adjugate of the pencil generated by the adjugate of \( x_0 \) and Cayley’s absolute \( C(\infty) \subset \mathbb{CP}^n \) in the hyperplane at infinity; since Cayley’s absolute encodes the Euclidean structure of \( \mathbb{C}^{n+1} \) (it is the set invariant under rigid motions and homotheties of \( \mathbb{C}^{n+1} := \mathbb{CP}^{n+1} \setminus \mathbb{CP}^n \)) the mixed metric-projective character of the confocal family becomes clear.
For quadrics with center spec$(A)$ is unambiguous (does not change under rigid motions $(R, t) \in O_{n+1}(\mathbb{C}) \times \mathbb{C}^{n+1}$) but for (isotropic) quadrics without center it may change with $(p + 1)$-roots of unity for the block of $(f_1$ in $A$ being $J_p$) $e_{n+1}$ in $A$ being $J_1$ even under rigid motions which preserve the canonical form, so it is unambiguous up to $(p + 1)$-roots of unity; for simplicity we make a choice and work with it.

We have the diagonal quadrics with(out) center respectively, for $A = \sum_{j=1}^{n+1} a_j^{-1} e_j e_j^T$, $A = \sum_{j=1}^{n} a_j^{-1} e_j e_j^T$; the diagonal isotropic quadrics without center come in different flavors, according to the block of $f_1 : A = J_p + \sum_{j=p+1}^{n+1} a_j^{-1} e_j e_j^T$; in particular, if $A = J_{n+1}$, then spec$(A) = \{0\}$ is unambiguous. General quadrics are those for which all eigenvalues have geometric multiplicity 1; equivalently, each eigenvalue has an only corresponding symmetric Jordan block; in this case the quadric also admits elliptic coordinates.

There are continuous groups of symmetries which preserve the symmetric Jordan canonical form for more than one symmetric Jordan block corresponding to an eigenvalue, so from a metric point of view a metric classification according to the elliptic coordinates and continuous symmetries may be a better one.

With $R_z := I_{n+1} - zA$, $z \in \mathbb{C} \setminus \text{spec}(A)^{-1}$ the family of quadrics $\{x_z\}_z$ confocal to $x_0$ is given by

$$Q_z(x_z) = x_z^T A R_z^{-1} x_z + 2(R_z^{-1} B)^T x_z + C + z B^T R_z^{-1} B = 0.$$

For $z \in \text{spec}(A)^{-1}$ we obtain singular confocal quadrics; those with $z^{-1}$ having geometric multiplicity 1 admit a singular set which is an $(n-1)$-dimensional quadric projectively equivalent to $C(\infty)$, so they will play an important role in the discussion of homographies $H \in \text{PGL}_{n+1}(\mathbb{C})$ taking a confocal family into another one, since $H^{-1}(C(\infty))$, $C(\infty)$ respectively $C(\infty)$, $H(C(\infty))$ will suffice to determine each confocal family. Such homographies preserve all metric-projective properties of confocal quadrics (including the good metric properties of the Ivory affine transformation) and thus, all integrable systems whose integrability depends only on the family of confocal quadrics (the resulting involutary transformation between integrable systems is called Hazidakis by Bianchi for the integrable system in discussion being the problem of isometrically deforming quadrics). While the spectrum ($z$’s of $Q_z$) of a family of confocal quadrics is not well defined, the relative spectrum (difference of $z$’s) is; thus, we can consider $Q_z$ as $Q_0$ for any $z \in \mathbb{C}$ by a translation of $\mathbb{C}$ which brings $z$ to 0.
2.1. Some classical metric properties of confocal quadrics and of the Ivory affine transformation

**Definition 2.3.** The Ivory affine transformation is an affine correspondence between confocal quadrics and having good metric properties: it is given by

\[ x_z = \sqrt{R_z}x_0 + C(z), \quad C(z) := -\left( \frac{1}{2} \int_0^z (\sqrt{R_w})^{-1} dw \right) B. \]

Note that \( C(z) = 0 \) for quadrics with center, \( = \frac{\hat{z}}{2}en_{n+1} \) for quadrics without center; for isotropic quadrics without center it is the Taylor series of \( \int_0^z (\sqrt{1-w})^{-1} dw \) at \( z = 0 \) with each monomial \( z^{k+1} \) replaced by \( z^{k+1}J_p^k \bar{f}_1 \), where \( J_p \) is the block of \( f_1 \) in \( A \) and thus, a polynomial of degree \( p \) in \( z \). Note

\[ AC(z) + (I_{n+1} - \sqrt{R_z})B = 0 = (I_{n+1} + \sqrt{R_z})C(z) + zB \]

(both are \( 0 \) for \( z = 0 \) and do not depend on \( z \)). Applying \( d \) to \( Q_z(x_z) = 0 \) we get \( dx_z^T R_z^{-1}(Ax_z + B) = 0 \), so the unit normal \( N_z \) is proportional to \( \hat{N}_z := -2\partial_z x_z \). If \( \mathbb{C}^{n+1} \ni x \in x_{z_1}, x_{z_2} \), then \( \hat{N}_{z_j} = R_{z_j}^{-1}(Ax + B) \); using

\[ R_z^{-1} - I_{n+1} = zAR_z^{-1}, \quad z_1R_{z_1}^{-1} - z_2R_{z_2}^{-1} = (z_1 - z_2)R_{z_1}^{-1}R_{z_2}^{-1} \]

we get

\[ 0 = Q_{z_1}(x) - Q_{z_2}(x) = (z_1 - z_2)\hat{N}_{z_1}^T\hat{N}_{z_2}, \]

so two confocal quadrics cut each other orthogonally (Lamé). For general quadrics the polynomial equation \( Q_z(x) = 0 \) has degree \( n + 1 \) in \( z \) and it has multiple roots if and only if \( 0 = \partial_z Q_z(x) = |\hat{N}_z|^2 \); thus, outside the locus of isotropic normals elliptic coordinates (given by the roots \( z_1, ..., z_{n+1} \) of the said equation) give a parametrization of \( \mathbb{C}^{n+1} \) suited to confocal quadrics. With \( x_0^0, x_1^0 \in x_0, V_0^1 := x_z^1 - x_0^0, \) etc. the preservation of length of segments between confocal quadrics (Ivory Theorem) becomes

\[ |V_0^1|^2 = |x_0^0 + x_1^1 - C(z)|^2 - 2(x_0^0)^T (I_{n+1} + \sqrt{R_z}) x_0^1 + zC = |V_1^0|^2; \]

the preservation of lengths of rulings (Henrici):

\[ w_0^T A w_0 = w_0^T \hat{N}_0 = 0, \quad w_z = \sqrt{R_z} w_0 \Rightarrow w_z^T w_z = |w_0|^2 - zw_0^T A w_0 = |w_0|^2; \]

the symmetry of the tangency configuration (Bianchi):

\[ (V_0^1)^T \hat{N}_0^0 = (x_0^0)^T A \sqrt{R_z} x_0^1 - B^T (x_0^1 + x_z^1 - C(z)) + C = (V_1^0)^T \hat{N}_0^1; \]

the preservation of angles between segments and rulings (Bianchi):

\[ (V_0^1)^T w_0^0 + (V_1^0)^T w_z^0 = -z(\hat{N}_0^0)^T w_0^0 = 0; \]

the preservation of angles between rulings (Bianchi):

\[ (w_0^0)^T w_z^1 = (w_0^0)^T \sqrt{R_z} w_0^1 = (w_z)^T w_0^1; \]
the preservation of angles between polar rulings:

\[(w_0^0)^T A\hat{w}_0^0 = 0 \Rightarrow (w_z^0)^T \hat{w}_z^0 = (w_0^0)^T \hat{w}_0^0 - z(w_0^0)^T A\hat{w}_0^0 = (w_0^0)^T \hat{w}_0^0,\]
equivalent cases.

We also have the Chasles-Jacobi result (that of Jacobi’s being an inspiration for that of Chasles’s).

(Jacobi) The tangent lines to a geodesic on \(x_0\) remain tangent to \(n - 1\) other confocal quadrics.

(Chasles) The common tangents to \(n\) confocal quadrics form a normal congruence and envelope geodesics on the \(n\) confocal quadrics.

### 3. CHASLES’S AND GRAVES’S RESULTS

Let \(a_1 > a_2 > 0\) and consider the confocal ellipses

\[x_z := [\sqrt{a_1 - z} \cos \theta \sqrt{a_2 - z} \sin \theta]^T, \quad z < 0, \quad \theta \in \mathbb{R}\]

outside the initial ellipse \(x_0\).

Chasles’s result about polygons circumscribed to a given ellipse and inscribed in a given set of ellipses confocal to the given one roughly states:

**Theorem 3.1 (Chasles).** Given a set of ellipses \(x_{z_j}, \quad z_j < 0, \quad j = 1, ..., n\) confocal to the given one \(x_0\), if a ray of light tangent to \(x_0\) at a point \(x_0^1 \in x_0\) be reflected in \(x_z^1\), then it remains tangent to \(x_0\) at \(x_0^2 \in x_0\) and if after successive reflections further in \(x_z^2, ..., x_z^n\) it returns to being tangent to \(x_0\) at \(x_0^1\), then this property and the perimeter of the obtained polygon is independent of the position of \(x_0^1\) on \(x_0\).

In fact, Chasles proved also a dual result related to a ray of light reflecting on \(x_0\) and tangent (possibly outside the segment of the actual trajectory) to confocal hyperbolas \(x_{z_j}, \quad a_1 > z_j > a_2, \quad j = 1, ..., n\); should one of the \(x_{z_j}\) be an ellipse with \(a_2 > z_j > 0\) then the ray of light becomes entrapped and will always be tangent to \(x_{z_j}\) (the Poncelet Theorem).

The fact that Chasles’s result is an immediate consequence of the next

**Theorem 3.2 (Graves).** If a thread longer than the perimeter of an ellipse \(x_0\) is passed around \(x_0\) and stretched with a pen, then the tip of the pen will describe an ellipse \(x_z, \quad z < 0\) confocal with the given one \(x_0\)

is straightforward, Chasles’s result being a particular configuration of \(n\) threads (or equivalently of a single thread stretched with \(n\) pens) such that their rectilinear parts are in continuation one of the other.
Conversely, Chasles’s result implies the previous by taking $z_1 := z < 0$ fixed, $z_j < 0$, $j = 2, ..., n$ infinitesimally close to 0 and letting $n \to \infty$.

If we give up the requirement that the rectilinear parts are in continuation one of the other, then we get more general thread configurations of the ellipse: a closed thread around $x_0$ and stretched with $n$ pens whose tips are situated on ellipses $x_{z_1}, ..., x_{z_n}$ such that consecutive rectilinear parts are in continuation one of the other or are joined by an arc of the ellipse $x_0$; we may even allow an ideal construction that consecutive rectilinear parts may not be in continuation one of the other and are joined by an arc of the ellipse but with cusps at both ends, in which case the arc of the ellipse joining their ends at $x_0$ has to be subtracted to get constant perimeter.

The Ivory affine transformation between confocal ellipses is an affine correspondence given by

$$x_z = \sqrt{R_z} x_0, \quad R_z := I_2 - zA, \quad A := \text{diag}[a_1^{-1}, a_2^{-1}]$$

as $z$ varies $x_z$ describes orthogonal curves of the family of confocal ellipses (the hyperbolas of the confocal family).

It has good metric properties, among which are the preservation of lengths of segments between confocal ellipses (if $x_0^0, x_0^1 \in x_0$ and by use of the Ivory affine transformation we get $x_z^0, x_z^1 \in x_z$, then $|x_z^1 - x_z^0| = |x_0^1 - x_0^0|$) and of the tangency configuration ($x_z^1 - x_0^0$ is tangent to $x_0$ at $x_0^0$ if and only if $x_z^0 - x_0^1$ is tangent to $x_0$ at $x_0^1$).

We have, thus, reduced our investigation to:

**Proposition 3.3 (Ivory affine transformation approach for Graves’s vertex configuration).** Given $z < 0$ and three points $x_0^0, x_0^1, x_0^2 \in x_0$ in this order and by the Ivory affine transformation the corresponding $x_z^1, x_z^0, x_z^2 \in x_z$, then $x_z^0 - x_0^1, x_z^0 - x_0^2$ reflect in $x_z$ at $x_z^0$ if and only if $x_z^1 - x_0^0, x_z^2 - x_0^0$ reflect in $x_0$ at $x_0^0$. 
Further in this case, by the preservation of the tangency configuration under the Ivory affine transformation, \( x_z^0-x_z^1 \) is tangent to \( x_0 \) at \( x_0^1 \) if and only if \( x_z^0-x_z^2 \) is tangent to \( x_0 \) at \( x_0^2 \) and further in this case the excess \( |x_z^0-x_z^1| \)-length, \( x_0^1(x_0^0x_0^2) \) does not depend on the position of \( x_0^0 \) on \( x_0 \).

**Proof.** The condition that \( x_0^1-x_0^0, x_0^2-x_0^0 \) reflect in \( x_z \) at \( x_0^0 \) becomes

\[
0 = (dx_z^0)^T \left( \frac{x_0^1-x_z^0}{|x_0^1-x_0^0|} + \frac{x_0^2-x_z^0}{|x_0^2-x_0^0|} \right) = (dx_0^0)^T \left( \frac{x_0^1-x_0^0}{|x_0^1-x_0^0|} + \frac{x_0^2-x_0^0}{|x_0^2-x_0^0|} \right);
\]

the tangency configuration follows because the segment \([x_z^1, x_z^2]\) must be tangent to \( x_0 \) at \( x_0^0 \). Simple variational arguments show that the reflection property is enough to infer the constant excess property in the tangency configuration (the gradient lines of the thread length function are the confocal hyperbolas of the confocal family); however, we shall explicitly state the constant excess property as an identity involving an elliptic integral and show how the reflection property is used at the analytic level also.

With \( x_0^j = [\sqrt{a_1} \cos \theta_j \  \sqrt{a_2} \sin \theta_j], \ j = 0, 1, 2 \) we have the tangency configuration

\[
(3.1) \quad \cos \theta_0 \sqrt{1-za_1^{-1}} \cos \theta_{1,2} + \sin \theta_0 \sqrt{1-za_2^{-1}} \sin \theta_{1,2} = 1,
\]

so

\[
(3.2) \quad \cos \theta_j = \frac{\cos \theta_0 \sqrt{1-za_1^{-1}} - (-1)^j \sin \theta_0 \sqrt{1-za_2^{-1}} \sqrt{-z(a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)}}{1-z(a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)},
\]

\[
\sin \theta_j = \frac{\sin \theta_0 \sqrt{1-za_2^{-1}} + (-1)^j \cos \theta_0 \sqrt{1-za_1^{-1}} \sqrt{-z(a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)}}{1-z(a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)},
\]

\( j = 1, 2. \)

We now need

\[
0 = \frac{d}{d\theta_0} \left[ \frac{|x_z^1-x_z^2|}{\sqrt{a_1a_2}} - \int_{\theta_1}^{\theta_2} \sqrt{a_2^{-1} \sin^2 \theta + a_1^{-1} \cos^2 \theta} \, d\theta \right]
= \frac{1}{z} \frac{d}{d\theta_0} \frac{2\sqrt{-z(1-za_1^{-1})(1-za_2^{-1})}}{1-z(a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)} - \sqrt{a_2^{-1} \sin^2 \theta_2 + a_1^{-1} \cos^2 \theta_2} \frac{d\theta_2}{d\theta_0}
+ \sqrt{a_2^{-1} \sin^2 \theta_1 + a_1^{-1} \cos^2 \theta_1} \frac{d\theta_1}{d\theta_0},
\]

or, using (3.1) for \( \frac{d\theta_j}{d\theta_0} \),
\[
\sin \theta_0 \cos \theta_0 (a_2^{-1} - a_1^{-1}) \sqrt{-z} \left( \frac{4 \sqrt{(1 - z a_1^{-1})(1 - z a_2^{-1})}}{1 - z (a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)} - \sum_{j=1}^{2} \frac{\sqrt{a_2^{-1} \sin^2 \theta_j + a_1^{-1} \cos^2 \theta_j}}{\sqrt{a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0}} \right)
\]

\[
- \sqrt{(1 - z a_1^{-1})(1 - z a_2^{-1})} \sum_{j=1}^{2} (-1)^j \sqrt{a_2^{-1} \sin^2 \theta_j + a_1^{-1} \cos^2 \theta_j} = 0.
\]

Now, from the reflection property, we have

\[
0 = \left( \frac{dx^0_z}{d\theta_0} \right)^T \sum_{j=1}^{2} \frac{x_j^0 - x_z^0}{|x_j^0 - x_z^0|}
\]

\[
\sum_{j=1}^{2} \frac{(a_1 - a_2) \sin \theta_0 \cos \theta_0 + a_2 \sin \theta_j \cos \theta_0 \sqrt{1 - z a_2^{-1} - a_1 \cos \theta_j \sin \theta_0 \sqrt{1 - z a_1^{-1}}}}{a_1 \left( \cos \theta_j - \cos \theta_0 \sqrt{1 - z a_1^{-1}} \right)^2 + a_2 \left( \sin \theta_j - \sin \theta_0 \sqrt{1 - z a_1^{-1}} \right)^2},
\]

or, using (3.2):

\[
\sum_{j=1}^{2} \frac{\sin \theta_0 \cos \theta_0 (a_2^{-1} - a_1^{-1}) \sqrt{-z} + (-1)^j \sqrt{(a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)(1 - z a_1^{-1})(1 - z a_2^{-1})}}{\sqrt{a_2^{-1} \sin^2 \theta_j + a_1^{-1} \cos^2 \theta_j} = 0,
\]

so we need

\[
\sqrt{a_2^{-1} \sin^2 \theta_j + a_1^{-1} \cos^2 \theta_j}
\]

\[
= \sqrt{(a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)(1 - z a_1^{-1})(1 - z a_2^{-1}) + (-1)^j \sin \theta_0 \cos \theta_0 (a_2^{-1} - a_1^{-1}) \sqrt{-z}}
\]

\[
1 - z (a_2^{-1} \sin^2 \theta_0 + a_1^{-1} \cos^2 \theta_0)
\]

\[
j = 1, 2,
\]

which is straightforward. \(\square\)

**4. DARBOUX’S AND STAUDE’S RESULTS**

Here, we shall reproduce computations mainly from Darboux ([4], Vol 2, Livre IV, Ch. XIV) and Salmon ([11], §421 a,b) concerning elliptic coordinates on \(\mathbb{R}^3\) and their corresponding applications (see also Bianchi ([2], §419–§427)).
4.1. Elliptic coordinates on $\mathbb{R}^3$

Let $a_1 > a_2 > a_3$ constants and $a_1 > u^1 > a_2 > u^2 > a_3 > u^3$ be the elliptic coordinates on

$$\mathbb{R}^3 \setminus \{x^1 x^2 x^3 = 0\} \ni x = [x^1 \ x^2 \ x^3]^T, \ (x^j)^2 = \frac{\prod_k (a_j - u^k)}{\prod_{k \neq j} (a_j - a_k)}, \ j = 1, 2, 3.$$ 

Because both the numerator and the denominator in each term on the right hand side contain the same number of negative terms, this provides good definition.

Also, we have

$$\sum_j \frac{(x^j)^2}{a_j - u} - 1 = \frac{\prod_j (u - u^j)}{\prod_j (a_j - u)}, \quad (4.1)$$

namely this relation is separately linear in $u^k$, $k = 1, 2$, so it is sufficient to verify it for two different values of an $u^k$; of course we take $u^k := a_k$ and $u^k := a_{k+1}$, in which case we have reduced what we had to prove from dimension 3 to dimension 2 and we have a backward induction.

We have the linear element of $\mathbb{R}^3 \setminus \{x^1 x^2 x^3 = 0\}$ in elliptic coordinates

$$ds^2 = |dx|^2 = \sum_k \frac{\prod_{j \neq k} (u^k - u^j)}{4 \prod_j (a_j - u^k)} (du^k)^2. \quad (4.2)$$

To see this first we need $\sum_j \frac{a_j - u}{\prod_{k \neq j} (a_j - a_k)} = 0$ (which follows by letting $u := a_1, a_2$) and then differentiate (4.1) with respect to $u$ and let $u := u^k$.

From (4.1) for $a_1 > u^1 > a_2$ constant we get the hyperboloids with two sheets of the confocal family, for $a_2 > u^2 > a_3$ constant we get the hyperboloids with one sheet of the confocal family and for $a_3 > u^3 > -\infty$ constant we get the ellipsoids of the confocal family.

Note also that the singular cases $u^k = a_k$, $a_{k+1}$ are allowed and make sense, both at the level of elliptic coordinates and linear element, by a limiting argument $u^k \nearrow a_k$, $u^k \searrow a_{k+1}$ (of course for each such singular value of $u^k$, we have a similar 2-dimensional discussion for the remaining parameters).

For $u^1 \nearrow a_1$ we get the doubly covered plane $\{x^1 = 0\}$ with elliptic coordinates

$$\begin{bmatrix} \pm 0 & \pm \sqrt{\frac{(a_2 - u^2)(a_2 - u^3)}{a_2 - a_3}} & \pm \sqrt{\frac{(a_3 - u^2)(a_3 - u^3)}{a_3 - a_2}} \end{bmatrix}^T, \quad a_2 > u^2 > a_3 > u^3$$

as the limit of a fattening hyperboloid with two sheets whose sheets tend to the point 0 and this forces them to elongate themselves along the plane $\{x^1 = 0\}$. 
For \( u^1 \searrow a_2 \) we get the doubly covered convex region of the hyperbola 
\[
\frac{(x^1)^2}{a_1 - a_2} + \frac{(x^3)^2}{a_3 - a_2} = 1 \text{ in the plane } \{x^2 = 0\} \text{ with elliptic coordinates }
\]
\[
\left[ \pm \sqrt{\frac{(a_1 - u_1)(a_1 - u_3)}{a_1 - a_3}} \pm 0 \pm \sqrt{\frac{(a_3 - u_2)(a_3 - u_3)}{a_3 - a_1}} \right]^T, \ a_2 > u_2 > a_3 > u_3
\]
as the limit of a thinning hyperboloid with two sheets which tends to the plane \( \{x^2 = 0\} \).

For \( u^2 \nearrow a_2 \) we get the doubly covered concave region of the hyperbola 
\[
\frac{(x^1)^2}{a_1 - a_2} + \frac{(x^3)^2}{a_3 - a_2} = 1 \text{ in the plane } \{x^2 = 0\} \text{ with elliptic coordinates }
\]
\[
\left[ \pm \sqrt{\frac{(a_1 - u_1)(a_1 - u_3)}{a_1 - a_3}} \pm 0 \pm \sqrt{\frac{(a_3 - u_1)(a_3 - u_3)}{a_3 - a_1}} \right]^T, \ a_1 > u_1 > a_2, \ a_3 > u_3
\]
as the limit of a thinning (without hole) hyperboloid with one sheet which tends to the plane \( \{x^2 = 0\} \); thus, for \( u^1 \searrow a_2 \searrow u^2 \) we get the focal hyperbola
\[
\left[ \pm \sqrt{\frac{(a_1 - a_2)(a_1 - u_3)}{a_1 - a_3}} \pm 0 \pm \sqrt{\frac{(a_3 - a_2)(a_3 - u_3)}{a_3 - a_1}} \right]^T
\]
covered once from each side.

For \( u^2 \searrow a_3 \) we get the doubly covered concave region of the ellipse 
\[
\frac{(x^1)^2}{a_1 - a_3} + \frac{(x^2)^2}{a_2 - a_3} = 1 \text{ in the plane } \{x^3 = 0\} \text{ with elliptic coordinates }
\]
\[
\left[ \pm \sqrt{\frac{(a_1 - u_1)(a_1 - u_3)}{a_1 - a_2}} \pm \sqrt{\frac{(a_2 - u_1)(a_2 - u_3)}{a_2 - a_1}} \pm 0 \right]^T, \ a_1 > u_1 > a_2, \ a_3 > u_3
\]
as the limit of a thinning (with hole) hyperboloid with one sheet which tends to the plane \( \{x^3 = 0\} \).

For \( u^3 \nearrow a_3 \) we get the doubly covered convex region of the ellipse 
\[
\frac{(x^1)^2}{a_1 - a_3} + \frac{(x^2)^2}{a_2 - a_3} = 1 \text{ in the plane } \{x^3 = 0\} \text{ with elliptic coordinates }
\]
\[
\left[ \pm \sqrt{\frac{(a_1 - u_1)(a_1 - u_2)}{a_1 - a_2}} \pm \sqrt{\frac{(a_2 - u_1)(a_2 - u_2)}{a_2 - a_1}} \pm 0 \right]^T, \ a_1 > u_1 > a_2 > u_2 > a_3
\]
as the limit of a thinning ellipsoid which tends to the plane \( \{x^3 = 0\} \); thus, for
$u^2 \setminus a_3 \setminus u^3$ we get the focal ellipse

$$\left[ \pm \sqrt{\frac{(a_1 - u^1)(a_1 - a_3)}{a_1 - a_2}} \pm \sqrt{\frac{(a_2 - u^1)(a_2 - a_3)}{a_2 - a_1}} \right]^T$$

covered once from each side.

For $u^3 \setminus -\infty$ the ellipsoid tends to infinity and in shape closer to a sphere.

Thus, through each point of $\mathbb{R}^3$ pass 3 quadrics of the confocal family which (excluding the focal ellipse and hyperbola) meet orthogonally (Lamé).

We now fix a hyperboloid with one sheet $\{u^2 = u^2_0\}$ and an ellipsoid $\{u^3 = u^3_0\}$; we are further interested in the region of the space outside both of them: $a_1 > u^1 > a_2 > u^2_0 > u^2 > a_3 > u^3_0 > u^3$.

We are interested in the congruence (2-dimensional family) of lines tangent to both $\{u^2 = u^2_0\}$ and $\{u^3 = u^3_0\}$ and in the linear elements of these lines in elliptic coordinates; according to the Chasles-Jacobi result this congruence is normal (the lines are the normals to an 1-dimensional family of surfaces) and its developables envelope geodesics on the two quadrics.

These surfaces are given in elliptic coordinates by

$$(4.3) \Phi := \frac{1}{2} \sum_k \varepsilon_k \int \sqrt{\frac{(u^k - u^2_0)(u^k - u^3_0)}{\prod_j (a_j - u^k)}} \, du^k = ct, \, \varepsilon_k := \pm 1, \, k = 1, 2, 3.$$

We have

$$|\nabla \Phi|^2 = \sum_k \frac{4 \prod_j (a_j - u^k)}{\prod_{j \neq k} (u^k - u^j)} \left( \frac{\partial \Phi}{\partial u^k} \right)^2 = \sum_k \frac{(u^k - u^2_0)(u^k - u^3_0)}{\prod_{j \neq k} (u^k - u^j)} = 1$$

(again, we can let for example $u^2_0 := u^1, u^2$), so these surfaces are indeed parallel.

Differentiating this with respect to $u^2_0, u^3_0$ we obtain

$$(\nabla \Phi)^T \nabla \frac{\partial \Phi}{\partial u^2_0} = (\nabla \Phi)^T \nabla \frac{\partial \Phi}{\partial u^3_0} = 0;$$

keeping account also of

$$\left( \nabla \frac{\partial \Phi}{\partial u^2_0} \right)^T \nabla \frac{\partial \Phi}{\partial u^3_0} = \sum_k \frac{4 \prod_j (a_j - u^k)}{\prod_{j \neq k} (u^k - u^j)} \frac{\partial^2 \Phi}{\partial u^k \partial u^2_0} \frac{\partial^2 \Phi}{\partial u^k \partial u^3_0} = \sum_k \frac{1}{4 \prod_{j \neq k} (u^k - u^j)} = 0$$

we get the fact that the families of surfaces $\Phi = ct, \, \frac{\partial \Phi}{\partial u^2_0} = ct, \, \frac{\partial \Phi}{\partial u^3_0} = ct$ form a triply orthogonal system and since $\Phi = ct$ are parallel we obtain that the last two families of surfaces are the two families of developables of the congruence of normals to the first family.
With
\[ \Delta(u) := (u - u_0^2)(u - u_0^3) \prod_j (a_j - u) \]
and applying \( d \) to \( \frac{\partial \Phi}{\partial u_0^2} = ct \) we get
\[ 0 = d \frac{\partial \Phi}{\partial u_0^2} = \frac{1}{4} \sum_k (-1)^k \epsilon_k (u^k - u_0^3) \frac{du^k}{\sqrt{\Delta(u^k)}}, \]
so for \( u^2 \nearrow u_0^2 \) we have \( du^2 = 0 \); thus, the developables \( \frac{\partial \Phi}{\partial u_0^2} = ct \) are tangent to \( \{ u^2 = u_0^2 \} \) and similarly, the developables \( \frac{\partial \Phi}{\partial u_0^3} = ct \) are tangent to \( \{ u^3 = u_0^3 \} \).

The developables \( \frac{\partial \Phi}{\partial u_0^2} = ct \) being tangent to \( \{ u^2 = u_0^2 \} \), have their lines of striction on \( \{ u^3 = u_0^3 \} \) which are geodesics (this is true for any normal congruence); thus, we get Jacobi’s equations of geodesics on \( \{ u^3 = u_0^3 \} \) with tangents tangent to \( \{ u^2 = u_0^2 \} \) (depending on two constants \( u_0^2, c \)):

\[ \epsilon_1 \int \frac{(u^1 - u_0^3)du^1}{\sqrt{\Delta(u^1)}} - \epsilon_2 \int \frac{(u^2 - u_0^3)du^2}{\sqrt{\Delta(u^2)}} = c. \quad (4.4) \]

Note also that the 1-dimensional family of geodesics corresponding to the same value of the constant \( u_0^2 \) (which can have here any value between \( a_1 \) and \( a_3 \)) are tangent to the same lines of curvature on the ellipsoid \( \{ u^3 = u_0^3 \} \). For example, for \( u^2 = u_0^2 \) we have \( du^2 = 0 \), so the geodesics are tangent to the two lines of curvature \( (u^2, u^3) = (u_0^2, u_0^3) \) on the ellipsoid \( \{ u^3 = u_0^3 \} \) (the geodesic bounces between them); if one such geodesic is closed, then all are closed and of the same length.

It is this result of Jacobi’s from 1838 (published in 1839) that opened the whole area of research on confocal quadrics and (hyper-)elliptic integrals of the XIX\textsuperscript{th} century mentioned in the preface of Salmon [11].

Finally, we are able now to discuss the differential equation of a line tangent to both \( \{ u^2 = u_0^2 \} \) and \( \{ u^3 = u_0^3 \} \) and its linear element in elliptic coordinates: clearly, such a line is the intersection of two developables \( \frac{\partial \Phi}{\partial u_0^2} = ct, \frac{\partial \Phi}{\partial u_0^3} = ct \) and its arc-length can be taken to be \( s := \Phi \).

We, thus, have
\[ 0 = \sum_k (-1)^k \epsilon_k \frac{(u^k - u_0^3)du^k}{\sqrt{\Delta(u^k)}} = \sum_k (-1)^k \epsilon_k \frac{(u^k - u_0^2)du^k}{\sqrt{\Delta(u^k)}}, \]
\[-2ds = \sum_k (-1)^k \epsilon_k \frac{(u^k - u_0^2)(u^k - u_3^2)du^k}{\sqrt{\Delta(u^k)}} \]
\[\Rightarrow 0 = \sum_k \frac{(-1)^k \epsilon_k du^k}{\sqrt{\Delta(u^k)}} = \sum_k u^k \frac{(-1)^k \epsilon_k du^k}{\sqrt{\Delta(u^k)}}, \]
\[-2ds = \sum_k (u^k)^2 \frac{(-1)^k \epsilon_k du^k}{\sqrt{\Delta(u^k)}}, \]
from where we get
\[
\frac{\epsilon_1 du^1}{(u^2 - u^3)\sqrt{\Delta(u^1)}} = \frac{\epsilon_2 du^2}{(u^1 - u^3)\sqrt{\Delta(u^2)}} = \frac{\epsilon_3 du^3}{(u^1 - u^2)\sqrt{\Delta(u^3)}} = \frac{2ds}{(u^1 - u^2)(u^1 - u^3)(u^2 - u^3)}. \tag{4.5}
\]

In particular, for \(u^3 = u_0^3\) we have \(du^3 = 0\) and we obtain the differential equation of geodesics on \(\{u^3 = u_0^3\}\) with tangents tangent to \(\{u^2 = u_0^2\}\) and for \((u^2, u^3) = (u_0^2, u_0^3)\) we have \(du^2 = du^3 = 0\) and we obtain the differential equation of lines of curvature common to both \(\{u^2 = u_0^2\}\) and \(\{u^3 = u_0^3\}\).

So far, we have not paid attention to the signs appearing in the formulae, since we have not made much use of them. However, in order to keep a precise accounting of the elliptic coordinates along threads we must now keep a good accounting of the signs: as \(s\) increases \(\epsilon_k du^k\) are always positive, so \(\epsilon_k\) changes when the variable \(u^k\) changes monotonically.

The four choices of sign \(\epsilon_1 \epsilon_2, \epsilon_1 \epsilon_3\) in the differential equation (4.5) of the common tangent correspond at the level of the geometric picture to the intersection of tangent cones from a point in space to the two confocal quadrics being four lines; these lines are obtained by reflecting one of them in the tangent planes of the three confocal quadrics that pass through that point and if for example, we want to pass from one of them to the one symmetric with respect to the tangent plane to the \(\{u^j = ct\}\) quadric, then we change the sign of \(du^j\). These simple remarks allow the generalization of Chasles's result.

### 4.2. Darboux’s generalization of Chasles’s result

We have

**Theorem 4.1 (Darboux).** Consider a ray of light tangent to two given confocal quadrics \(\{u^j = ct\}, \{u^k = ct\}, j \neq k\) and which reflects consecutively in \(n\) other quadrics of the confocal family. If after a certain number of reflections the ray of light returns to its original position, then this property and the
perimeter of the obtained polygon is independent of the original position of the ray of light.

Proof. For simplicity, we assume that the two initial confocal quadrics are the hyperboloid with one sheet \( \{u^2 = u_0^2\} \) and the ellipsoid \( \{u^3 = u_0^3\} \); further that \( n = 1 \) and that this quadric (on which the vertices of the polygon are situated) is an ellipsoid \( \{u^3 = u_1^3\} \); thus, \( u_0^3 > u_1^3 \), \( \{u^3 = u_0^3\} \) lies inside \( \{u^3 = u_1^3\} \), the points of tangency with \( \{u^3 = u_0^3\} \) (\( \{u^2 = u_0^2\} \)) are situated on (and possibly outside) the actual light trajectory segments and on the actual segments we have \( a_1 > u_1 > a_2 > u_0^2 > u_2 > a_3 > u_0^3 > u_3 > u_1^3 \).

Consider the polygon \( P \) with vertices \( A_1, A_2, \ldots, A_m \) situated on \( \{u^3 = u_1^3\} \); since

\[
\sum_k \int_{\mathcal{P}} \frac{(-1)^{k+1} \epsilon_k \, du^k}{\sqrt{\Delta(u^k)}} = ct
\]

we have

\[
\sum_k \int_{\mathcal{P}} \frac{(-1)^{k+1} \epsilon_k \, du^k}{\sqrt{\Delta(u^k)}} = 0.
\]

As we move from \( A_1 \) towards \( A_2 \), \( u^3 \) increases from \( u_1^3 \) to \( u_0^3 \) until we reach the point of tangency with \( \{u^3 = u_0^3\} \), then it decreases to \( u_1^3 \); thus, \( du^3 \) changes sign from + to − at \( \{u^3 = u_0^3\} \) and the third integral on \( A_1A_2 \) is

\[
2 \int_{u_1^3}^{u_0^3} \frac{du^3}{\sqrt{\Delta(u^3)}};
\]

on the whole polygon \( P \) it is

\[
2m \int_{u_1^3}^{u_0^3} \frac{du^3}{\sqrt{\Delta(u^3)}}.
\]

For the first two integrals we don’t have any change in the sign of the variations \( du^1, du^2 \) at the vertices of \( P \) (assumed not to be on the planes of coordinates and on \( \{u^2 = u_0^2\} \)), so they vary freely within their domain of variation.

Thus, we obtain

\[
\int_{\mathcal{P}} \frac{\epsilon_1 \, du^1}{\sqrt{\Delta(u^1)}} = 2n \int_{a_1}^{a_2} \frac{du^1}{\sqrt{\Delta(u^1)}},
\]

\( n \) being the number of times the value \( a_2 \) is taken by \( u^1 \), that is the number of points of intersections of \( P \) with the plane \( \{x^2 = 0\} \), and thus, an even number (for each such occurrence we double the multiplicity since the value \( u^1 = a^2 \) is taken from both sides of the plane; note also that \( P \) cuts the planes \( \{x^1 = 0\}, \{x^2 = 0\} \), alternatively).

Similarly

\[
- \int_{\mathcal{P}} \frac{\epsilon_2 \, du^2}{\sqrt{\Delta(u^2)}} = -2n' \int_{a_3}^{a_0^2} \frac{du^2}{\sqrt{\Delta(u^2)}},
\]

\( n' \) being again an even number (\( P \) touches the hyperboloid with one sheet \( \{u^2 = u_0^2\} \) and cuts the plane \( \{x^3 = 0\} \) alternatively, since between any two passes through the plane \( \{x^3 = 0\} \) the other extreme value of \( u^2 \) is achieved smoothly and thus, at a point of tangency with a hyperboloid with one sheet which is forced to be \( \{u^2 = u_0^2\} \).
Thus, we have
\[ n \int_{a_2}^{a_1} \frac{du^1}{\sqrt{\Delta(u^1)}} - n' \int_{a_3}^{u_0^2} \frac{du^2}{\sqrt{\Delta(u^2)}} + m \int_{u_1^3}^{u_0^3} \frac{du^3}{\sqrt{\Delta(u^3)}} = 0, \]
\[ n \int_{a_2}^{a_1} \frac{u^1 du^1}{\sqrt{\Delta(u^1)}} - n' \int_{a_3}^{u_0^2} \frac{u^2 du^2}{\sqrt{\Delta(u^2)}} + m \int_{u_1^3}^{u_0^3} \frac{u^3 du^3}{\sqrt{\Delta(u^3)}} = 0, \]
(4.6) \[ 2n \int_{a_2}^{a_1} \frac{(u^1)^2 du^1}{\sqrt{\Delta(u^1)}} - 2n' \int_{a_3}^{u_0^2} \frac{(u^2)^2 du^2}{\sqrt{\Delta(u^2)}} + 2m \int_{u_1^3}^{u_0^3} \frac{(u^3)^2 du^3}{\sqrt{\Delta(u^3)}} = \text{perimeter}(P) \]
the last two relations being obtained by similar computations. The first two relations impose certain rationality conditions on hyper-elliptic integrals involving \( u_0^2, u_0^3, u_1^3 \) (so, in general, such configurations do not exist); the last relation shows that the perimeter of \( P \) is the same for all such polygons \( P \).

All that remains to check is that the existence of such polygons \( P \) is an open condition.

Consider \( B_1 \) infinitesimally close to \( A_1 \) on the ellipsoid \( \{u^3 = u_1^3\} \); through it we draw a line tangent to both \( \{u^2 = u_0^2\} \) and \( \{u^3 = u_0^3\} \) and infinitesimally close to \( A_1 A_2 \) to obtain \( B_2 \in \{u^3 = u_1^3\} \) infinitesimally close to \( A_2 \); etc. Thus, we obtain the polygonal line \( P' : B_1, B_2, ..., B_m, B_1' \) and we want \( B_1' = B_1 \).

The computations from the first two relations of (4.6) still hold, except for a small deficit in the variations of the first two elliptic coordinates near the coordinates \( (u^1, u^2) \) of \( B_1 \) and \( (u^1, u^2) \) of \( B_1' \):
\[
\frac{du^1}{\sqrt{\Delta(u^1)}} - \frac{du^2}{\sqrt{\Delta(u^2)}} = \frac{du^1}{\sqrt{\Delta(u^1)}} - \frac{du^2}{\sqrt{\Delta(u^2)}},
\]
\[
\frac{u^1 du^1}{\sqrt{\Delta(u^1)}} - \frac{u^2 du^2}{\sqrt{\Delta(u^2)}} = \frac{u^1 du^1}{\sqrt{\Delta(u^1)}} - \frac{u^2 du^2}{\sqrt{\Delta(u^2)}}.
\]

This can be interpreted as a non-degenerate differential system in \( (u^1, u^2) \); since for \( B_1 = A_1 \) it has the initial condition \( (u^1, u^2) = (u_0^1, u_0^2) \), we conclude that the obvious solution \( (u^1, u^2) = (u_0^1, u_0^2) \) is unique. \( \square \)

Remark 4.2. Note that if we allow the ellipsoid \( \{u^3 = u_0^3\} \) in Darboux’s result to become degenerated (that is \( u_0^3 = a_3 \)) and one of the vertices of the polygon is situated in the \( \{x^3 = 0\} \) plane, then we get Chasles’s result; in this vein, Darboux’s generalization of Chasles’s result is a statement about closed geodesics on a degenerated 3-dimensional ellipsoid in \( \mathbb{R}^4 \).

4.3. Staude’s generalization of Graves’s result

Note that the computations in the last part of (4.6) referring to arc-length are valid also for line of curvature \( (u^2, u^3) = (u_0^2, u_0^3) \) segments and for geodesics
on \{u^3 = u_0^3\} with tangents tangent to \{u^2 = u_0^2\} segments.

Staude's thread construction of confocal ellipsoids roughly states

**Theorem 4.3 (Staude).** If a thread \( P \) passed around an ellipsoid \( \{u^3 = u_0^3\} \) and outside the hyperboloid with one sheet \( \{u^2 = u_0^2\} \) be stretched with a pen at a point \( P \) such that it touches once both visible parts of the hyperboloid with one sheet \( \{u^2 = u_0^2\} \), then the point \( P \) will move on an ellipsoid \( \{u^3 = u_1^3\} \) confocal to and outside \( \{u^3 = u_0^3\} \) and \( \{u^2 = u_0^2\} \).

**Remark 4.4.** The thread consists either of two rectilinear segments from \( P \) tangent to both \( \{u^2 = u_0^2\} \) and \( \{u^3 = u_0^3\} \), two geodesic segments on \( \{u^3 = u_0^3\} \) with tangents tangent to \( \{u^2 = u_0^2\} \) and in continuation of the two rectilinear segments, two line of curvature \( (u^2, u^3) = (u_0^2, u_0^3) \) segments in continuation of the previous two geodesic segments and another geodesic segment on \( \{u^3 = u_0^3\} \) with tangents tangent to \( \{u^2 = u_0^2\} \) and in continuation of and joining the two line of curvature \( (u^2, u^3) = (u_0^2, u_0^3) \) segments, or, if a rectilinear segment of the thread touches the hyperboloid with one sheet \( \{u^2 = u_0^2\} \) before the ellipsoid \( \{u^3 = u_0^3\} \), then it is not required to touch the corresponding line of curvature branch.

**Proof.** Note that with \( n, n', m \) being the same as in Darboux's result we have \( n = n' = 2 \), \( m = 1 \), since the thread cuts alternatively the planes \( \{x^1 = 0\}, \{x^2 = 0\} \) twice, touches the hyperboloid with one sheet \( \{u^2 = u_0^2\} \) and cuts the plane \( \{x^3 = 0\} \) alternatively twice (here we use the fact that the thread touches once each visible part of the hyperboloid with one sheet \( \{u^2 = u_0^2\} \)) and touches the ellipsoid \( \{u^3 = u_1^3\} \) on which \( P \) is situated only once (at \( P \)).

For the length of the thread \( P \) we get (similarly to the last relation of (4.6))

\[
4 \int_{a_1}^{a_2} \frac{(u^1 - u_0^2)(u^1 - u_0^3)du^1}{\sqrt{\Delta(u^1)}} - 4 \int_{a_3}^{u_0^3} \frac{(u^2 - u_0^2)(u^2 - u_0^3)du^2}{\sqrt{\Delta(u^2)}} + 2 \int_{u_1^3}^{u_0^3} \frac{(u^3 - u_0^2)(u^3 - u_0^3)du^3}{\sqrt{\Delta(u^3)}},
\]

so it depends only on the third elliptic coordinate \( u_1^3 \) of \( P \). \( \Box \)
Remark 4.5. For $u_0^2 = a_2$ or $u_0^3 = a_3$ (or both) the line of curvature segments become geodesic segments and we have a genuine variational problem. For $(u_0^2, u_0^3) = (a_2, a_3)$ the thread configuration as it appears in Hilbert-Cohn-Vossen ([10], §4) and Salmon ([11], §421 a,b) changes in the sense that one of the rectilinear segments is not continued until it reaches $\{u_0^3 = a_3\}$ after touching $\{u_0^2 = a_2\}$, but it is broken to reach the focus of the corresponding branch of the focal hyperbola (there is no well defined normal for the degenerated hyperboloid with one sheet $\{u_0^2 = a_2\}$ at the points of its singular boundary-hyperbola, so this construction makes sense from a geometric point of view). The two points of view are equivalent and are explained by the thread construction of an ellipse with the thread passing through its foci being able to be replaced with the same construction, but the foci being replaced with any point of the corresponding branch of the hyperbola which is orthogonal focal curve of the ellipse.

5. THREAD CONFIGURATIONS FOR ELLIPSOIDS

Just as Darboux, in order to simplify the presentation we assume that the vertices of the thread configuration move on the same ellipsoid confocal to the given one.

Since the geodesic segments in the thread are in continuation of rectilinear segments in the thread, in order to allow variations of vertices on their corresponding confocal ellipsoids we invert the point of view: rectilinear segments are in continuation of geodesic segments. Thus, we need to consider the intersection of a forward and a backward part of tangent surfaces of two geodesic segments on $\{u_3 = u_0^3\}$ with tangents tangent to $\{u_2 = u_0^2\}$: this intersection is a curve situated on a confocal ellipsoid and the length of a thread fixed at two points on the two geodesic segments and stretched with a pen situated on this curve is constant.

Conversely, if the forward part of the tangent surface of a geodesic segment on $\{u_3 = u_0^3\}$ with tangents tangent to $\{u_2 = u_0^2\}$ is reflected in a confocal ellipsoid $\{u_3 = u_1^3 < u_0^3\}$, then we obtain the backward part of the tangent surface of a geodesic segment on $\{u_3 = u_0^3\}$ with tangents tangent to $\{u_2 = u_0^2\}$.

Thus, we are led to the basic local result which allows thread configurations for ellipsoids and which appears (implicitly) in both Darboux’s and Staude’s results:

**Proposition 5.1** (Staude’s vertex configuration). The forward and backward parts of the tangent surfaces of two geodesic segments on $\{u_3 = u_0^3\}$ with tangents tangent to $\{u_2 = u_0^2\}$ meet along a confocal ellipsoid $\{u_3 = u_1^3 > u_0^3\}$; moreover, the length of the thread measured from a fixed point on the first
geodesic to a fixed point on the second geodesic and via the vertex on the confocal ellipsoid \( \{u^3 = u_0^3\} \) is constant.

Proof. We have such tangent surfaces

\[
\frac{\partial \Phi}{\partial u_0^3} = \frac{1}{4} \sum_k (-1)^k \int \frac{(u^k - u_0^3)\epsilon_k du^k}{\sqrt{\Delta(u^k)}} = ct, \quad \epsilon_k du^k > 0;
\]

\( \epsilon_1 \) changes sign at the intersection with the planes of coordinates \( \{x^1 = 0\}, \{x^2 = 0\}, \epsilon_2 \) changes sign at the intersection with the plane of coordinate \( \{x^3 = 0\} \) and at the points of tangency with \( \{u^2 = u_0^2\} \) and \( \epsilon_3 \) changes sign from \( - \) on the backward part of the tangent surface to \( + \) on the forward part of the tangent surface along the geodesic on \( \{u^3 = u_0^3\} \). If we reflect the forward part of the tangent surface along the ellipsoid \( \{u^3 = u_1^3\} \), then \( \epsilon_3 \) changes sign from \( + \) to \( - \) and we get the backward part of a similar tangent surface. The length of the thread measured from a fixed point on the first geodesic to a fixed point on the second one is constant since \( u^1, u^2 \) vary between given values and \( u^3 \) varies twice between \( u_1^3 \) and \( u_0^3 \).

Since the computations in Darboux’s result are mostly valid also for geodesics on \( \{u^3 = u_0^3\} \) with tangents tangent to \( \{u^2 = u_0^2\} \), they remain valid if we allow such geodesic segments in the thread, in which case (4.6) is replaced with

\[
(5.1) \quad n \int_{a_2}^{u_1^1} \frac{(u^1 - u_0^3)du^1}{\sqrt{\Delta(u^1)}} - n' \int_{a_3}^{u_0^2} \frac{(u^2 - u_0^3)du^2}{\sqrt{\Delta(u^2)}} + m \int_{u_1^3}^{u_0^3} \frac{(u^3 - u_0^3)du^3}{\sqrt{\Delta(u^3)}} = 0,
\]

\[
2n \int_{a_2}^{u_1^1} \frac{u^1(u^1 - u_0^3)du^1}{\sqrt{\Delta(u^1)}} - 2n' \int_{a_3}^{u_0^2} \frac{u^2(u^2 - u_0^3)du^2}{\sqrt{\Delta(u^2)}} + 2m \int_{u_1^3}^{u_0^3} \frac{u^3(u^3 - u_0^3)du^3}{\sqrt{\Delta(u^3)}} = \text{length}(\mathcal{P}).
\]

Note that (4.6) implies (5.1): as the sides \( A_1A_2, A_2A_3, \ldots, A_mA_1 \) move on their subjacent tangent surfaces of geodesic segments on \( \{u^3 = u_0^3\} \) the polygon \( \mathcal{P} \) closes and its length remains constant. Conversely, if the polygon fails to close, but \( A_m, A_1 \) still can be joined by a thread formed by two rectilinear segments and a geodesic segment on \( \{u^3 = u_0^3\} \) with tangents tangent to \( \{u^2 = u_0^2\} \) in continuation of the two rectilinear segments and joining them, then only the rationality condition from (5.1) remains valid among the rationality conditions (4.6). The length of the geodesic segment in the thread \( A_mA_1 \) can be liberally distributed among the segments \( A_1A_2, \ldots A_{m-1}A_m \) (thus, making them pieces of thread); if we allow consecutive rectilinear segments in the thread \( \mathcal{P} \)}
to be joined by geodesic segments but with cusps at both ends, then we can prescribe any desired length to all but one of the geodesic segments in the thread $\mathcal{P}$ (of course the integrals along the geodesic segments with cusps at ends have to be subtracted).

Summing up, the rationality condition of (5.1) is the condition that after several iteration of the local result (the backward part of the tangent surface obtained by reflection in the ellipsoid $\{u^3 = u_0^3\}$ of the forward part of another tangent surface being continued to obtain the forward part of the tangent surface and thus, allowing the iteration) the process closes up. Since we have only an 1-dimensional family of geodesic segments under consideration, only a rationality condition is required as closing condition, so the first equation of (5.1) provides also a sufficient condition for an open set of such thread configurations.

In particular, if we have no vertices ($m = 0$), then the first equation of (5.1) becomes the rationality condition for the existence of closed geodesics on $\{u^3 = u_0^3\}$ with tangents tangent to $\{u^2 = u_0^2\}$ and the last equation gives the length of such closed geodesics.

Thus, Darboux’s result (having no geodesic segments) and closed geodesics (having only geodesic segments) are the two extremes of thread configurations formed by rectilinear and geodesic segments.

For Staude’s result there is still the question of the thread closing up at $P$; thus, we need to consider the question for thread without line of curvature segments; from the first equation of (5.1) this is equivalent to

$$\int_{a_2}^{a_1} \frac{(u^1 - u_0^3)}{\sqrt{\Delta(u^1)}} du^1 - \int_{a_3}^{u_0^3} \frac{(u^2 - u_0^3)}{\sqrt{\Delta(u^2)}} du^2 - \frac{1}{2} \int_{u_0^3}^{u_0^3} \frac{(u^3 - u_0^3)}{\sqrt{\Delta(u^3)}} du^3 > 0. \tag{5.2}$$

We are interested in this relation since it has to do with the variation of elliptic coordinates on geodesics: consider a geodesic segment on $\{u^3 = u_0^3\}$ with tangents tangent to $\{u^2 = u_0^2\}$ touching once each branch of the line of curvature $(u^2, u^3) = (u_0^2, u_0^3)$ (the geodesic segment begins and ends at such touching points). From (5.1) this happens in more than half a turn around the $e_3$-axis (that is the plane through the $e_3$-axis and passing through one of the touching points cuts again the considered geodesic segment) if and only if we have (5.2) (note that for $u_0^3 = a_2$ equality is obtained in (5.2) since (4.4) forces geodesics passing through an umbilic of an ellipsoid to also pass through its opposite one). If we extend this geodesic segment at each end with similar geodesic segments, then the forward (backward) half of the tangent surfaces of these new two geodesic segments will cut along a curve on a confocal ellipsoid; for $P$ inside this ellipsoid Staude’s thread construction is not possible (of course one can correct this by allowing a longer line of curvature segment, in which case we have $n = 4$ or by allowing an ideal construction with a smaller line
of curvature segment with cusps at both ends and thus, its length must be subtracted); for $P$ outside (on) this ellipsoid the thread construction is possible and does (not) contain line of curvature segments.

If in (5.2) we have the opposite inequality (for $u_0^3$ close to $a_3$ such a geodesic segment looks close enough to the two tangent segments from Graves’s result), then the story changes completely: Staude’s thread construction is possible for all confocal ellipsoids and it always contains line of curvature segments.

If the rationality condition from (5.1) is not satisfied, then one can make again liberal use of line of curvature $(u^2, u^3) = (u^2_0, u^3_0)$ segments in the thread (including with cusps at both ends) when the geodesic segments of the thread touch such lines of curvature; all but one such line of curvature segments can be arbitrarily prescribed and the length of the thread $P$ is

$$
2n \int_{a_2}^{a_1} \frac{(u^1 - u_0^2)(u^1 - u_0^3)du^1}{\sqrt{\Delta(u^1)}} - 2n' \int_{a_3}^{u_0^3} \frac{(u^2 - u_0^2)(u^2 - u_0^3)du^2}{\sqrt{\Delta(u^2)}} + m \int_{u_1^3}^{u_3} \frac{(u^3 - u_0^2)(u^3 - u_0^3)du^3}{\sqrt{\Delta(u^3)}}.
$$

6. DARBOUX’S THEOREM IN GENERAL SETTING

Since quadrics with distinct non-zero eigenvalues of the quadratic part defining the quadric form an open dense set in the set of all quadrics, by a continuity argument one can infer that Darboux’s generalization of Chasles’s result is valid for all quadrics.

However, while elliptic coordinates a-priori are a must for geodesics and lines of curvature on quadrics, they should not be necessary for straight line segments and should be replaced by purely algebraic computations; locally, the Ivory affine transformation provides such a venue and Darboux’s constant perimeter property concerning moving polygons circumscribed to a given set of $n$ and inscribed in arbitrarily many $n$-dimensional confocal quadrics provides the needed global arguments.

6.1. An Ivory affine transformation approach for the vertex configuration

We have now

**PROPOSITION 6.1** (Ivory affine transformation approach for the vertex configuration). Given three points $x_0^0, x_0^1, x_0^2 \in x_0$ and by the Ivory affine transformation the corresponding points $x_z^0, x_z^1, x_z^2 \in x_z$, then $V_1^0, V_2^0$ reflect in $x_z$ at $x_z^0$ if and only if $V_1^1, V_2^0$ reflect in $x_0$ at $x_0^0$. Further in this case, by the preservation of the tangency configuration under the Ivory affine transformation, we
conclude that $V_1^0$ is tangent to $x_0$ at $x_0^1$ if and only if $V_2^0$ is tangent to $x_0$ at $x_0^2$ and further in this case $x_0^1, x_0^0, x_0^2$ are co-linear. Also $V_1^1, V_1^0$ are tangent to the same set of quadrics confocal to the given one $x_0$.

\[ \begin{array}{c}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\circ & \circ & \circ \\
\ast & \ast & \ast
\end{array}
\end{array} \]

Remark 6.2. Note that excepting totally real cases one loses the orientation of the real numbers, so from a complex point of view refractions must also be considered as reflections.

Proof. Note $\sqrt{R_z}V_1^0 = -V_1^0 - z\hat{N}_0^0$ (here we use $\hat{N}_0^0 = Ax_0^0 + B$, $(I_{n+1} + \sqrt{R_z})C(z) + zB = 0$).

We have

\[
(dx_0^0)^T \left( \frac{V_0^0}{|V_0^0|} \pm \frac{V_2^0}{|V_2^0|} \right) = (dx_0^0)^T \sqrt{R_z} \left( \frac{V_0^0}{|V_0^0|} \pm \frac{V_2^0}{|V_2^0|} \right) = -(dx_0^0)^T \left( \frac{V_0^0}{|V_0^0|} \pm \frac{V_2^0}{|V_2^0|} \right). \]

Thus, $V_1^0, V_2^0$ reflect in $x_z$ at $x_0^1$ if and only if $V_0^1, V_0^2$ reflect in $x_0$ at $x_0^0$ and the tangency part follows from this.

The vector $V_0^1$ is tangent to $x_{z'}$ if and only if the quadratic equation in $t$

\[ 0 = Q_{z'}(x_0^0 + tV_0^1) = (V_0^1)^T AR_{z'}^{-1}V_0^1 t^2 + 2(V_0^1)^T R_{z'}^{-1} \hat{N}_0^0 t + z'(\hat{N}_0^0)^T R_{z'}^{-1} \hat{N}_0^0 \]

has double root and we would like the same to hold for $Q_{z'}(x_0^1 + tV_1^0) = 0$, that is we want the discriminant

\[ \Delta := \left[ (V_0^1)^T R_{z'}^{-1} \hat{N}_0^0 \right]^2 - z'(V_0^1)^T A R_{z'}^{-1} V_0^1 (\hat{N}_0^0)^T R_{z'}^{-1} \hat{N}_0^0 \]

to be symmetric in $x_0^0, x_0^1$. This is so because $\sqrt{R_z}V_0^1 = -V_1^0 - z\hat{N}_1^0$, so

\[ z^2 \Delta = \left[ (V_0^1)^T R_{z'}^{-1} \left( V_0^1 + \sqrt{R_z} V_0^0 \right) \right]^2 \]

\[ + (V_0^1)^T (I_{n+1} - R_{z'}^{-1}) V_0^1 (V_0^1 + \sqrt{R_z} V_0^0)^T R_{z'}^{-1} \left( V_0^1 + \sqrt{R_z} V_0^0 \right) \]

\[ = \left[ (V_0^1)^T R_{z'}^{-1} \sqrt{R_z} V_0^1 \right]^2 + |V_0^1|^2 \left[ 2(V_0^1)^T R_{z'}^{-1} \sqrt{R_z} V_0^1 \right] \]

\[ + \left( 1 - \frac{z'}{z} \right) \left[ (V_0^1)^T R_{z'}^{-1} V_0^1 + (V_1^0)^T R_{z'}^{-1} V_0^1 \right] \]

\[ - \left( 1 - \frac{z'}{z} \right) \left( V_0^1 \right)^T R_{z'}^{-1} V_0^1 (V_0^1)^T R_{z'}^{-1} V_0^1 + \frac{z}{z'} |V_0^1|^2 |V_0^1|^2 \]
and from the Ivory Theorem we have $|V_0^1|^2 = |V_0^0|^2$. □

Now, we are able to derive the vertex configuration of Darboux’s Theorem: assume that $V_1^0$ is tangent to $x_0$ at $x_0^0$; by reflection in $x_z$ at $x_0^0$ we get $V_2^0$ tangent to $x_0$ at $x_0^2$. If $V_1^0$ is tangent to $x_z'$, then so is $V_0^1$; since $V_0^1$ and $V_2^0$ are co-linear, the same statement holds for $V_0^2$, so we finally conclude that $V_2^0$ is tangent to $x_z'$.

6.2. Darboux’s Theorem in general setting

**Theorem 6.3** (Darboux in general setting). Consider a ray of light tangent to $n$ given confocal quadrics in $\mathbb{C}^{n+1}$ and which reflects consecutively in $p$ other given quadrics of the confocal family. If after a minimal number $p$ of reflections the ray of light returns to its original position, then this property and the perimeter of the obtained polygon is independent of the original position of the ray of light.

*Proof.* Consider the polygonal line $\mathcal{P}$ with vertices $x_{zj}^j \in Q_{zj}$, $j = 0, 1, 2, \ldots, p$ and which is obtained as follows: pick arbitrarily $x_0^0 \in Q_{z0}$, then $x_{z1}^1 \in Q_{z1}$ such that $x_{z1}^1 - x_0^0$ is tangent to the $n$ given confocal quadrics $Q_{z_1}', \ldots, Q_{z_n}'$ at points $y_{zj}^j$, $j = 1, \ldots, n$ (according to Chasles-Jacobi’s result and an observation of Darboux’s there are choices of the direction $x_{z1}^1 - x_0^0$ given by the intersections of the tangent cones to $Q_{z_1}', \ldots, Q_{z_n}'$ through $x_0^0$; note also that for each choice of direction there are in general two choices of $x_{z1}^1$); now choose $x_{z2}^2 \in Q_{z2}$ such that $x_{z1}^1 - x_0^0$, $x_{z2}^2 - x_{z1}^1$ reflect in $Q_{z1}$ at $x_{z1}$ (note that for $z_2 \neq z_0$ there are in general two choices of $x_{z2}^2$ such that $x_{z2}^2 \neq x_0^0$, since $x_{z2}^2 - x_{z1}^1$ will not be tangent in general to $Q_{z2}$), etc.

The polygonal line $\mathcal{P}$ closes to a Darboux $p$-polygon if and only if $z_0 = z_p$, $x_{z0}^0 = x_{zp}^p$, $x_{zp}^p - x_{zp-1}^{p-1}$, $x_{z1}^1 - x_0^0$ reflect in $Q_{z0}$ at $x_0^0$ and $p$ is minimal (a picture with $p = 3$ should clarify the issues under discussion).

Now by Chasles-Jacobi’s result and Proposition 6.1 in general each segment $x_{zj}^j - x_{zj-1}^{j-1}$, $j = 1, \ldots, p$ of the polygonal line $\mathcal{P}$ is tangent to same $n$ quadrics $Q_{z_1}', \ldots, Q_{z_n}'$ at points $y_{zk}^j$, $k = 1, \ldots, n$.

The condition that $\mathcal{P}$ closes to a Darboux $p$-polygon is not just the condition that $x_{z0}^0 - x_{zp-1}^{p-1}$ is tangent to the $n$ quadrics $Q_{z_1}', \ldots, Q_{z_n}'$ at points $y_{zk}^j$, $k = 1, \ldots, n$: by Chasles-Jacobi’s such a line is uniquely determined by being tangent to $n$ quadrics, having a passing point and a direction chosen among the $(n + 1)$ directions (reflections in the principal spaces passing through that point; they are decided by the differentials of the $(n + 1)$ points moving on the $(n + 1)$ quadrics passing through that point); knowing this direction among the


\((n + 1)\) choices will force the reflection property needed to close the Darboux \(p\)-polygon.

The data about the polygonal line \(\mathcal{P}\) we have is

\[
(6.1) \quad (dx^j_{x_j})^T \left( \frac{\varepsilon_j x^j_{x_j} - x^{j-1}_{x_{j-1}}}{|x^j_{x_j} - x^{j-1}_{x_{j-1}}|} - \varepsilon_{j+1} \frac{x^{j+1}_{x_{j+1}} - x^j_{x_j}}{|x^{j+1}_{x_{j+1}} - x^j_{x_j}|} \right) = 0,
\]

\[\varepsilon_j := \pm 1, \; j = 1, \ldots, p - 1\]

and we get \(\mathcal{P}\) closing to a Darboux \(p\)-polygon if and only if

\[z_0 = z_p, \; x^{0}_{z_0} = x^p_{z_p}, \; \varepsilon_0 = \varepsilon_p := \pm 1, \; p \text{ minimal},\]

\[
(6.2) \quad (dx^p_{z_p})^T \left( \varepsilon_p \frac{x^p_{z_p} - x^{p-1}_{z_{p-1}}}{|x^p_{z_p} - x^{p-1}_{z_{p-1}}|} - \varepsilon_1 \frac{x^1_{z_1} - x^0_{z_0}}{|x^1_{z_1} - x^0_{z_0}|} \right) = 0.
\]

Similarly to Darboux’s original proof, what we need now to prove is that (6.1), (6.2) and the perimeter

\[
(6.3) \quad \sum_{j=1}^{p} \varepsilon_j |x^j_{x_j} - x^{j-1}_{x_{j-1}}| = c_0
\]

of \(\mathcal{P}\) are independent of the choice of \(x^{0}_{z_0} \in \mathcal{Q}_{z_0}, \; x^{1}_{z_1} \in \mathcal{Q}_{z_1}\) respectively, near and instead of \(x^{0}_{z_0} \in \mathcal{Q}_{z_0}, \; x^{1}_{z_1} \in \mathcal{Q}_{z_1}\) and such that \(z_1, \ldots, z_p, \; \varepsilon_1, \ldots, \varepsilon_p, \; z'_1, \ldots, z'_n\) remain fixed.

If we apply \(d\) to (6.3), then we get the sum of (6.1) and (6.2):

\[
d \sum_{j=1}^{p} \varepsilon_j |x^j_{x_j} - x^{j-1}_{x_{j-1}}| = \sum_{j=1}^{p} (dx^j_{x_j})^T \left( \varepsilon_j \frac{x^j_{x_j} - x^{j-1}_{x_{j-1}}}{|x^j_{x_j} - x^{j-1}_{x_{j-1}}|} - \varepsilon_{j+1} \frac{x^{j+1}_{x_{j+1}} - x^j_{x_j}}{|x^{j+1}_{x_{j+1}} - x^j_{x_j}|} \right).
\]

The same behavior applies to (6.1)’ (which is obtained by construction and induction on \(j = 1, \ldots, p - 1\)) and (6.2)’: together are equivalent to (6.3)’ being constant (by continuity presumably the constant of (6.3)).

Similarly to Darboux’s original proof and by a continuity argument we need only prove \(x^p_{z_p} = x^{0}_{z_0}\) and (6.2)’; using this and (6.1)’ we get (6.3)’.

Summing up: given \(x^j_{x_j}, \; y^j_{x_j}, \; j = 0, \ldots, p, \; k = 1, \ldots, n\) as before (to satisfy including (6.1), (6.2), (6.3)) and ’ quantities constructed from \(x^{0}_{z_0} \in \mathcal{Q}_{z_0}, \; x^{1}_{z_1} \in \mathcal{Q}_{z_1}\) respectively near and instead of \(x^{0}_{z_0} \in \mathcal{Q}_{z_0}, \; x^{1}_{z_1} \in \mathcal{Q}_{z_1}\) and such that \(z_1, \ldots, z_p, \; \varepsilon_1, \ldots, \varepsilon_p, \; z'_1, \ldots, z'_n\) remain fixed, then all ’ quantities will be near their corresponding counterpart quantities and will satisfy, except for \(x^p_{z_p}\), the relations satisfied by their counterparts. But now, by continuity Chasles-Jacobi’s result will be precise enough to choose from among the choices of \(x^p_{z_p}\) the correct one such that \(x^p_{z_p}\) will be near \(x^p_{z_p} = x^{0}_{z_0}\).

Now, we have the two differential systems

\[
(\text{Differential System 1})
\]

\[
(\text{Differential System 2})
\]
\[(dx_{zp})^T \left( \epsilon_p \frac{x_{zp}^p - x_{zp-1}^p}{|x_{zp}^p - x_{zp-1}^p|} - \epsilon_1 \frac{x_{z1}^1 - x_{z0}^0}{|x_{z1}^1 - x_{z0}^0|} \right) = 0, \]

\[(dx_{zp})^T \left( \epsilon_p \frac{x_{zp}^p - x_{zp-1}^p}{|x_{zp}^p - x_{zp-1}^p|} - \epsilon_1 \frac{x_{z1}^1 - x_{z0}^0}{|x_{z1}^1 - x_{z0}^0|} \right) = 0. \]

Since the first system is Pfaff non-exact with unique solution \(x_{zp}^p = x_{z0}^0\) near \(x_{z0}^0\) (should it be exact in some variables, we would have a continuous \((1 \leq n - 1)\)-dimensional family of solutions), by continuity in parameters and nearness of computations we deduce that also the second system is Pfaff non-exact with unique solution \(x_{zp}^p = x_{z0}^0\). □

6.3. An Ivory affine transformation approach for newly obtained Darboux \(p\)-polygons and its iteration

Take any quadric (for example \(Q_{z_k}^i\); we can consider \(z_1^i := 0\)) of the \(n\) ones \(Q_{z_k}^i\), \(k = 1, \ldots, n\). By the Ivory affine transformation as in Proposition 6.1 one can reverse the role of the vertices of the polygonal line \(P\) and of the points of tangency \(y_{z_1}^1, \ldots, y_{z_p}^p\) with \(Q_{z_1}^i\), thus, getting a-priori a new Darboux \(p\)-polygon \(\tilde{P}\) with vertices \(y_{z_1}^1, \ldots, y_{z_p}^p\) (note that in order to apply Proposition 6.1 we need the points \(x_{z_1}^1, x_{z_2}^1, \ldots, x_{z_p}^{p-1}\) obtained by the Ivory affine transformation), the poligonal segments tangent to \(Q_{z_1}^i\) at \(x_{z_1}^1, \ldots, x_{z_p}^p\) and to \(Q_{z_k}^i\) at points \(\tilde{y}_{z_k}^j\), \(j = 1, \ldots, p, k = 2, \ldots, n\) different from the original ones (because they lie on different lines). These Darboux \(p\)-polygons are actually closed because the original one \(P\) is closed and because we have reflection properties at all vertices; also, they have the same perimeter as that of \(P\) for \(z_1 = \ldots = z_p\) (because of the Ivory Theorem on preservation of lengths of segments between confocal quadrics); for the remaining general case they have the same perimeter by the general form of the Darboux Theorem we just proved (note however that the Darboux polygon \(P\) with the same perimeter may be different from the original one, as there are many choices in its construction).

This process can be of course iterated. Does it stop after a certain number of iterations? If yes, then how many do we actually get?

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REFERENCES


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