The concept of ideal in a lattice was introduced by Stone [15] in 1934, as
stated by Birkhoff ([3], page 21), who cites also the earlier papers by Tarski [18]
and Moisil [12]. In 1953, Dubreil-Jacotin, Lesieur and Croisot ([5], page 196)
worked with ideals defined in join semilattices. Ever since then the theory of
ideals and filters in lattices, semilattices and related structures has tremen-
dously developed. There are also several papers which generalize ideals and
filters to arbitrary posets.

As a matter of fact, it can be said that this generalized ideal theory fits
the spirit of the Romanian geometer and algebraist Dan Barbilian, who used
to analyse the role of each hypothesis of an important theorem; sometimes he
decomposed a theorem into several theorems obtained by introducing in turn
the hypotheses of the original theorem; cf. Rudeanu [14]. The present paper is
in the same line, dealing with ideal (and filter) theory in arbitrary posets. We offer an overview of the state of the art and several new results.

The duality between ideals and filters in lattices is inherited by their generalizations. So, in this paper we adopt the ideal language without loss of generality.

Generalizing to posets the concept of a (semi)lattice ideal means generalizing property (i)&(ii), where

(i) if \( x \in I \) and \( y \leq x \) then \( y \in I \), and

(ii) if \( x, y \in I \) then \( x \lor y \in I \).

But (i) already makes sense in posets as it stands and the posets satisfying it are called *down-sets*; cf. [8]. Thus, the generalizations which have been devised in the literature keep condition (i) and suggest various generalizations of (ii).

Accordingly, in Section 1, after a few preliminaries we follow the work of Venkatanarasimhan [20, 21], whose sketched an ideal-like theory in posets, for the non-empty down-sets, known as *order ideals*. In Section 2 we offer an overview of four generalizations of lattice ideals that have been suggested in the literature, and we point out several common features of them. Section 3 is devoted to one of these generalizations, the V-ideals. We present a remarkable unpublished work of Katriňák [9], who studied the lattice of V-ideals, in arbitrary posets and in the class of K-distributive posets, introduced by himself. The main result is that a poset is K-distributive if and only if its lattice of V-ideals is distributive. In Section 4 we prove several properties of BD-ideals, which are the generalization closest to the conventional definition of ideals. Among other results we obtain an intrinsic description of the ideal generated by a set and we prove that in the class of distributive posets (more general than semilattices) every maximal ideal is prime and the ideal lattice is distributive. Several examples and counter-examples fit the Barbilian line mentioned above. In a short section of Conclusions we summarize a thorough study by Erné [6] of various types of distributivity for posets and we suggest further possible surveys in this field.

We assume the reader is familiar with the fundamentals of lattice theory.

### 1. PRELIMINARIES AND ORDER IDEALS

In this section, we establish a few poset prerequisites, then we concentrate on the sets satisfying property (i) above, called *down-sets*; cf. [8]. After a few remarks on them, we present prime order ideals, introduced by Venkatanarasimhan [20, 21].
Let \((P, \leq)\) be a poset. For every \(X \subseteq P\), we denote by
\[
X^- = \{ z \in P \mid z \leq x \text{ for all } x \in X \},
\]
\[
X^+ = \{ z \in P \mid x \leq z \text{ for all } x \in X \},
\]
the set of all lower bounds / upper bounds of \(X\), respectively.

Since \(x \in \emptyset \implies z \leq x\) (ex falso quodlibet, e.f.q.), it follows that \(\emptyset^- = P\) and similarly \(\emptyset^+ = P\). It is well known that \(+\) and \(-\) establish a Galois connection between \(2^P\) and \(2^P\), therefore \(+\) and \(-\) are closure operators.

A subset \(A \subseteq P\) is called a down-set (an upper-set) if for every \(z \in P\) and every \(a \in A\), if \(z \leq a\) then \(z \in A\) (if \(a \leq z\) then \(z \in A\)). These two concepts are dual to each other, and it is immediately seen that a subset \(A \subseteq P\) is a down-set (an upper-set) if \(P \setminus A\) is an upper-set (a down-set). So, in the sequel we will refer to down-sets without loss of generality.

Clearly \(P\) is a down-set and \(\emptyset\) is a down-set by e.f.q. More generally:

**Proposition 1.1.** A subset \(A \subseteq P\) is a down-set if and only if either \(A^- \subseteq A\) or \(A^\emptyset = \emptyset\).

**Proof.** Sufficiency is obvious. To prove necessity, suppose \(A\) is a non-empty down-set. If \(z \in A^-\), take \(a \in A\). Then \(z \leq a\), hence \(z \in A\). Thus \(A^- \subseteq A\). \(\square\)

**Remark 1.1.** The family \(\mathcal{D}(P)\) of down-sets of a poset \(P\) is a complete sublattice of the complete lattice \((2^P, \cup, \cap)\).

Now for every \(X \subseteq P\) we define
\[
\downarrow X = \{ z \in P \mid z \leq x \text{ for some } x \in X \}.
\]
If \(X \neq \emptyset\) then \(X \cup X^- \subseteq \downarrow X\), while \(\downarrow \emptyset = \emptyset\). Note also that \(\downarrow \{a\} = (a)\).

**Proposition 1.2.** \(\downarrow X\) is the down-set generated by \(X\), that is, the least down-set which includes \(X\).

**Proof.** The statement is true for \(\emptyset = \downarrow \emptyset\), because \(\emptyset\) is a down-set. Now assume \(X \neq \emptyset\). Clearly \(\downarrow X\) is a down-set and \(X \subseteq \downarrow X\). Let \(Y\) be a down-set such that \(X \subseteq Y\). If \(z \in \downarrow X\), take \(x \in X\) such that \(z \leq x\); it follows that \(x \in Y\), hence \(z \in Y\). Thus \(\downarrow X \subseteq Y\). \(\square\)

**Corollary 1.1.** A subset \(S \subseteq P\) is a down-set if and only if \(X = \downarrow X\).

**Proposition 1.3.** \(\downarrow\) is a join-complete Kuratowski operator, that is, it satisfies (i) \(\downarrow\emptyset = \emptyset\), (ii) \(X \subseteq \downarrow X\), (iii) \(\downarrow (\bigcup_i X_i) = \bigcup_i \downarrow X_i\), (iv) \(\downarrow\downarrow X = \downarrow X\).

**Proof.** (i) and (ii) were already noted.

(iii) \(z \in \bigcup_i \downarrow X_i\) iff \(z \leq x\) for some \(x\) in some \(X_i\) iff \(z \leq x\) for some \(x \in \bigcup X_i\) iff \(z \in \downarrow \bigcup X_i\).
is a proper order ideal, if there is a prime order ideal $M$ of $P$ such that $x, y \in I$ implies $b \cup I \subseteq \{ x \} = (x \wedge y)$ and $z \subseteq I \iff z \in I$.

A proper order ideal $I$ of a poset $P$ is said to be maximal if for every proper ideal $J$, if $I \subseteq J$ then $J = I$ or $J = P$.

**Proposition 1.4.** Every maximal order ideal of a poset is prime.

*Proof.* Let $I$ be a maximal order ideal of $P$. If $a, b \in P$ satisfy $(a, b) \subseteq I$ but $(a) \not\subseteq I$, we will prove that $b \in I$. For $I \subseteq I \cup \{a\} \subseteq (I \cup \{a\}) = I \cup \downarrow \{a\} = I \cup (a)$, hence $I \cup (a) = P$ by the maximality of $I$, therefore $b \in I \cup (a)$, whence $b \in I$ or $b \in (a)$. In the latter case $(b) = (b) \subseteq I$, hence again $b \in I$. \qed

**Proposition 1.5.** If $I$ is an order ideal of a poset $P$ and $a \in P \setminus I$, then there is a prime order ideal $M$ of $P$ such that $I \subseteq M$ and $a \notin M$.

*Proof.* Let $I$ be the non-empty family of all order ideals $J$ such that $I \subseteq J$ and $a \notin J$. It follows by Zorn’s lemma that $I$ has a maximal member $M$. Since $M$ is a proper order ideal, if $M$ is not prime then it fails to satisfy (1.5). So there are $x, y \in P$ such that $(x) \subseteq M$ but $x \notin M$ and $y \notin M$. Note that $a \notin M$ because $M \in I$, hence we cannot have both $a \in (x)$ and $a \notin (y)$ because this would imply $a \in M$. Therefore $a \notin (x)$ or $a \notin (y)$, say $a \notin (x)$, which implies $a \notin M \cup (x) = I \cup \{x\}$ (cf. Remark 1.1). But the maximality of $M$ in $I$ implies $I \cup \{x\} = P$, a contradiction. \qed

**Proposition 1.6.** In a poset with $1$ every order ideal is included in a maximal order ideal.

*Proof.* If $I$ is a proper order ideal then $1 \notin I$, hence there is an order ideal $M$ which is maximal with the property $I \subseteq M$ and $1 \notin M$. This maximality implies that if $M \subseteq J$ where $J$ is an order ideal, then $J$ fails to satisfy $1 \notin J$, so that $1 \in J$, that is, $J = P$. Thus $M$ is a maximal order ideal. \qed
Besides the above results, Venkatanarasimhan studied a topology on the set of all prime order ideals of \( P \) and proved, among other results, that the lattice of open sets of this topological space is isomorphic to the lattice \( \mathcal{O}(P) \) of a poset \( P \) with 0.

Belding [2] proved that a poset \( P \) is characterized to within poset isomorphism by the lattice \( \mathcal{O}(P) \) and also that \( P \) is characterized to within poset isomorphism by the set \( \mathcal{O}(P) \) endowed with set-theoretical union \( \cup \) as semigroup operation.

2. FOUR DEFINITIONS OF IDEALS IN POSETS

We have found in the literature four generalizations of ideals (and filters) to arbitrary posets, which we present here in chronological order, which turns out to coincide with the decreasing order of their generality. We discuss in some detail the adequacy of these definitions, pointing out a few common features which they share with the ideals of a lattice, including the fact that the ideals form an algebraic lattice \( \mathcal{I}(P) \).

These concepts are the following:

1) **Closed ideals** or **normal ideals**, introduced by Birkhoff ([3], p. 59), who gives credit to Stone [16] for the case of Boolean algebras and also to Tarski [19]. This term designates the sets \( I \) satisfying \( I^{+−} \subseteq I \) (or equivalently, \( I = I^{+−} \), since \( ^{+−} \) is a closure operator).

2) A variant of closed ideals was suggested by Frink [7]: the sets \( I \) such that every finite subset \( F \subseteq I \) satisfies \( F^{+−} \subseteq I \). We will call them **Frink ideals**.

3) The **Venkataranarasimhan ideals** or **V-ideals** for short are the lower sets \( I \) such that for every finite non-empty subset \( F \subseteq I \), if \( \sup F \) exists then \( \sup F \in I \); cf. [20, 21].

4) The ideals suggested by Balbes and Dwinger [1], which we will call **BD-ideals**, are the lower sets \( I \) such that for every \( x, y \in I \), if \( x \vee y \) exists then \( x \vee y \in I \).

The above chronological order turns out to coincide with the decreasing order of their generality, as shown in Proposition 2.1 below.

Remark 2.1. Note first that \( \{x\}^{+−} = [x] = (x) = [x] = (x)^{+−} \). Then Frink ideals are also lower sets: for \( y \leq x \in I \) implies \( (y) \subseteq (x) = \{x\}^{+−} \subseteq I \), hence \( y \in I \).

In the sequel it will be convenient to denote \( \sup\{x_1, \ldots, x_n\} \), which may or may not exist, by \( \bigvee_{i=1}^{n} x_i \).
Proposition 2.1. The following implications hold, where none of them is an equivalence:

\[ \text{closed ideal} \implies \text{Frink ideal} \implies \text{V-ideal} \implies \text{BD-ideal}. \]

Proof. closed ideal \implies \text{Frink ideal}: For every finite subset \( F \subseteq I \) we have \( F^+ \subseteq I^+ \subseteq I \).

Frink ideal \implies \text{V-ideal}: Every Frink ideal \( I \) is a lower set by Remark 2.1. If \( x_1, \ldots, x_n \in I \) and \( \bigvee_{i=1}^n x_i \) exists, then setting \( F = \{x_1, \ldots, x_n\} \), we get \( F^+ = [\bigvee_{i=1}^n x_i] \), hence \( \bigvee_{i=1}^n x_i \in F^+ \subseteq I \), that is, \( \bigvee_{i=1}^n x_i \in I \).

V-ideal \implies \text{BD-ideal}: Trivial.

An example of a Frink ideal which is not a closed ideal is given in Frink [7].

Now take the poset in Fig. 3 from Section 4. We have \( \{a\}^- = \{a\} \), hence \( \{a\} \) is a lower set, and the only finite non-empty subset of \( a \) is \( \{a\} \) itself. Therefore \( \{a\} \) is a V-ideal. But \( \{a\}^{+-} = \{c, d\}^- = \{a, b\} \), hence \( \{a\} \) is not a Frink ideal.

In the poset \( P \cup \{(0,0,0,0,0)\} \), where \( P \) is the poset in Example 4.2 from Section 4, the set \( I = \{(0,0,0,0,0), a, b, c\} \) is a BD-ideal which is not a V-ideal. □

At first glance it seems trivial to say that a generalization of a concept from lattice theory should reduce to the original concept if applied to lattices. However a slight refinement can be introduced.

Given a concept \( C \) defined for a class \( \mathcal{K} \) of mathematical objects, we will say that a concept \( \overline{C} \) defined for a larger class \( \overline{\mathcal{K}} \) of mathematical objects is a good generalization or simply a generalization (a strong generalization, a weak generalization) of the concept \( C \), if the concept \( \overline{C} \) applied to the class \( \mathcal{K} \) coincides with \( C \) (is stronger than \( C \), is weaker than \( C \), respectively).

Thus, e.g., we have seen in Section 1 that the definition of prime lower sets is a good generalization, in the above sense, of the definition of prime ideals in lattices.

Let us examine the above definitions of ideals from this point of view.

The closed ideals are widely used in lattice theory (e.g., in the McNeille completion of posets) and it is well known that this concept is stronger than that of a lattice ideal. In other words, the closed ideals in the role of generalized ideals represent a strong generalization.

In a lattice every ideal \( I \) is a Frink ideal, because if \( F \) is a finite non-empty subset then \( \sup F \in I \) and \( F^+ = [\sup F]^- = (\sup F) \subseteq I \), while \( \varnothing^+ \) is \( \{0\} \) or \( \varnothing \) according as the poset \( P \) has least element 0 or not. Conversely, if \( I \) is a Frink ideal, then \( I \) is a lower set by Remark 2.1, and if \( a, b \in I \) then \( a \lor b = \min(\{a, b\}^+) \in \{a, b\}^{+-} \subseteq I \), that is, \( a \lor b \in I \).
In a lattice $\bigvee_{i=1}^{n} x_i$ exists for every $n \geq 1$ and the associativity of join implies that V-ideals coincide with lattice ideals, or equivalently, with BD-ideals, therefore the last two generalizations in our list are good ones.

Summarizing, generalization 1) is a strong generalization, while 2), 3) and 4) are good generalizations.

Frink [7] says that a generalization to posets of the concept of lattice ideal should satisfy three conditions: to reduce to the conventional concept of ideal in the case of lattices, the entire set $P$ should be a generalized ideal, and the principal ideals $(x]$ should be generalized ideals.

We have just discussed the first requirement. It is plain that $P$ is an ideal according to either of the four definitions. It follows from Remark 2.1 that the principal ideals are closed ideals, hence they are also Frink ideals, V-ideals and BD-ideals.

Now let us recall the concepts and results involved in the theorem we have announced in the introduction of this Section.

An element $c$ of a poset $P$ is said to be compact if whenever $c \leq \sup F$, where $F \subseteq P$, there is a finite subset $F_0 \subseteq F$ such that $c \leq \sup F_0$. A complete lattice $L$ is said to be algebraic if it is compactly generated, that is, for every $x \in L$ there is a set $C$ of compact elements of $L$ such that $x = \sup C$.

A subset $\Lambda$ of a poset $P$ (possibly $\Lambda = P$) is called upward directed or simply directed, if for every $x, y \in \Lambda$ there is $z \in \Lambda$ such that $x, y \leq z$. This property is immediately extended from $\{x, y\}$ to any finite subset $F \subseteq \Lambda$. A family $(A_\lambda)_{\lambda \in \Lambda}$ of subsets of a set $M$ is said to be directed if the index set is a directed poset $(\Lambda, \leq)$ and for every $\lambda, \mu \in \Lambda$, if $\lambda \leq \mu$ then $A_\lambda \subseteq A_\mu$.

A Moore family or a closure system of subsets of a set $M$ is a subset $C \subseteq 2^M$ such that every intersection of members of $C$ belongs to $C$; in particular $M \in C$ (the empty intersection). If $C$ is a Moore family, then the map

$$X \mapsto \overline{X} = \bigcap \{Y \in C \mid X \subseteq Y\}$$

is a closure operator ($X \subseteq \overline{X}$, $\overline{\overline{X}} = \overline{X}$ and $X \subseteq Y \implies \overline{X} \subseteq \overline{Y}$) and the poset $(C, \subseteq)$ is a complete lattice in which the complete meet operation is set-theoretical intersection and the complete join operation is $\bigvee X = \overline{\bigcap X}$.

A closure system $C$ is said to be algebraic if the complete lattice $(C, \cap, \bigvee)$ is isomorphic to the lattice of all subalgebras of a certain algebra (in the sense of universal algebra). One of several characterizations of algebraic systems $C$ within closure systems is the property that the union of any directed family of members of $C$ belongs to $C$.

If $C$ is an algebraic system, then the lattice $(C, \cap, \bigvee)$ is algebraic, the compact elements being the closures of the finite sets.
Theorem 2.1 (Katriňák [9]). In a poset with 0, for either of Frink ideals, V-ideals and BD-ideals, the family $\mathcal{I}(P)$ of ideals is an algebraic lattice in which the principal filters are compact.

Proof. Every intersection of lower sets is a lower set by Proposition 1.0. Besides, it is easily seen that in each of the three cases, every intersection of sets satisfying the specific condition in the definition of the corresponding type of ideals, satisfies also that condition. The role of 0 is to ensure that every such intersection is non-empty. Therefore $\mathcal{I}(P)$ is a Moore family.

If $(I_\lambda)_{\lambda \in \Lambda}$ is a directed family of ideals then $\bigcup I_\lambda$ is a lower set by Proposition 1.0. Besides, for every finite subset $F \subseteq \bigcup I_\lambda$ there is $\lambda_0 \in \Lambda$ such that $F \subseteq I_{\lambda_0}$. In the case of Frink ideals it follows that $F^{+-} \subseteq I_{\lambda_0} \subseteq \bigcup I_\lambda$. In the case of V-ideals, if $\sup F$ exists then $\sup F \in I_{\lambda_0}$, hence $\sup F \in \bigcup I_\lambda$. In particular in the case of BD-ideals we take $F = \{x, y\}$ and it follows that $x \vee y \in \bigcup I_\lambda$. We have thus proved that $\bigcup I_\lambda$ is an ideal. Therefore $\mathcal{I}(P)$ is an algebraic lattice.

Finally note that $x \in (x)$, $(x)$ is an ideal in each sense, and if $x \in I$ then $(x) \subseteq I$ by Remark 2.1 for Frink ideals and by definition for V-ideals and BD-ideals. This shows that $(x) = \overline{(x)}$, therefore $(x)$ is a compact element of $\mathcal{I}(P).$ □

Open question. Is every compact element of $\mathcal{I}(P)$ a principal ideal? In join semilattices this is easily proved using the existence of $\bigvee x_i$.

Of course, there are properties specific to each generalization. Thus, De Barros [4] studies set-collections of finite character, and obtains as applications properties of filters, meaning dual Frink ideals. For instance, he obtains a characterization of maximal filters and a necessary and sufficient condition in order that every principal filter $(x)$ be an intersection of maximal filters.

The next two sections are devoted to properties of V-ideals and of BD-ideals, respectively.

3. MORE ABOUT V-IDEALS

Theorem 2.1 of the previous Section and the entire Section 3, except Lemma 3.1, are contained in a remarkable unpublished work of Katriňák [9].

In this section the ideals are in fact V-ideals and the symbol $\bigvee$ applied to ideals designates the complete join operation in the lattice $\mathcal{I}(P)$. Two descriptions of the structure of the ideals $\bigvee I_\lambda$ from $\mathcal{I}(P)$, in arbitrary posets $P$ and in the class of K-distributive posets introduced by Katriňák, are provided. The
main theorem says that a poset $P$ is K-distributive if and only if the lattice $\mathcal{I}(P)$ is distributive.

Lemma 3.1 will be used in the proof of Theorem 3.1.

**Lemma 3.1** (Venkataranasinghan [20, 21]). For every $n \geq 1$ and every $x_1, \ldots, x_n \in P$ the following properties are equivalent:

(i) $\bigvee_{i=1}^{n} x_i$ exists;
(ii) $\left(\bigvee_{i=1}^{n} x_i\right) = \bigvee_{i=1}^{n} (x_i)$;
(iii) $\bigvee_{i=1}^{n} (x_i)$ is a principal ideal.

**Proof.** (i)$\implies$(ii): For all $i$ we have $x_i \leq \bigvee x_i$ hence $(x_i) \subseteq (\bigvee x_i)$, and also $x_i \in (x_i) \subseteq (\bigvee x_i)$ hence $x_i \in (\bigvee x_i)$. From these we infer $(\bigvee x_i) \subseteq (\bigvee x_i)$ and $\bigvee x_i \in (\bigvee x_i)$, respectively. The latter relation implies $(\bigvee x_i) \subseteq (\bigvee x_i)$.

(ii)$\implies$(iii): Trivial.

(iii)$\implies$(i): Suppose $\bigvee (x_i) = (x)$. Then $x_i \in (x_i) \subseteq (x)$, hence $x_i \leq x$, for all $i$. If $x_i \leq y$ for all $i$ then $(x_i) \subseteq (y)$ for all $i$, hence $(x) = \bigvee (x_i) \subseteq (y)$, therefore $x \leq y$. This proves that $x = \bigvee x_i$. $\square$

In the sequel, we will use the following

**Notation.** If the formula defining a set involves joins, it is understood that the formula holds for all existing joins, so that the running variables take all the values for which the corresponding joins exist.

Now for every $Y \subseteq P$ set

$$D(Y) = \{ y \in P \mid y \leq \bigvee_{i=1}^{n} y_i ; \ y_1, \ldots, y_n \in Y, \ n \geq 1 \},$$

$$D^1(Y) = D(Y) , \quad D^{n+1}(Y) = D(D^n(Y)) .$$

**Proposition 3.1.** For every $(I_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{I}(P)$ we have

$$\bigvee_{\lambda \in \Lambda} I_\lambda = \bigcup_{n \geq 1} D^n \left( \bigcup_{\lambda \in \Lambda} I_\lambda \right) .$$

**Proof.** Clearly $D(Y)$ are lower sets and $Y \subseteq D(Y)$ hence $D^n(Y) \subseteq D^{n+1}(Y)$, showing that the sequence $(D^n(Y))_{n \geq 1}$ is always increasing.

It follows that $\bigcup D^n (\bigcup I_\lambda)$ is a lower set by Proposition 1.0 and it includes $\bigcup I_\lambda$.

Now suppose $y_1, \ldots, y_n \in \bigcup D^n (\bigcup I_\lambda)$ and $\bigvee y_i$ exists. Then there is $m \geq 1$ such that $y_1, \ldots, y_n \in D^m (\bigcup I_\lambda)$, therefore $\bigvee y_i \in D^{m+1} (\bigcup I_\lambda)$, hence $\bigvee y_i \in \bigcup D^n (\bigcup I_\lambda)$. Therefore $\bigcup D^n (\bigcup I_\lambda)$ is an ideal.

If $I$ is an ideal such that $\bigcup I_\lambda \subseteq I$ then $D(\bigcup I_\lambda) \subseteq I$ because all $y_i$ and $y$ occurring in $D(\bigcup I_\lambda)$ are in $I$, and similarly if $D^n(\bigcup I_\lambda) \subseteq I$ then $D^{n+1}(\bigcup I_\lambda) \subseteq I$. We have thus proved that $\bigcup D^n (\bigcup I_\lambda) \subseteq I$. $\square$
Let us say that a poset $P$ is Katriňák-distributive or $K$-distributive for short, if for every $n \geq 1$ and every $x, x_1, \ldots, x_n \in P$, if $\bigvee_{i=1}^{n} x_i$ exists and $x \leq \bigvee_{i=1}^{n} x_i$, then there exist $y_1, \ldots, y_n \in P$ such that $x = \bigvee_{i=1}^{n} y_i$ and each $y_i$ has an upper bound among $x_1, \ldots, x_n$.

For the meaning of this definition the reader is referred to the distributivity dealt with in the next Section.

**Proposition 3.2.** If the poset $P$ is $K$-distributive, then for every $(I_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{I}(P)$ we have $\bigvee_{\lambda \in \Lambda} I_\lambda = D(\bigcup_{\lambda \in \Lambda} I_\lambda)$, that is,

$$
\tag{3.2}
\bigvee_{\lambda \in \Lambda} I_\lambda = \{ y \in P \mid y \leq \bigvee_{i=1}^{n} y_i ; y_1, \ldots, y_n \in \bigcup_{\lambda \in \Lambda} I_\lambda, \ n \geq 1 \} .
$$

**Proof.** It was shown in the proof of Proposition 3.1 that $D(\bigcup I_\lambda)$ is a lower set which includes $\bigcup I_\lambda$ and if $I$ is an ideal which includes $\bigcup I_\lambda$ then $D(\bigcup I_\lambda) \subseteq I$. Therefore, if we prove that $D(\bigcup I_\lambda)$ is closed with respect to existing $\bigvee y_i$, it will follow that $D(D(\bigcup I_\lambda)) = D(\bigcup I_\lambda)$.

If $y_1, \ldots, y_n \in D(\bigcup I_\lambda)$ and $\bigvee y_i$ exists then

$$
y_i \leq \bigvee_{k=1}^{n(i)} y_{ik}, \text{ where } y_{ik} \in \bigcup_{\lambda \in \Lambda} I_\lambda \ (k = 1, \ldots, n(i)) \ (i = 1, \ldots, n),
$$

hence $y_i = \bigvee_{k=1}^{n(i)} z_{ik}$, where each $z_{ik}$ has an upper bound among $y_{i1}, \ldots, y_{in(i)}$ for $(k = 1, \ldots, n(i))$ and $(i = 1, \ldots, n)$. Therefore

$$
\tag{3.3}
\bigvee_{i=1}^{n} y_i = \bigvee_{i=1}^{n} \bigvee_{k=1}^{n(i)} z_{ik} .
$$

But $\bigcup I_\lambda$ is a lower set by Proposition 1.0, hence all $z_{ik} \in \bigcup I_\lambda$. Besides, since the joins in (3.3) do exist, it follows that the right side of (3.3) is the l.u.b. of the $z_{ik}$'s. Therefore $\bigvee_{i=1}^{n} y_i \in D(\bigcup I_\lambda)$. \[\square]\n
**Proposition 3.3.** If the poset $P$ is $K$-distributive, then the lattice $\mathcal{I}(P)$ is distributive.

**Proof.** Take $I, J, H \in \mathcal{I}(P)$. We have to prove that $(I \lor J) \cap H \subseteq (I \cap H) \lor (J \cap H)$.

Take $y \in (I \lor J) \cap H$. Then $y \in H$ and $y \leq \bigvee_{i=1}^{n} y_i$ for some $n \geq 1$ and some $y_1, \ldots, y_n \in I \cup J$, by Proposition 3.2. It follows that $y = \bigvee_{i=1}^{n} z_i$ for some $z_1, \ldots, z_n$ such that each $z_i$ has an upper bound among $y_1, \ldots, y_n$. Therefore all $z_i \in I \cup J$. Besides, for all $i$ we have $z_i \leq y$, hence $z_i \in H$. It follows that

$$
z_i \in (I \cup J) \cap H = (I \cap H) \cup (J \cap H) \ (i = 1, \ldots, n),
$$

hence $y \in (I \cap H) \lor (J \cap H)$, again by Proposition 3.2. \[\square\]
Theorem 3.1. A poset $P$ is K-distributive if and only if the lattice $I(P)$ is distributive.

Proof. It remains to prove that if the lattice is distributive then $P$ is K-distributive. So assume that $I(P)$ is distributive.

Suppose $x, x_1, \ldots, x_n \in P$ are such that $\bigvee_{i=1}^{n} x_i$ exists and $x \leq \bigvee_{i=1}^{n} x_i$. Then using Lemma 3.1 and distributivity we obtain $(x) = (x) \cap (\bigvee x_i) = (x) \cap \bigvee (x) \cap (x_i)$. Setting $(x) \cap (x_i) = X_i$ for each $i$ and using Proposition 3.1 we obtain $x \in (x) = \bigvee_{i=1}^{n} X_i = \bigcup_{m \geq 1} D^m(\bigcup X_i)$. Let $p$ be the minimum index such that $x \in D^p(\bigcup X_i)$.

Note that $\bigcup X_i \subseteq (x)$, hence the elements $y$ and $y_1, \ldots, y_n$ occurring in the expression of $D(\bigcup X_i)$ are in $(x)$. So $D(\bigcup X_i) \subseteq (x)$ and it follows by induction that $D^m(\bigcup X_i) \subseteq (x)$ for all $m \geq 0$, where we have set $D^0(Y) = Y$.

If $p > 1$ then $x \leq \bigvee_{j=1}^{m} y_j$ for some $m \geq 1$ and some $y_1, \ldots, y_m \in D^{p-1}(\bigcup X_i)$, where $p - 1 \geq 1$, hence each $y_j \leq \bigvee_{k=1}^{n(j)} z_{jk}$, for some $z_{jk} \in D^{p-2}(\bigcup_{i=1}^{n} X_i) \subseteq (x)$. If all $z_{jk} \leq z$ then all $y_j \leq z$, hence $x \leq \bigvee_{j=1}^{m} y_j \leq z$. Thus $x$ is l.u.b. of all $z_{jk}$, hence $x \in D^{p-1}(\bigcup_{i=1}^{n} X_i)$, in contradiction with the choice of $p$.

It follows that $p = 1$, that is, $x \in D(\bigcup X_i)$. Therefore $x \leq \bigvee_{j=1}^{m} y_j$ for some $m \geq 1$ and some $y_1, \ldots, y_m \in \bigcup X_i$. But all $y_j \leq x$, hence $y = \bigvee_{j=1}^{m} y_j$. On the other hand for each $j$ there is some $i$ such that $y_j \in X_i$, which implies $y_j \leq x_i$ for that $i$. We have thus proved that $P$ is K-distributive. \qed

4. MORE ABOUT BD-IDEALS

The concept of a Balbes-Dwinger ideal, the most restrictive among the four generalizations dealt with in Section 2, is also the most natural one: it is obtained by just adding “if $x \vee y$ exists” to the conventional definition. In this section the term “ideal” means BD-ideal, and we introduce also strong ideals, which are a strong generalization in the sense of Section 2. We prove that the family of strong ideals of a poset with 0 is an algebraic lattice, like the family of ideals (cf. Theorem 2.1); yet the principal ideals are not strong.

The main concern of this Section is about ideals and prime ideals. Among other results we obtain an intrinsic characterization of the ideal generated by a subset and we prove that in the class of distributive posets (more general than semilattices) every maximal ideal is prime and the ideal lattice is distributive. Several examples and counter-examples clarify the exact relationships between the concepts dealt with in this Section, just in the line of Barbilian mentioned in the Introduction of this paper.
By a strong ideal of a poset \((P, \leq)\) we mean a non-empty lower set \(I\) such that for every \(x, y \in I\), all existing upper bounds of \(\{x, y\}\) belong to \(I\).

The total set \(P\) is a strong ideal, hence an ideal: the improper one. The other ideals and strong ideals are said to be proper.

A weak generalization in the sense defined in Section 2 would be the following: define a weak ideal as a non-empty directed and lower set of the poset \(P\).

Nevertheless we will not deal with this definition, because according to it the total set \(P\) may fail to be an ideal, which is unacceptable in Frink’s opinion (and ours).

The following general properties are illustrated in the next example. Every isolated point (cf. Example 1.1) is a strong ideal. In a poset without proper joins (meaning \(x \lor y\) with \(x\) incomparable with \(y\)), lower sets coincide with ideals.

**Example 4.1.** The poset in Fig. 1 has a single proper join, \(c\) is an isolated point, \(\{a, b, d, u\}\) is a proper strong ideal, while \(\{a, b, d\}\) is just a proper ideal.

![Fig. 1](image)

The following analogue of Theorem 2.1 holds for strong ideals:

**Theorem 4.1.** In a poset with 0 the family of strong ideals is an algebraic lattice.

**Proof.** Similar to that of Theorem 2.1. The strong ideals form a Moore family and the union of a directed family of strong ideals is a strong ideal by a standard argument. □

In view of Theorem 2.1, for every subset \(X\) of a poset with 0 there exists the ideal \((X]\) generated by \(X\), that is, the smallest ideal that includes \(X\). It is obtained as the intersection of all ideals that include \(X\). Yet we are interested in obtaining an intrinsic description of \((X]\). It is plain that \((\emptyset]\) = \(\{0\}\). For \(X \neq \emptyset\) we will use formula (1.3) which constructs the lower set \(\downarrow X\) generated by \(X \neq \emptyset\), which, like the next Proposition, doesn’t require the existence of 0.

**Proposition 4.1.** Let \(P\) be an arbitrary poset and \(\emptyset \neq X \subseteq P\). Set \(X_0 = X\) and \(X_{n+1} = \{z \in P \mid z \leq x \lor y ; x, y \in X_n\}\) for all \(n \geq 0\). Then \((X] = \bigcup_{n \geq 1} X_n\).
Proof. The identity \( x = x \lor x \) shows that the sequence \( (X_n)_{n \geq 0} \) is increasing, hence \( X \subseteq \bigcup_{n \geq 1} X_n \neq \emptyset \).

This also implies that if \( x, y \in \bigcup_{n \geq 1} X_n \), then \( x, y \in X_p \) for some \( p \), hence if \( x \lor y \) exists then \( x \lor y \in X_{p+1} \subseteq \bigcup_{n \geq 1} X_n \). Besides, all \( X_n \) with \( n \geq 1 \) are lower sets, therefore \( \bigcup_{n \geq 1} \) is a lower set by Remark 1.1 via the characterization \( \downarrow A = A \) of lower sets \( A \).

We have thus proved that \( \bigcup X_n \) is an ideal which includes \( X \). If \( I \) is an ideal such that \( X \subseteq I \), then \( X_1 = \{ z \in P \mid x \leq x \lor y ; x, y \in X \} \subseteq I \) and it immediately follows by induction that all \( X_n \subseteq I \), therefore \( \bigcup_{n \geq 1} X_n \subseteq I \). \( \square \)

Formula

\[
(1.2) \quad [x] = \{ z \in P \mid z \leq x \}
\]

from Section 1 can be also proved using Proposition 4.1, namely we get in turn \( X = \{ x \} = X_0, X_1 = \{ z \in P \mid z \leq x \}, X_2 = X_1 \), hence \( X_n = X_1 \) for all \( n \geq 1 \).

In the case of join semilattices it is well known that

\[
(4.2) \quad [X] = \{ z \mid z \leq x_1 \lor \ldots \lor x_n, x_1, \ldots, x_n \in X, n \geq 1 \},
\]

but, unlike what happens in Proposition 3.2, it seems there is little hope to extend formula (4.2) to arbitrary posets. Indeed, consider the following examples concerning the existence of \( a \lor b \lor c \) and the existence of \( b \lor c \) or \( a \lor c \) or \( a \lor b \).

Example 4.2. Let \( P \) be the subset of the unitary cube \( \{0, 1\}^5 \) that consists of the following elements: \( a = (1, 0, 0, 0, 0), b = (0, 1, 0, 0, 0), c = (0, 0, 1, 0, 0), m = (0, 1, 1, 0, 1), n = (0, 1, 1, 1, 0), p = (1, 0, 1, 0, 1), q = (1, 0, 1, 1, 0), r = (1, 1, 0, 0, 1), s = (1, 1, 0, 1, 0), u = (1, 1, 1, 1, 1) \). The upper bounds of \( b, c \) are \( m, n, u \), those of \( a, c \) are \( p, q, u \), while the upper bounds of \( a, b \) are \( r, s, u \). So there are no \( b \lor c, a \lor c, a \lor b \), although \( a \lor b \lor c \) does exist.

Example 4.3. In the opposite sense consider the subset \( \{ a = (1, 0, 0), b = (0, 1, 0), c = (0, 0, 1), b \lor c = (0, 1, 1), c \lor a = (1, 0, 1), a \lor b = (1, 1, 0) \} \) of \( \{0, 1\}^3 \); there is no \( a \lor b \lor c \).

In the sequel an important role will be played by the principal ideals \( (x) \). Since these ideals are not strong, from now on we are forced to disregard strong ideals.

Now, we wish to adopt a definition of prime ideals in posets, which should be a good generalization, in the sense of Section 2, of prime lattice ideals. But it was noted in Section 1 that the definition of prime lower sets in posets is such a good generalization; that is why we adopt it for prime ideals of posets as well. That is to say, we define a prime ideal of a poset as a proper ideal \( I \).
which satisfies the condition 
\[(4.3) \quad (x] \cap (y] \subseteq I \implies x \in I \text{ or } y \in I;\]
by duality, a prime filter will be a proper filter \(F\) satisfying the condition 
\[(4.3') \quad [x) \cap [y) \subseteq F \implies x \in F \text{ or } y \in F.\]

In order to generalize the relationship between prime ideals and prime filters in a lattice, we need the following lemma:

**Lemma 4.1.** In every poset we have 
\[(4.4) \quad \text{if } x \land y \text{ exists then } (x] \cap (y] = (x \land y),\]
\[(4.4') \quad \text{if } x \lor y \text{ exists then } [x) \cap [y) = [x \lor y) \text{ and } (x] \lor (y] = (x \lor y).\]

**Proof.** If \(x \land y\) exists then (4.4) follows from 
\[z \in (x] \cap (y) \iff z \leq x \text{ and } z \leq y \iff z \leq x \land y \iff z \in (x \land y)\]
and the first part of (4.4') is obtained by duality, which doesn’t alter set-theoretical operations. Besides, if \(x \lor y\) exists then \((x], (y] \subseteq (x \lor y]\) and if \(I\) is an ideal such that \((x], (y] \subseteq I\) then \(x, y \in I\), hence \(x \lor y \in I\), i.e., \((x \lor y] \subseteq I\). \(\square\)

The discrepancy between (4.4) and (4.4') is due to the fact that the (semi)lattice of ideals is not self-dual; cf. Remark 4.1 below.

**Proposition 4.2.** In every poset \(P\), if \(I\) is a prime ideal then \(P \setminus I\) is a proper filter.

**Proof.** We already know that \(P \setminus I\) is an upper set. Then \(P \setminus I \neq \emptyset\) because the ideal \(I\) is proper and \(P \setminus I \neq P\) because \(I \neq \emptyset\).

Now suppose that \(x, y \in P \setminus I\) and the g.l.b. \(x \land y\) exists. Then \(x \land y \in I\) would imply \((x] \cap (y] \subseteq I\) by (4.4), hence \(x \in I\) or \(y \in I\) by (4.3), a contradiction. Thus \(x \land y \in P \setminus I\) and \(P \setminus I\) is a proper filter. \(\square\)

The following corollary generalizes Exercise I.3.17 in Grätzer [8] from lattices to join semilattices.

**Corollary 4.1.** In a join semilattice \(P\),
\(I\) is a prime ideal \(\iff\) \(P \setminus I\) is a prime filter.

**Proof.** \(\implies\): We prove that the proper filter \(P \setminus I\) obtained in Proposition 4.2 is in fact a prime filter. Indeed, if \((x] \cap (y] \subseteq P \setminus I\) then \(x \lor y \in P \setminus I\) by (4.4'). It follows that \(x \in P \setminus I\) or \(y \in P \setminus I\), otherwise \(x \in I\) and \(y \in I\) would imply \(x \lor y \in I\), a contradiction.

\(\iff\): Write the dual of \(\implies\) for \(F = P \setminus I\). \(\square\)
The concept of a maximal ideal (maximal filter) is verbatim the same in all contexts dealing with ideals and/or filters, namely a proper ideal (proper filter) $S$ such that if $S \subseteq T$ where $T$ is a proper ideal (proper filter), then $S = T$.

It is well known that in a distributive lattice every maximal ideal is prime. A natural expectation would be that a generalization of this property holds in a special class of semilattices. In fact, several concepts of distributive semilattices have been suggested in the literature. One of them says that a distributive join semilattice is a join semilattice which satisfies the following property: for every $x, y, z$,

$$x \leq y \vee z \implies \exists y_1 \exists z_1 \ y_1 \leq y \wedge z_1 \leq z \wedge x = y_1 \vee z_1$$

(see e.g. Rhodes [13]). This definition is justified by the fact that the above property is satisfied in distributive lattices: take $y_1 = x \wedge y$ and $z_1 = x \wedge z$. [We don’t know whether a lattice satisfying this property is necessarily distributive, in other words whether the distributivity of join semilattices is a good generalization or a strong generalization of distributive lattices].

By a distributive poset we mean a poset such that for every $x, y, z$,

$$\exists (y \vee z) \& x \leq y \vee z \implies \exists y_1 \exists z_1 \ y_1 \leq y \wedge z_1 \leq z \wedge x = y_1 \vee z_1. \tag{4.5}$$

Every poset without proper joins (defined before Example 4.1) is distributive, because in (4.5) we have $x \leq \min(y, z)$ and the decomposition $x = x \vee x$ does the job. For instance, so is the poset in Example 4.7. Also, a $K$-distributive poset with 0 is distributive, because if $x \leq y \vee z$ and $x = y_1 \vee z_1$ with both $y_1, z_1 \leq y$ then $x \leq y$, so that the decomposition $x = x \vee 0$ fits (4.5).

Note that in every poset property (4.5) holds for certain particular cases of inequalities $x \leq y \vee z$. So, (4.5) is trivially satisfied if $x = y \vee z$. Also, the decomposition $x = x \vee x$ fits (4.5) if $y = z$ or if $x \leq y, z$. Therefore, in order to prove that a join semilattice is distributive, it suffices to check (4.5) for the triples $x, y, z$ satisfying $x < y \vee z$, $y \neq z$ and $x$ is not a common lower bound of $y$ and $z$.

Example 4.4. The poset in Example 4.1 is not distributive, because $a < a \vee b$ but the only join decomposition of $a$ of the form $a = a_1 \vee a_2$ is $a = a \vee a$. The join semilattice in Fig. 2 is immediately seen to be distributive.

![Fig. 2](image-url)
The next aim of this paper is to generalize the theorem saying that every maximal ideal is prime, from distributive lattices to distributive posets.

The beginning of the proof, given in the following lemma, is exactly the same as for distributive lattices.

**Lemma 4.2.** Let $I$ be a maximal ideal of a poset $P$, and $a, b \in P$ two elements such that $(a) \cap (b) \subseteq I$ and $a \notin I$. Then $b \in (I \cup (a))$.

**Proof.** We have $I \subseteq I \cup (a) \subseteq (I \cup (a))$. Since $I$ is maximal, it follows that $(I \cup (a)) = P$, therefore $b \in (I \cup (a))$. □

In distributive lattices the proof of the theorem is easily completed using the fact that $(I \cup (a)) = \{z \mid z \leq i \lor \alpha, i \in I, \alpha \in (a)\}$. This formula is not valid in (distributive) posets; instead, we will use universal-algebraic techniques. More exactly, introducing the partial algebra $(P, \lor)$, we will prove that $(I \cup (a))$ is in fact the subalgebra of $P$ generated by $I \cup (a)$. Then universal algebra will enable us to prove that the element $b$ of this subalgebra belongs in fact to $I$.

**Notation.** In the sequel $I, a, b$ are the ideal and the elements from Lemma 4.2, while $E$ is the subalgebra of the partial algebra $(P, \lor)$ generated by $I \cup (a)$.

In other words, $E$ is the smallest subset of $P$ such that $I \cup (a) \subseteq E$ and if $x, y \in E$ and $x \lor y$ exists then $x \lor y \in E$.

**Lemma 4.3.** If the poset $P$ is distributive, then $E$ is a lower set.

**Proof.** If $f \leq i \in I$ then $f \in I \subseteq E$. If $f \leq \alpha \in (a)$ then $f \in (a) \subseteq E$. If $f \leq e_1 \lor e_2$ where $e_1, e_2 \in E$ and the lower bounds of either of $e_1, e_2$ are in $E$, then $f = f_1 \lor f_2$ for some $f_1 \in E$ and some $f_2 \in E$, hence $f = f_1 \lor f_2 \in E$. □

**Lemma 4.4.** If the poset $P$ is distributive, then $E = P$.

**Proof.** The definition of $E$ and Lemma 4.3 show that $I \cup (a) \subseteq E$ and $E$ is an ideal, therefore $P = (I \cup (a)) \subseteq E$. □

**Theorem 4.2.** In a distributive poset every maximal ideal is prime.

**Proof.** It follows from Lemma 4.4 that $b \in E$, therefore $b$ can be computed as prescribed by universal algebra.

Recall that each element of the subalgebra generated by a set is obtained from finitely many generators by successive applications of the operations of the algebra, and this construction can be represented by a tree. In our case the construction of the element $b$ is represented by a binary tree labelled with elements of $P$. Each application of $e = e_1 \lor e_2$ is represented by a vertex labelled $e$ together with two sons labelled $e_1$ and $e_2$. The root is labelled $b$ and the leaves are labelled by generators from $I \cup (a)$. 
A well-known property of trees says that from each leaf there is a (unique) path to the root. For each generator \( \alpha \in \{a\} \) this path yields an increasing chain of the corresponding labels, from \( \alpha \) to \( b \). Therefore \( \alpha \leq b \), hence \( \alpha \in \{a\} \cap \{b\} \subseteq I \) according to the hypothesis of Lemma 4.2. We have thus proved that all the generators used in the construction of \( b \) belong to \( I \). Since the other elements occurring in the construction of \( b \) are obtained by rule \( e = e_1 \lor e_2 \), it follows immediately by induction that all of them belong to \( I \); in particular \( b \in I \). (This includes the particular case when \( b \) itself is a generator). \( \square \)

In the next two examples, the non-distributivity follows from the fact that \( a < b \lor c \) while \( a \) has only trivial decompositions like \( a = a \lor a \).

**Example 4.5.** Distributivity is essential in Theorem 4.2. Consider the tree \( \{a, b, c, 1\} \) in which the elements \( a, b, c \) are pairwise incomparable and 1 is greatest element. The proper ideals of this non-distributive join semilattice are \( \{a\}, \{b\}, \{c\} \); each of them is maximal but not prime. For instance, we have \( \{a\} \cap \{b\} = \emptyset \subset \{c\} \), but \( a \notin \{c\} \) and \( b \notin \{c\} \).

**Example 4.6.** The converse of Theorem 4.2 does not hold. Indeed, consider the non-distributive join semilattice \( \{a, b, c, 1\} \), in which \( a < b < 1 \) and \( c < 1 \) while \( a, c \) and \( b, c \) are incomparable. However we are going to prove that the maximal ideals are still prime.

Clearly, the maximal ideals are \( \{a, b\} \) and \( \{c\} \) and it remains to prove they are prime. If \( I \) stands for any of these ideals then \( 1 \notin I \), hence if \( x = y \) condition (4.3) reduces to \( \{y\} \subseteq I \implies y \in I \) and it is satisfied. Therefore it suffices to examine the cases \( x \neq 1 \& y \neq 1 \& x \neq y \). These cases are: (i) \( x = a, y = b \) or vice versa, which implies \( \{x\} \cap \{y\} = \{a\} \); (ii) \( x = a, y = c \) or vice versa, which implies \( \{x\} \cap \{y\} = \emptyset \); (iii) \( x = b, y = c \), or vice versa, which implies \( \{x\} \cap \{y\} = \emptyset \). Cases (i), (ii), (iii) show that the ideal \( \{a, b\} \) is prime, while cases (ii), (iii) show that the ideal \( \{c\} \) is prime.

Note that the semilattices in Examples 4.5 and 4.6 are obtained by deleting the least element 0 from the standard non-distributive lattices diamond and pentagone, respectively.

**Open question.** Is it true that a join semilattice is non-distributive if and only if it does not include a subsemilattice isomorphic to one of the semilattices in Examples 4.5 and 4.6 ?

The last example somehow summarizes the topics discussed so far in this Section.

**Example 4.7.** Consider the poset \( P \) in Fig. 3.
$P$ is self-dual, that is, for every property of $P$ relative to ideals, the dual property about filters is also valid.

$P$ has no proper joins (i.e., $x \lor y$ with $x$ incomparable with $y$). Therefore lower sets coincide with ideals and $P$ is distributive.

The maximal ideals are $\{a, b, c\}$ and $\{a, b, d\}$, while $\{a\}$, $\{b\}$ and $\{a, b\}$ are just ideals. There are no strong ideals.

Since condition (4.3) is satisfied for $x = y$ in any poset, in order to prove that a proper ideal $I$ is prime, it suffices to check (4.3) for $x \neq y$. In the present poset $P$ we have $[a] = \{a\}$, $[b] = \{b\}$, $[c] = [d] = \{a, b\}$, hence $[a] \cap [b] = \emptyset$, $[a] \cap [c] = [a] \cap [d] = \{a\}$, $[b] \cap [c] = [b] \cap [d] = \{b\}$ and $[c] \cap [d] = \{a, b\}$. Therefore the ideals $\{a\}$, $\{b\}$, $\{a, b, c\}$ and $\{a, b, d\}$ are prime, while $\{a, b\}$ is not prime.

This shows that in a distributive poset a non-maximal prime ideal may be prime or not prime. In the non-distributive join semilattice from Example 4.6 maximal ideals coincide with prime ideals, since $\{a\}$ is not prime.

Another interesting property is the following.

**Theorem 4.3.** If $P$ is a distributive poset with 0, the ideal lattice $\mathcal{I}(P)$ is distributive.

**Proof.** Let $I, J, K$ be ideals; we will prove that $I \cap (J \cup K) \subseteq (I \cap J) \cup (I \cap K)$. Let $J \cup K = \bigcup_{n \geq 1} X_n$ and $(I \cap J) \cup (I \cap K) = \bigcup_{n \geq 1} Y_n$ be the developments obtained by applying Proposition 4.1. Then $I \cap (J \cup K) = \bigcup_{n \geq 1} (I \cap X_n)$ and we shall prove that $I \cap X_n \subseteq Y_n$ for all $n$.

If $z \in I \cap X_1$ then $z \in I$ and $z \leq \alpha \lor \beta$ for some $\alpha \in J$ and $\beta \in K$. Hence $z = \alpha_1 \lor \beta_1$ for some $\alpha_1 \leq \alpha$ and $\beta_1 \leq \beta$. It follows that $\alpha_1 \in J$ and $\beta_1 \in K$; besides, $\alpha_1, \beta_1 \leq z$, hence $\alpha_1, \beta_1 \in I$, therefore $\alpha_1 \in I \cap J$ and $\beta_1 \in I \cap K$, so that $z \in Y_1$, proving that $I \cap X_1 \subseteq Y_1$.

For the inductive step assume $I \cap X_n \subseteq Y_n$. If $z \in I \cap X_{n+1}$ then $z \in I$ and $z \leq x \lor y$ for some $x, y \in X_n$. Hence $z = x_1 \lor y_1$ for some $x_1 \leq x$ and $y_1 \leq y$. But the $X_n$'s in Proposition 4.1 are lower sets, therefore $x_1, y_1 \in X_n$. Besides, $x_1, y_1 \leq z$, hence $x_1, y_1 \in I \cap X_n \subseteq Y_n$, therefore $z \in Y_{n+1}$. This proves that $I \cap X_{n+1} \subseteq Y_{n+1}$. \( \square \)
Open question. Prove or disprove the converse of Theorem 4.3.

Remark 4.1. The following principles of duality hold. A) The dual of a property valid for the ideals of an arbitrary poset is a property valid for the filters of an arbitrary poset. B) Let us refer to property (4.5) and its dual as join distributivity and meet distributivity, respectively. Then the dual of a property valid for the ideals of a join-distributive poset is a property valid for the filters of a meet-distributive poset.

Remark 4.2. Unlike what happens for lattices, join distributivity and meet distributivity are not equivalent for posets. For instance, the tree \{a, b, c, 1\} in Example 4.5 is not join-distributive, as \(a < b \lor c\) but \(a\) has no proper join decompositions. Yet the dual tree \{a, b, c, 0\} is join-distributive because it has no proper joins, therefore its dual, that is the tree \{a, b, c, 1\}, is meet-distributive.

Conclusions. In this paper, we have focused on several definitions of ideals (and filters) in posets and the corresponding ideal lattices, and we culminated with a theorem maximal \(\Rightarrow\) prime. The results established in this paper are valid in any poset or in any poset which is distributive in the sense we have explained.

In order to keep the paper within reasonable space limitations, we have omitted the properties of ideals in pseudocomplemented posets, which deserve to be the subject of a separate paper.

Besides, the ideal theory of posets includes also theorems which generalize other basic results from ideal theory in lattices but which are not valid for arbitrary posets, not even for distributive posets in the sense defined above. For this quite extended field we refer the reader to the papers mentioned below and to the literature indicated therein.

According to Erné [6], the following are basic theorems: (1) there is a prime ideal, every proper ideal is included\(^2\) in a prime ideal, every proper ideal is an intersection of prime ideals, every ideal disjoint from a filter is included in a prime ideal still disjoint from the filter; (2) the statements obtained from the above by replacing “prime” with “maximal”, and of course (3), the duals of the above theorems. The paper [6] is a thorough study of the relationships between the generalizations of the above theorems to many classes of posets satisfying various types of distributivity that have been introduced in the literature. As the author says, the result is that “many known algebraic, lattice theoretical or topological facts concerning prime and maximal ideals have been extended to

\(^2\) We disagree with the common practice of saying “contained”, which means ∈, instead of “included”, which means \(\subseteq\).
the setting of posets.”. A synoptic table “Distributive laws and pseudocomplementation. Their role in prime and maximal ideal theorems”, included in the paper, is noteworthy. The paper has 31 references to the literature.

The paper [17], not quoted in [6] and also unavailable to us, is reviewed as follows in MR 84h:06004: “Distributive elements in a general poset are defined and studied. The main theorem establishes some necessary and sufficient conditions for a poset to be distributive.” The papers [10, 11], which cite [7, 17] and two other articles, are hard to follow and not self-contained.

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