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Boundary Element Tearing and Interconnecting Dual–Primal Method

by

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Abstract

The aim of this paper is to introduce the *dual primal boundary element tearing and interconnecting* (BETI-DP) method with Dirichlet and hypersingular boundary integral operator preconditioners. In previous articles BETI and coupled FETI/BETI methods were introduced. As a natural continuation we present here the BETI-DP method and discuss few general choices of the dual spaces for the three dimensional case. We show that the condition number of the system matrix equiped with the Dirichlet and with the hypersingular boundary integral operator preconditioner in the FETI-DP method.

Key words and phrases: FETI-DP, BETI-DP, Steklov-Poincaré operator, Schur complement, preconditioners, hypersingular boundary integral operator.

1 Introduction

The classical *finite element tearing and interconnecting* (FETI) and its boundary element counterpart *boundary element tearing and interconnecting* (BETI) methods are domain decomposition methods of iterative substructuring type. The local finite and boundary element spaces are given on each substructure separately. The global continuity is enforced by using Lagrange multipliers, resulting a saddle point problem which can be solved iteratively via its dual problem using a special type of preconditioned conjugate gradient method. The *dual primal finite element tearing interconnecting method* (FETI-DP) was introduced by Farhat, Lesoinne, Le Tallec, Pierson and Rixen [2]. The term *dual-primal* refers to the idea of enforcing some continuity constraints across the interface between the subdomains as in a primal method, while all the other constraints are enforced using Lagrange multipliers as in the dual methods. The tearing part is identical for FETI and BETI as well as for FETI-DP and BETI-DP, the major differences appear in the interconnecting part. An important contribution in the analysis of the two dimensional case of second and fourth order elliptic problems was brought by Mandel and Tezaur [8] and also by Brenner [1]. In the three dimensional case we mention here the works of Farhat, Lesoinne and Pierson [3] and Klawonn, Widlund and Dryja [4]. Recently, Mandel and Tezaur have given a pure algebraic formulation of FETI-DP which is independent of the studied problem [9].

In [5] and [6] the BETI and coupled FETI/BETI methods were introduced. These results are based on the fact that the finite element Schur complement as well as the discrete boundary element Steklov-Poincaré

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operator are both discrete versions of the same Dirichlet-Neumann map. Spectral equivalence inequalities were given, which ensure the spectral equivalence between the finite element Schur complement matrix, the discrete boundary element Steklov-Poincaré matrix and the discrete hypersingular integral operator matrix. Note that all constants are independent of the discretisation. With the help of this result (Lemma 3.1 in [5]) the hypersingular boundary integral operator preconditioner was introduced and all the convergence results were transfered to the BETI and coupled FETI/BETI methods.
Following those ideas and adapting them for the *dual-primal* case we introduce in this paper the BETI-DP concept.
The rest of this paper is organized as follows: in the next section we present the BETI-DP formulation. Section 3 is dedicated to the presentation of the preconditioners. In Section 4 a brief analysis of the introduced preconditioners is given. Section 5 presents some numerical results and finally we sketch some conclusions in Section 6.

2 BETI-DP formulation

2.1 Model Problem - Boundary Element Formulation

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with the boundary $\Gamma = \partial\Omega$ which is assumed to be polygonal . We consider the Dirichlet boundary value problem

$$-\mathrm{div}[\alpha(x)\nabla u(x)] = 0 \quad \text{for } \ x \in \Omega, \quad u(x) = g \quad \text{for } \ x \in \Gamma. \tag{1}$$

Let us assume that there is given a nonoverlapping decomposition of Ω satisfying

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i,\ \Omega_i \cap \Omega_j = \emptyset \ \text{ for } \ i \neq j,\ \Gamma_i = \partial\Omega_i,\ \Gamma_{ij} = \Gamma_i \cap \Gamma_j,\ \Gamma_S = \bigcup_{i=1}^{p} \Gamma_i.$$

In what follows we assume that the coefficient function α is piecewise constant, i.e.,

$$\alpha(x) = \alpha_i > 0 \quad \text{for } \ x \in \Omega_i,\ i = 1, \dots, p.$$

Thus, instead of the global boundary value problem (1) we have to solve now the local boundary value problems

$$-\alpha_i \Delta u_i(x) = 0 \quad \text{for } \ x \in \Omega_i, \quad u_i(x) = g(x) \quad \text{for } \ x \in \Gamma_i \cap \Gamma, \tag{2}$$

along with the transmission conditions on the internal coupling boundaries

$$u_i(x) = u_j(x), \quad \alpha_i \frac{\partial}{\partial n_i} u_i(x) + \alpha_j \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } \ x \in \Gamma_{ij}, \tag{3}$$

where n_i is the unit outward normal vector with respect to Γ_i.
The solutions of the local problems (2) can be written by using the representation formulae (see [10])

$$u_i(x) = \int_{\Gamma_i} U^*(x,y) \frac{\partial}{\partial n_i} u_i(y) ds_y - \int_{\Gamma_i} \frac{\partial}{\partial n_{i(y)}} U^*(x,y) u_i(y) ds_y \quad \text{for } \ x \in \Omega_i, \tag{4}$$

where $U^*(x,y)$ is the fundamental solution of the Laplace operator:

$$U^*(x,y) = \; -\tfrac{1}{2\pi}\log(|x-y|). \tag{5}$$

On the boundary Γ_i the solution verifies the Cauchy-Calderon equation

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_i & V_i \\ D_i & \frac{1}{2}I + K_i' \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix} \tag{6}$$

where $t_i = \frac{\partial}{\partial n_i} u_i$ is the normal derivative on Γ_i, and the boundary integral operators are given as, the single layer potential $(V_i t_i)(x)$,the double layer potential $(K_i u_i)(x)$, the adjoint double layer potential $(K_i' t_i)(x)$ and the the hypersingular boundary integral operator $(D_i u_i)(x)$
The properties of all boundary integral operators are wellknown (see for example [10]). In particular, the local single layer potential V_i is positive definite in the two dimensional case when we assume $diam(\Omega_i) < 1$.
From (6) we obtain the local Dirichlet-Neumann map

$$t_i(x) := [D_i + (\frac{1}{2}I + K_i')V_i^{-1}(\frac{1}{2}I + K_i)]u_i(x) =: (S_i u_i)(x) \quad \text{for} \;\; x \in \Gamma_i, \tag{7}$$

where $S_i : H^{1/2}(\Gamma_i) \longrightarrow H^{-1/2}(\Gamma_i)$ denotes the local Steklov-Poincaré operator.

2.2 Tearing

Let us consider the trace space $H^{1/2}(\Gamma_S) := \{u_{|\Gamma_s} : u \in H^1(\Omega)\}$ on the skeleton Γ_S and its subspace

$$H_0^{1/2}(\Gamma_S, \Gamma) := \{v \in H^{1/2}(\Gamma_S) \;:\; v(x) = 0 \quad for\; x \in \Gamma\}.$$

Let $\hat{g} \in H^{1/2}(\Gamma_S)$ be an arbitrary but fixed extension of the given Dirichlet datum $g \in H^{1/2}(\Gamma)$.
Now we consider the transmission conditions of the functions u_i and of the conormal derivate $\alpha_i t_i$ along Γ_i. One possibility is to find a global function $\hat{u} \in H_0^{1/2}(\Gamma_S, \Gamma)$ such that $u_i := \hat{u} + \hat{g}$ on Γ_i and

$$\alpha_i(S_i u_i)(x) + \alpha_j(S_j u_j)(x) = 0 \quad for\; x \in \Gamma_{ij} \tag{8}$$

are satisfied on all local coupling boundaries Γ_{ij}. This leads us to the variational problem: Find $\hat{u} \in H_0^{1/2}(\Gamma_s, \Gamma)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i(S_i\hat{u})(x)v(x)ds_x = -\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i(S_i\hat{g})(x)v(x)ds_x \tag{9}$$

for all $v \in H_0^{1/2}(\Gamma_S, \Gamma)$. Due to the implicit definition of the local Dirichlet-Neumann map (7) it is in general not possible to discretize the variational problem (9) in an exact manner. Thus we have to approximate the local Dirichlet boundary value problems which occur in the definition of the local Dirichlet-Neumann map.

For $v_i \in H^{1/2}(\Gamma_i)$ the application of S_i is given by

$$(S_i v_i)(x) = (D_i v_i)(x) + (\frac{1}{2}I + K_i')w_i(x) \quad \text{for } x \in \Gamma_i, \tag{10}$$

where $w_i \in H^{-1/2}(\Gamma_i)$ is the solution of the variational problem

$$\langle V_i w_i, \tau_i \rangle_{\Gamma_i} = \langle (\frac{1}{2}I + K_i)v_i, \tau_i \rangle_{\Gamma_i} \quad \text{for all } \ \tau_i \in H^{-1/2}(\Gamma_i). \tag{11}$$

Let $Z_{i,h} = \text{span}\{\psi_k^i\}_{k=1,N_i} \subset H^{-1/2}(\Gamma_i)$ be a conformal trial space, for example the space of piecewise constant functions with respect to a local quasi uniform and regular boundary mesh with average mesh size h_i. The Galerkin variational problem of (11) is to find $w_{i,h} \in Z_{i,h}$ such that

$$\langle V_i w_{i,h}, \tau_{i,h} \rangle_{\Gamma_i} = \langle (\frac{1}{2}I + K_i)v_i, \tau_{i,h} \rangle_{\Gamma_i} \quad \text{for all } \ \tau_{i,h} \in Z_{i,h}. \tag{12}$$

Hence we can define an approximate Steklov-Poincaré operator as

$$(\tilde{S}_i v_i)(x) = (D_i v_i)(x) + (\frac{1}{2}I + K_i')w_{i,h}(x) \quad \text{for } x \in \Gamma_i. \tag{13}$$

Now the perturbed variational problem is to find $\hat{u} \in H_0^{1/2}(\Gamma, \Gamma_s)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i (\tilde{S}_i \hat{u})(x) v(x) ds_x + \sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i (\tilde{S}_i \hat{g})(x) v(x) ds_x = 0 \tag{14}$$

for all $v \in H_0^{1/2}(\Gamma_S, \Gamma)$. Let W_h be a boundary element subspace of $H_0^{1/2}(\Gamma, \Gamma_S)$, e.g.

$$W_h = \text{span}\{\varphi_n\}_{n=1}^{M} \subset H_0^{1/2}(\Gamma, \Gamma_S)$$

of piecewise linear basis functions with respect to a quasi uniform and regular mesh with mesh size h_S.

The space $W_{i,h} = \text{span}\{\varphi_n^i\}_{n=1}^{M_i}$ denotes the restriction of W_h to Γ_i. The resulting Galerkin variational formulation of (14) is to find $u_h \in W_h$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i (\tilde{S}_i u_h)(x) v(x) ds_x + \sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i (\tilde{S}_i \hat{g})(x) v(x) ds_x = 0 \tag{15}$$

for all $v \in W_h$. This equation has a unique solution in W_h.

The corresponding algebraic system of the Galerkin variational problem (15) is

$$\sum_{i=1}^{p} \alpha_i A_i^\top \tilde{S}_{i,h} A_i \underline{u} = \sum_{i=1}^{p} A_i^\top \underline{f}_i \tag{16}$$

where we used the isomorphism $\underline{v} \in \mathbf{R}^M \longleftrightarrow v_h = \sum_{j=1}^{M} v_j \varphi_j \in W_h$. A_i denote the connectivity matrices that map the vectors $\underline{v}$ originating from the global discretization on Γ_S into their local components $\underline{v}_i$ corresponding to the local discretization on Γ_i.

In (16) the discrete approximate Steklov-Poincaré operator is

$$\tilde{S}_{i,h} = D_{i,h} + (\frac{1}{2}M_{i,h}^T + K_{i,h}^T)V_{i,h}^{-1}(\frac{1}{2}M_{i,h} + K_{i,h}) \tag{17}$$

with the boundary element matrices

$$\begin{array}{rcl} V_{i,h}[l,k] &=& \langle V_i\psi_k^i, \psi_l^i\rangle_{\Gamma_i}, \\ D_{i,h}[m,n] &=& \langle D_i\varphi_n^i, \varphi_m^i\rangle_{\Gamma_i}, \\ K_{i,h}[m,k] &=& \langle K_i\varphi_k^i, \psi_m^i\rangle_{\Gamma_i}, \\ M_{i,h}[m,k] &=& \langle \varphi_k^i, \psi_m^i\rangle_{\Gamma_i}, \end{array} \tag{18}$$

for $k,l = 1,...,M_i$ and $m,n = 1,...,N_i$. The linear system (16) is equivalent to the solution of the minimisation problem

$$\tilde{J}(\underline{u}) = \min_{\underline{v}\in\mathbf{R}^M} \tilde{J}(\underline{v}) \tag{19}$$

where the linear functional is given by

$$\tilde{J}(\underline{v}) := \sum_{i=1}^{p}\left[\frac{1}{2}(\alpha_i\tilde{S}_{i,h}A_i\underline{v}, A_i\underline{v}) - (f_i, A_i\underline{v})\right]. \tag{20}$$

By introducting the local vectors $\underline{v}_i = A_i\underline{v}$ we obtain

$$\bar{J}(\underline{v}_1,...,\underline{v}_p) := \sum_{i=1}^{p}\left[\frac{1}{2}(\alpha_i\tilde{S}_{i,h}\underline{v}_i, \underline{v}_i) - (\underline{f}_i, \underline{v}_i)\right] \tag{21}$$

to be minimised subject to the continuity constraints across the interface. Let us denote $W = \Pi_{i=1}^p W_i$ and $S = \mathrm{diag}(\alpha_i\tilde{S}_{i,h})_{i=1...p}$. Then we have to find the minimum of

$$J(\underline{w}) := \frac{1}{2}(S\underline{w}, \underline{w}) - (\underline{f}, \underline{w}) \longrightarrow min \quad \text{where } \underline{w} := \begin{bmatrix} \underline{v}_1 \\ \vdots \\ \underline{v}_p \end{bmatrix} \in W \tag{22}$$

subject to same continuity constraints across the interface.

2.3 Interconnecting

It remains to impose the constraints that correspond to the continuity across the interface, i.e. $u_i(x) = u_j(x)$ for $x \in \Gamma_{ij}$.
Every vertex (i.e. the endpoints of each edge of Γ_s) is called *corner point*. The basic idea of BETI-DP is to consider the degrees of freedom corresponding to the corners as global degrees of freedom. Let

$$\underline{u}_c = \begin{pmatrix} u_c^1 \\ \vdots \\ u_c^{M_c} \end{pmatrix} \tag{23}$$

be the vector of the degrees of freedom corresponding to the corner points, where M_c is the total number of the corner points.
Let us consider W^c the space of the corners degrees of freedom.
Let $R_c^i : W^c \longrightarrow W_i$ be the matrix operator between the euclidian spaces W_c and W_i in such a way that $R_c^i \underline{u}_c = \underline{u}_{c,i}$. Subscript "c" designates the degrees of freedom on $\partial\Omega_i$ corresponding to the corners. Subscript "r" designates the degrees of freedom on $\partial\Omega_i$ others than the corners: the remainders.
After reordering he have

$$\underline{u}_i = \begin{bmatrix} \underline{u}_{r,i} \\ \underline{u}_{c,i} \end{bmatrix} \tag{24}$$

Now the continuity conditions have to be enforced only on the remainder degrees of freedom We do that with the help of B_r matrix. The matrix $B_r = [B_r^1, \ldots, B_r^p]$ is constructed with $\{-1, 0, 1\}$ as entries. Each row of the matrix B_r is connected with a pair of matching remainder nodes across the interface. The entries of such a matrix are 1 and -1 for the indices corresponding to the matching nodes and 0 elsewhere. So we have

$$B_r \underline{u}_r = 0 \ ,$$

where

$$\underline{u}_r = \begin{bmatrix} \underline{u}_{r,1} \\ \vdots \\ \underline{u}_{r,p} \end{bmatrix} \tag{25}$$

is the vector of the remainder degrees of freedom.
We have to solve the following constrained minimisation problem :
Find $\underline{u} \in W$ such that :

$$\left\{ \begin{array}{c} J(\underline{u}) = \frac{1}{2}\langle S\underline{u}, \underline{u}\rangle - \langle \underline{\tilde{f}}, \underline{u}\rangle \longrightarrow min \\ B_r \underline{u}_r = 0 \quad \text{and} \quad R_c^i \underline{u}_c = \underline{u}_{c,i} \ \text{ for } i = \overline{1,p} \end{array} \right. . \tag{26}$$

We denote now by $\hat{W}_\Pi$ the subspace of W spanned by the vectors which are 1 in each corner point and 0 in rest. This will be called the *primal space* and is the subspace generated by the degrees of freedom which correspond to the corner points. We denote by $\tilde{W}_\Delta$ the subspace of W generated by the vectors which vanish in all the points corresponding to the corner points.
This is called the *dual space* and is the subspace generated by the degrees of freedom which correspond to the remainder points. By $\tilde{W}_{\Delta,i}$ we denote the subspace generated by the remainder degrees of freedom of Ω_i. Now we introduce the subspace $\tilde{W} = \tilde{W}_\Delta \oplus \hat{W}_\Pi$ which is exactly the space where we are looking for the solution of problem (26) after we impose the corner points constraints. That means we have to solve now :
Find $\underline{u} \in \tilde{W}$ such that :

$$\left\{ \begin{array}{c} J(\underline{u}) = \frac{1}{2}\langle S\underline{u}, \underline{u}\rangle - \langle \underline{\tilde{f}}, \underline{u}\rangle \longrightarrow min \\ B_r \underline{u}_r = 0 \end{array} \right. . \tag{27}$$

The resulting algebraic system after introducing the Lagrange multiplier dual variables looks:

$$\begin{pmatrix} S_{rr}^1 & & & S_{rc}^1 R_c^1 & {B_r^1}^T \\ & \ddots & & \vdots & \vdots \\ & & S_{rr}^p & S_{rc}^p R_c^p & {B_r^p}^T \\ {R_c^1}^T {S_{rc}^1}^T & \dots & {R_c^p}^T {S_{rc}^p}^T & \sum_{s=1}^p {R_c^s}^T S_{cc}^s R_c^s & 0 \\ B_r^1 & \dots & B_r^p & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_{r,1} \\ \vdots \\ \underline{u}_{r,p} \\ \underline{u}_c \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{\tilde{f}}_{r,1} \\ \vdots \\ \underline{\tilde{f}}_{r,p} \\ \sum_{i=1}^p {R_c^i}^T \underline{\tilde{f}}_{c,i} \\ 0 \end{pmatrix}. \tag{28}$$

Or in a compact form :

$$\begin{pmatrix} S_{rr} & S_{rc}R_c & B_r^\top \\ (S_{rc}R_c)^\top & S_c & 0 \\ B_r & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_r \\ \underline{u}_c \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_r \\ \underline{f}_c \\ 0 \end{pmatrix}, \tag{29}$$

where

$$\alpha_i \tilde{S}_{i,h} = \begin{pmatrix} S_{rr}^i & S_{rc}^i \\ {S_{rc}^i}^T & S_{cc}^i \end{pmatrix}, \qquad \underline{f}_i = \begin{pmatrix} \underline{\tilde{f}}_{r,i} \\ \underline{\tilde{f}}_{c,i} \end{pmatrix}. \tag{30}$$

After eliminating the primal variables u_r and u_c the following system has to be solved:

$$F\underline{\lambda} = \underline{g} \tag{31}$$

where

$$F = B_r \tilde{S}^{-1} B_r^T \tag{32}$$

and

$$\tilde{S} = S_{rr} - (S_{rc}R_c)(S_{cc})^{-1}(S_{rc}R_c)^T. \tag{33}$$

F is a symmetric and positive definite (s.p.d.) matrix. Hence we can solve the system $F\underline{\lambda} = \underline{g}$ by using a preconditioned conjugate gradient method.

3 Preconditioning

As in the previous FETI-DP articles [2],[4] we introduce the **Dirichlet Preconditioner**:

$$M = \sum_{j=1}^{p} D_\Delta^j B_r^j S_{rr}^j (B_r^j)^T (D_\Delta^j)^T \tag{34}$$

Using the ideas and the spectral equivalences demonstrated in [6] we introduce the **Hypersingular Preconditioner** :

$$M = \sum_{j=1}^{p} D_\Delta^j B_r^j D_{rr}^j (B_r^j)^T (D_\Delta^j)^T \tag{35}$$

where

$$\alpha_i D_{i,h} = \begin{pmatrix} D_{rr}^i & D_{rc}^i \\ {D_{rc}^i}^T & D_{cc}^i \end{pmatrix} \tag{36}$$

is the boundary element matrix of the discrete local hypersingular boundary integral operator and D_Δ^j are diagonal scaling matrices as defined in [4].

4 Analysis

Let us generate a quasi-regular finite element mesh in every subdomain Ω_i starting from the subdomain boundary mesh. This is always possible since we assumed the boundary subdomain mesh quasi-regular as well with the discretisation parameter h_i. We assume a triangular mesh. Let us denote each subdomain finite element stiffness matrix by $K_{i,h}^{FEM}$. Numbering the unknowns corresponding to the boundary Γ_i first, then $K_{i,h}^{FEM}$ has the following blockstructure:

$$K_{i,h}^{FEM} = \begin{pmatrix} K_{\Gamma\Gamma,i} & K_{\Gamma I,i} \\ K_{I\Gamma,i} & K_{II,i} \end{pmatrix} \tag{37}$$

where the indices Γ and I denote the subdomain boundary and interior unknowns, respectively. The finite element Schur complement matrix arrising from the elimination of interior unknowns can be represented in the following form:

$$S_{i,h}^{FEM} = K_{\Gamma\Gamma,i} - K_{\Gamma I,i} K_{II,i}^{-1} K_{I\Gamma,i} \tag{38}$$

Lemma **1** *The local boundary element Schur complement matrix* $S_{i,h}^{BEM} = \tilde{S}_{i,h}$ *and the local finite element Schur complement matrix* $S_{i,h}^{FEM}$ *are spectrally equivalent to the exact Galerkin matrix* $S_{i,h}$ *of the Steklov-Poincaré operator* S_i *and to the boundary element matrix* $D_{i,h}$ *of the local hypersingular boundary integral operator* D_i*, i.e.,*

$$S_{i,h}^{BEM} \simeq S_{i,h}^{FEM} \simeq S_{i,h} \simeq D_{i,h}$$

for all i=1, ..., p, where $A \simeq B$ *means that the matrices A and B are spectrally equivalent (with spectral constants which are independent of discretization constants).*

It is useful to remark and easy at hand to prove that if we have A and B two s.p.d. spectral equivalent matrices, block partitioned (with the same dimension blocks) with c_1 and c_2 the spectral equivalence constants,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{39}$$

$$c_1(Bv, v) \leq (Av, v) \leq c_2(Bv, v)$$

then $S_{11}^A \sim S_{11}^B$ with the same constants c_1 and c_2, where
$S_{11}^A = A_{11} - A_{12}(A_{22})^{-1} A_{21}$ is the Schur's complement matrix of A that corresponds to A_{11} and analogue we define S_{11}^B .
Now if we take a look to the system (28) and to the original FETI-DP system (13) in Mandel & Tezaur [8] we observe that we have two algebraic systems block constructed with parts from Steklov-Poincaré BEM matrices (BETI-DP system) which is spectral equivalent with discrete Steklov-Poincaré FEM matrix (FETI-DP system). The connectivity matrices and B are identical in both methods.
For the two dimensional elliptic problem solved with FETI-DP method it was proven in [8] that the condition number of the system matrix (32) equiped with the Dirichlet preconditioner has the following upper boundary:

Theorem **1** *For d=2 it holds that:*

$$\frac{\lambda_{max}(MF)}{\lambda_{min}(MF)} \leq C\left(1 + log\frac{H}{h}\right)^2, \tag{40}$$

where C is a constant not dependent of H and h.

Using the above remarks and the spectral equivalence Lemma 1 it is obvious that the condition number of the BETI-DP dual system matrix (32) equiped with Dirichlet as well as with the hypersingular preconditioner has the same upper bound as the condition number of the FETI-DP dual system matrix equiped with Dirichlet preconditioner.

5 Numerical Results

For numerical results let us consider that the squared domain $\Omega = [0, 1]^2$ divided in 9 square subdomains as we can see in Fig.1. For this example we solve the problem

$$-div[\alpha(x)\nabla u(x)] = 0 \quad \text{for} \ \ x \in \Omega, \quad u(x) = g \quad \text{for} \ \ x \in \Gamma, \tag{41}$$

where $g(x) = x_1 + x_2$. We see that we have a floating subdomain and 4 corner points.

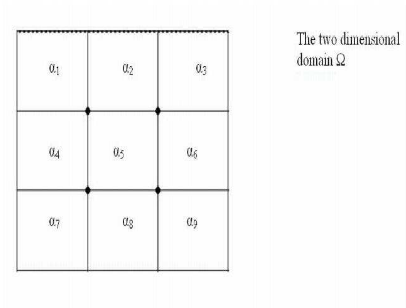

Figure 1: Model problem.

The local single layer potentials V_i, h were inverted by using direct solvers. In the tests, we used the Dirichlet preconditioner (DP) and the hypersingular preconditioner (HP). The bold numbers appearing in the first row of each table represent the number of discretization nodes on the boundary of each subdomain. As a stopping criteria we used $\varepsilon = 10^{-7}$.

No Jumps		**40**	**60**	**80**	**104**
	HP	12	12	13	13
BETI-DP	**DP**	8	8	9	9

Table 1: The number of iterations steps for no jumps, α =1 everywhere.

Small Jumps		40	60	80	104
BETI-DP	SDP	14	15	16	16
	SHP	17	17	18	19

Table 2: The number of iterations steps for small jumps, $\alpha_5 = 10^{-1}$ (in middle) and α_i=1 elsewhere.

Large Jumps		40	60	80	104
BETI-DP	DP	18	18	19	21
	HP	21	21	22	23

Table 3: The number of iterations steps for large jumps, $\alpha_5 = 10^{-3}$ in (in middle) and α_i=1 elsewhere.

6 Conclusions

The benefits of BETI-DP method are mainly the same as for the FETI-DP method, but of course we add those comming from using the boundary element method, namely :

- Parallelizable method.
- We have to deal only with invertible local subproblems.
- No need of characterization of the nullspaces of the local subproblems.
- It uses standard preconditioned conjugate gradient methods instead of projected preconditioned conjugate gradient method like in standard FETI(BETI).
- We may use the sparse representation for the boundary element matrices.
- It has the benefit of using the Hypersingular Preconditioner which involves only matrix by vector multiplication, less expensive than computing of the local solvers for the Dirichlet Preconditioner.
- One level FETI may be seen the light of the dual-primal spaces definitions as a degenerated FETI-DP method with a null primal space.

As maybe the reader already observed from the previous BETI and coupled FETI/BETI papers, the method formulation is strictly analytical involving Dirichlet-Neumann mappings. In the recent works of Mandel and Tezaur [9] the FETI and FETI-DP methods were given a strictly algebraic formulation totally independent of the solved problem. The advantage of BETI(BETI-DP) is that once we have proven the spectral equivalence between FEM and BEM matrices we can transport all the FETI-DP results to BETI-DP. Intuitively this spectral equivalence is valid due to the fact that both FEM and BEM matrices are discretisations of the same operator. This leads us to the further use of coupled FETI/BETI-DP as well as using of the Hypersingular Preconditioner for the FE domains also method already used with good numerical results for one level FETI/BETI as can be seen in [7].

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