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Optimal Control in Pursuit Problems

by
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Abstract

This paper focuses upon the elaboration of a method of optimization in problems of pursuit based on the results of the theory of optimal damping. Thus, we determine the function of control which minimizes the functional representing the index of performance, in the given conditions of differential and algebraic restrictions of the variables of state, respectively of command. We analyze the cases which suppose the linearization of the equations of movement and of the index of performance.

Key words and phrases: optimal pursuit, optimal command relative to the damping of the functional, terminal surface, smooth manifold.

1 Introduction

In space missions, one of the most important objectives is constituted by the pursuit and meeting of spaceships on orbits within terrestrial boundaries. The realization of such operations imposes the minimization of certain characteristic parameters: energy consumption, time, the distance between pursuer and the target.

Thus, in [1], we consider the pursuit in plane supposing the limited angular velocity of the pursuer. In [2] we determine the meeting time of two vehicles in the conditions of certain restrictions imposed upon velocities and in [3] we analyze the trajectories of pursuit in minimal time when the angular velocity and the acceleration are limited and the interception direction is free.

Reference [4] settles the necessary conditions of meeting in the case of constant velocities when the radius of curvature of the trajectory is submitted to a restriction. In [5] we determine the time of pursuit using the auxiliary problem of the distance between the two vehicles at the moment of touching the terminal surface S, which is supposed to be a convex set. It is also considered that the differential of distance as against time does not depend explicitly on the components of acceleration and what is obtained is the domain of the admissible states for the problem of optimum.

Reference [6] presents the pursuit on Lagrangian orbits with constraints on the components of acceleration and [7] determines, within the same hypotheses, the commands which minimize the time of pursuit using the conditions at the limit on the terminal surface.

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The passive optimal pursuit is dealt with in [8] and supposes a distance imposed on the terminal surface between pursuer and the target with a minimum consumption of fuel.
The class of the above-mentioned problems has been solved by using the principle of the Maximum [9].
A different method of solving the problems of pursuit is that of the differential games [10].
We have used elements of functional analysis [11] for the linearized model in problems of pursuit. This supposes minimizing the norm within the Hilbert space as long as the commanded dynamic system which describes the movement on a circular orbit of the pursuer as against the target is linear, invariant in time and controllable. Determining the optimal command within the Hilbert space $L^2[t_0, t_f]$, regarding minimizing the energy necessary to the orbital meeting demonstrates the existence and uniqueness of the solution for the stated problem of optimization. In what follows, we shall consider minimizing the energy consumption on the trajectories of pursuit within a given time $[0, T]$.
By using results of the theory of optimal damping for transitory answers [12], this paper develops a new method regarding optimizing the trajectories of pursuit. The objective is determining the control function and the trajectory of the system with stated initial data, which minimizes the J functional representing the cost index.

2 The Optimal Damping in a Transitory Regime

Let us consider the commanded system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \qquad \mathbf{x}(t_0) = \mathbf{x}^0, \tag{1}$$

where the variables of state $\mathbf{x} = (x_1, \ldots, x_n)^*$ and the command ones $\mathbf{u} = (u_1 \ldots, u_r)^*$ are submitted to the constraints

$$g_i(\mathbf{x}, \mathbf{u}, t) \leq 0, \qquad i = 1, \ldots, k. \tag{2}$$

The admissible commands $\mathbf{u}(t) \in \Omega$ are piecewise continuous in time vectorial functions to which the movements $\mathbf{x} = \mathbf{x}(t, \mathbf{u}, \mathbf{x}^0, t_0)$ correspond, in such a way that $\mathbf{x}$ and $\mathbf{u}$ should satisfy (1) and (2).
Let the final surface S be defined by:

$$S(t, x_1, \ldots, x_n) = 0. \tag{3}$$

The function $V(t, x_1, \ldots, x_n)$ represents the distance of the point in movement $x(t, \mathbf{u}, \mathbf{x}^0, t_0)$ of surface S.
Our final aim is that the solution of the commanded system (1) submitted to the restriction (2) should determine the minimizing of this distance.
If $f_0(\mathbf{x}, \mathbf{u}, t)$ is a function with the same properties as the second member of system (1), then the functional

$$L = V(\mathbf{x}(t, \mathbf{u}), t) + \int_{t_0}^{t} f_0(\mathbf{x}, \mathbf{u}, t) dt \tag{4}$$

is defined along the trajectories and commands of system (1), and for the given t, we obtain the value of the functional.
Determining a command for which functional (4) decreases as soon as possible from the point $(\mathbf{x}^0, t_0)$ of the associated curve imposes that dL/dt should be minimum.
Let us suppose that $\mathbf{u}^0(t)$ and the associated movement $\mathbf{x}^0(t)$ are optimal as regards the damping of the L functional. What results is that the decreasing velocity at each point of the optimal curve is maximal, expressed by

$$\frac{dV}{dt} + f_0(\mathbf{x}, \mathbf{u}^0, t) = \inf_{\mathbf{u}\in\Omega} \left(\frac{dV}{dt} + f_0(\mathbf{x}, \mathbf{u}, t) \right). \tag{5}$$

2.1 Formulating the Problem of Optimum

Let us determine the command $\mathbf{u}^0(t)$ and the associated movement $\mathbf{x}^0(t)$ for any $t \in [0, T]$ which satisfy the system (1) and the constraints (2) so that $\mathbf{u}^0(t)$ should minimize the functional

$$J = p(\mathbf{x}(T)) + \int_0^T f_0(\mathbf{x}, \mathbf{u}, t) dt \tag{6}$$

or

$$J(\mathbf{u}) \geq J(\mathbf{u}^0), \qquad \mathbf{u} \in \Omega. \tag{7}$$

For further developments, we shall use the results expressed by Theorems 1 and 2 [12].

Theorem 1 *The command $\mathbf{u}^0 = (u_1^0, \ldots, u_r^0)$ optimal as regards the damping of functional L with the boundary condition.*

$$V(t, \mathbf{x})|_S = p(\mathbf{x}(T)) \tag{8}$$

ensures, for the functional J, the smallest value as against the values obtained under the action of all the other commands which transfer $\mathbf{x}^0$ to $\mathbf{x}(T) \in S$ if $\mathbf{u}^0(t)$ and $\mathbf{x}^0(t)$ verify the equation

$$W = \frac{\partial V}{\partial t} + (\text{grad } V, \mathbf{f}(\mathbf{x}, \mathbf{u}, t)) + f_0 \equiv 0. \tag{9}$$

Theorem 2 *If the conditions are met*
1) *There is a function $V(t_1, \mathbf{x})$ defined for $t \in [0, T]$, $\mathbf{x} \in E_n$ twice continuously derivable in such a way that*

$$V(T, \mathbf{x}(T)) = 0; \tag{10}$$

2) *There is a function $\mathbf{u}^0(t, \mathbf{x})$ continuously derivable in such a way that $\mathbf{u}^0(t, \mathbf{x})$ is an inner point of Ω and*

$$\inf_{\mathbf{u}\in U} W(t, \mathbf{x}, \mathbf{u}) = W\left(t, \mathbf{x}^0(t), \mathbf{u}^0(t, \mathbf{x}^0(t))\right) \equiv 0; \tag{11}$$

3) *The command* $\mathbf{u} = \mathbf{u}^0(t, \mathbf{x})$ *transfers system* (1) *from the initial state to the final state* $\mathbf{x}(T)$ *then what results is that:*
There is an optimal command $\mathbf{u}^0(t)$*, an associated movement* $\mathbf{x}^0(t)$ *and a function* $\boldsymbol{\lambda}_j(t)$*,* $j = 0, 1, \ldots, n$*, in such a way in which it should verify the canonical differential system*

$$\begin{aligned} \dot{x}_j &= \frac{\partial H}{\partial \lambda_j}, \\ \dot{\lambda}_j &= -\frac{\partial H}{\partial x_i}, \end{aligned} \tag{12}$$

where the function

$$H(t, \mathbf{x}(t), \mathbf{u}, \boldsymbol{\lambda}(t)) \quad = \quad \boldsymbol{\lambda}^*(t)\mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{13}$$

meets the relation

$$H(t, \mathbf{x}^0(t), \mathbf{u}^0, \boldsymbol{\lambda}(t)) \quad = \quad \inf_{\mathbf{u} \in U}(t, \mathbf{x}^0(t), \boldsymbol{\lambda}, \mathbf{u}) = -\frac{\partial V(t, \mathbf{x}^0(t))}{\partial t}. \tag{14}$$

2.2 The Final State Situated on a Smooth Manifold

What is required is the minimization of the functional (6), in the hypothesis that the final state $\mathbf{x}(T)$ is situated on the smooth manifold

$$g_j(\mathbf{x}(T)) \quad = \quad 0, \qquad j = 1, 2, \ldots, k. \tag{15}$$

By replacing the condition $V(T, \mathbf{x}(T)) = 0$ of Theorem 2 by the condition

$$\begin{aligned} V(T, \mathbf{x}(T)) &= p(\mathbf{x}(T)), \\ g_j(\mathbf{x}(T)) &= 0 \end{aligned} \tag{16}$$

we shall keep the assertion of Theorem 2, but we only have $n+k$ conditions at the limit at our disposal to determine the functions $\lambda_j(t)$, $j = 0, 1, \ldots, n$, and $x_j^0(t)$, $j = 0, 1 \ldots, n$, taking into account the initial state of the system at $t = 0$ and k conditions for the final state (15).
It can effortlessly be proven that

$$\text{grad } V(T, \mathbf{x}(T)) \quad = \quad -\text{grad } p(\mathbf{x}(T)) \tag{17}$$

is contained in the hyper plane generated of the g_j degree $j = 1, 2, \ldots, k$.
Hence, there exist the constants $\ell_1, \ldots, \ell_k$ in such a way that

$$\text{grad } V \quad = \quad \text{grad } p + \sum_{j=1}^{k} \ell_j \, \text{grad } g_j. \tag{18}$$

Since we have

$$\lambda_j(t) = \frac{\partial V}{\partial x_j} \quad \text{for} \quad \mathbf{x} = \mathbf{x}^0(t), \quad j = 1, \ldots, n, \tag{19}$$

equality (18) becomes

$$\lambda_s(T) = \frac{\partial p(\mathbf{x}(T))}{\partial x_s} + \sum_{j=1}^{k} \ell_j \frac{\partial g_j}{\partial x_s}, \qquad s = 1, \ldots, n, \tag{20}$$

representing the conditions of transversality.
The $2n + k$ unknown constants for the functions $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$, ℓ_j, $j = 1, \ldots, k$, are determined out of the initial condition, the equation of the smooth manifold and the condition of transversality.

3 The Pursuit in Space

In an inertial system with the origin O we shall consider the movement of the centres of the mass points K and Z defined via the vectors of position

$$\begin{aligned} \mathbf{x}_K &= (x_{K_1}, x_{K_2}, x_{K_3}), \\ \mathbf{x}_Z &= (x_{Z_1}, x_{Z_2}, x_{Z_3}), \end{aligned} \tag{21}$$

representing the vehicle which pursues, respectively the target.
Accelerations are exercised on K and Z and their existence is due to the presence of a gravitational field with the potential related to the mass unity $U(x_1, x_2, x_3)$.
The pursuing vehicle K is actioned by an acceleration $\mathbf{u}(u_1, u_2, u_3)$.
The equations of movement of the two material points are given by

$$\ddot{x}_{Z_i} = -\frac{\partial U}{\partial x_i}(x_{Z_1}, x_{Z_2}, x_{Z_3}), \qquad i = 1, 2, 3, \tag{22}$$

and

$$\ddot{x}_{K_i} = -\frac{\partial U}{\partial x_i}(x_{K_1}, x_{K_2}, x_{K_3}) + u_i, \qquad i = 1, 2, 3. \tag{23}$$

We shall suppose that the potential U admits continuous partial differentials of a superior order ≥ 2 in the field of the analyzed movement.
The trajectory $x_Z(t)$ as a solution of the system (22) is the trajectory of reference for the description of the movement of K.
The functions $x_{Z_i}(t)$ are defined for any t and are continuously differentiable.
We shall introduce the coordinates

$$x_i = x_{K_i} - x_{Z_i}, \qquad i = 1, 2, 3. \tag{24}$$

It results

$$\begin{aligned}\ddot{x}_i &= -\frac{\partial U}{\partial x_i}\left[x_{Z_1}+x_1, x_{Z_2}+x_2, x_{Z_3}+x_3\right]+ \\ &+\frac{\partial U}{\partial x_i}\left[x_{Z_1}, x_{Z_2}, x_{Z_3}\right]+u_i, \qquad i=1,2,3.\end{aligned} \tag{25}$$

The differential equations (25) describe the relative movement of K as against Z.
We shall presuppose that the distance between the two mass centres is small enough as against the distance to the reference centre, so that the linear approximation should offer a good approximation.

4 Linear case

4.1 The Equation of Movement

Let us consider the terms of a superior order ≥ 2 negligible, so that we should have

$$\begin{aligned}\frac{\partial U}{\partial x_i}\left[x_{Z_1}(t)+x_1, x_{Z_2}(t)+x_2, x_{Z_3}(t)+x_3\right] &= \frac{\partial U}{\partial x_i}\left[x_{Z_1}(t), x_{Z_2}(t), x_{Z_3}(t)\right]+ \\ &+\sum_{j=1}^{3}\frac{\partial^2 U}{\partial x_i \partial x_j}\left[x_{Z_1}(t), x_{Z_2}(t), x_{Z_3}(t)\right].\end{aligned} \tag{26}$$

By introducing the state vector

$$\mathbf{x} = (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) \tag{27}$$

we shall obtain

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}+\mathbf{Q}\mathbf{u}, \tag{28}$$

where

$$\mathbf{P}(t) = \begin{pmatrix}\mathbf{O}_3 & \vdots & \mathbf{E}_3 \\ \dots & \vdots & \dots \\ \mathbf{A}(t) & \vdots & \mathbf{O}_3\end{pmatrix}, \qquad \mathbf{Q}=\begin{pmatrix}\mathbf{O}_3 \\ \dots \\ \mathbf{E}_3\end{pmatrix}, \tag{29}$$

where $\mathbf{O}_3$ – the null matrix of order 3, $\mathbf{E}_3$ – the unity matrix of order 3, $\mathbf{A}(t)$ – the symmetrical matrix of order 3

$$\mathbf{A}(t)=\mathbf{A}^*=(a_{ij})$$

in which

$$a_{ij} = -\frac{\partial^2 U}{\partial x_i \partial x_j}\left[x_{Z_1}(t), x_{Z_2}(t), x_{Z_3}(t)\right]. \tag{30}$$

4.2 The Expression of Energy

The energy between the pursuer and the target is given by

$$E = \sum_{j=1}^{3} \frac{\dot{x}_{Z_j}^2}{2} - \sum_{j=1}^{3} \frac{\dot{x}_{K_j}^2}{2} + U\left(x_{Z_1}, x_{Z_2}, x_{Z_3}\right) - U\left(x_{K_1}, x_{K_2}, x_{K_3}\right). \tag{31}$$

By developing the calculus and only keeping the linear component, we shall obtain

$$E_1 = -\sum_{j=1}^{3} \dot{x}_{Z_j}(t) x_j - \sum_{j=1}^{3} \frac{\partial U}{\partial x_j}\left[x_{Z_1}(t), x_{Z_2}(t), x_{Z_3}(t)\right] x_j. \tag{32}$$

By derivation, we have

$$\begin{aligned} \frac{dE_1}{dt} &= -\sum_{j=1}^{3} \left[\left(\ddot{x}_{Z_j} + \frac{\partial U}{\partial x_j}\left(x_{Z_i}\right)\right) \dot{x}_j + \frac{\partial^2 U}{\partial x_i \partial x_j}\left(x_{Z_i}\right) \dot{x}_{z_i} x_j + \dot{x}_{z_j} \ddot{x}_j\right] \\ &= -\sum_{j=1}^{3} \left[\frac{\partial^2 U}{\partial x_i \partial x_j}\left(x_{Z_i}\right) x_j + \ddot{x}_j\right] \dot{x}_{Z_j} = -\sum_{j=1}^{3} \dot{x}_{Z_j}(t) u_i \end{aligned} \tag{33}$$

so that the expression of energy is written

$$E = -\int_{t_0}^{t_1} \sum_{j=1}^{3} \dot{x}_{Z_j}(t) u_i dt = \int_{t_0}^{t_1} \mathbf{B}^*(t) \mathbf{u} dt, \tag{34}$$

where the sign * represents the transposed, in which we marked

$$\mathbf{B}^*(t) = \left(-\dot{x}_{Z_1}(t), -\dot{x}_{Z_2}(t), -\dot{x}_{Z_3}(t)\right) \tag{35}$$

and

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{36}$$

4.3 The Problem of Minimizing the Energy

Let us consider the commanded system of the equations of movement

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x} + \mathbf{Q}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}^0 \tag{37}$$

and the functional

$$J = \int_0^T \mathbf{B}^*(t) \mathbf{u} dt + \mathbf{C}^* \mathbf{x}^1, \tag{38}$$

where

$$C^* \quad = \quad (1,1,1,0,0,0). \tag{39}$$

What is requested is determining the optimal command $\mathbf{u}^0(t)$ relative to the functional (38), which transfers the system (37) from the given state $\mathbf{x} = \mathbf{x}^0$ into $\mathbf{x}^1 = \mathbf{x}(T)$ with the restriction on the command defined by

$$|u_i| \quad \leq \quad 1, \qquad i = 1,2,3. \tag{40}$$

The elements of the matrix $\mathbf{P}(t)$ and the elements of the vector $\mathbf{B}(t)$ are assumed to be piecewise continuous in time for $t \in [0,T]$.
Let us take

$$V(t,\mathbf{x}) \quad = \quad \boldsymbol{\lambda}^*(t)\mathbf{x} + \varphi(t) \tag{41}$$

and

$$z \quad = \quad V + \int_{t_0}^{t} \mathbf{B}^*(t)\mathbf{u}dt. \tag{42}$$

By solving the differential of z on the integral curve of the system (37), we shall obtain

$$\dot{z} \quad = \quad W = (\boldsymbol{\lambda}^* + \boldsymbol{\lambda}^*\mathbf{P})\mathbf{x} + \dot{\varphi} + (\boldsymbol{\lambda}^*\mathbf{Q} + \mathbf{B}^*)\mathbf{u}. \tag{43}$$

Function W takes the smallest value in the conditions of the constraint of command (40) if

$$u_j^0(t) \quad = \quad -\operatorname{sgn}\,(\boldsymbol{\lambda}^*\mathbf{Q} + H^*)_j, \qquad j = 1,2,3, \tag{44}$$

where j indicates the components of the vector $(\boldsymbol{\lambda}^*\mathbf{Q} + \mathbf{B}^*)$, which is given by

$$(\boldsymbol{\lambda}^*\mathbf{Q} + \mathbf{B}^*) \quad = \quad \left(\lambda_4(t) - \dot{x}_{Z_1}(t), \lambda_5(t) - \dot{x}_{Z_2}(t), \lambda_6(t) - \dot{x}_{Z_3}(t)\right). \tag{45}$$

Let us choose φ and $\boldsymbol{\lambda}$ in such that

$$W(t,\mathbf{x},\mathbf{u}^0) \quad = \quad \inf_{\mathbf{u}\in U} W(t,\mathbf{x},\mathbf{u}) = 0. \tag{46}$$

By explicating (46), we obtain

$$\dot{\boldsymbol{\lambda}}^* + \boldsymbol{\lambda}^*\mathbf{P} \quad = \quad 0, \tag{47}$$

and

$$\dot{\varphi} - \sum_{j=1}^{3} (\boldsymbol{\lambda}^*\mathbf{Q} + \mathbf{B}^*)_j \operatorname{sgn}\,(\boldsymbol{\lambda}^*\mathbf{Q} + \mathbf{B}^*)_j \quad = \quad 0, \tag{48}$$

whence

$$\dot{\varphi} - \sum_{k=1}^{3} |\lambda_{k+3}(t) - x_{Z_k}| \quad = \quad 0. \tag{49}$$

By imposing the boundary condition

$$\varphi(T) \quad = \quad 0 \tag{50}$$

it results that

$$\varphi(t) \quad = \quad \int_T^t \sum_{k=1}^{3} |\lambda_{k+3}(t) - \dot{x}_{Z_k}(t)|\, dt. \tag{51}$$

In the problem of optimum that we have formulated, we shall analyze the following cases:
1 – The final state is zero and, hence, $V(T, \mathbf{x}^1) = 0$;
2 – The final state is not specified and $V(T, \mathbf{x}^1) = p(\mathbf{x}^1)$;
3 – The final state is found on a smooth manifold $g_j(T, \mathbf{x}^1) = 0$, $j = 1, 2, 3$, $V(T, \mathbf{x}^1) = p(\mathbf{x}^1)$.
Although the presented theory is generally valid for the three cases we have formulated, the modification of the boundary conditions determine different solutions.

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