

*Dedicated to Dr. TOADER MOROZAN
on the occasion of his 70th birthday*

DYNAMICAL SYSTEMS MODELLING COMPUTATIONAL CIRCUITS

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Two ODE systems describing Winner Take All (WTA) computational electronic circuits with local and global feedback are considered. The results obtained show that the dynamic solutions converge toward appropriate equilibria which are locally and respectively globally asymptotically stable.

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1. INTRODUCTION

We are concerned with ODE systems (1) and (2) below:

$$(1) \quad \begin{cases} a\dot{x}_i = -bx_i - p \sum_{k=1}^n g(\lambda x_k) + d_i - p(n-1), \\ x_i(0) = 0, \quad i \in \overline{1, n}, \end{cases}$$

$$(2) \quad \begin{cases} \alpha\dot{y}_i = -by_i + bu + d_i, \\ a\dot{u} = -bu - p \sum_{k=1}^n g(\lambda y_k) - p(n-1), \\ y_i(0) = u(0) = 0, \quad i \in \overline{1, n}. \end{cases}$$

Here $n \in \mathbb{N}$, a, b, p, α are given positive parameters and g is an increasing function from $\mathbb{R}$ to $(-1, 1)$ controlled by the positive parameter λ .

The d_i , $i \in \overline{1, n}$ are the input of (1) and (2). Suppose that they are distinct and ordered as

$$(3) \quad d_{\sigma(1)} > d_{\sigma(2)} > \cdots > d_{\sigma(n)},$$

where $\sigma(\cdot)$ is an index permutation.

Our paper shows that if we know the density of the d_i , then we can properly choose positive numbers δ and λ such that the stationary states $\bar{x}$ of (1) and $\bar{y}$ of (2) fulfill

$$(4) \quad \bar{x}_{\sigma(1)} \geq \delta \geq -\delta \geq \bar{x}_{\sigma(2)} > \bar{x}_{\sigma(3)} > \cdots > \bar{x}_{\sigma(n)}$$

and similarly for $\bar{y}$. The fact that the $\sigma(1)$ -th output $\bar{x}_{\sigma(1)}$ (or $\bar{y}_{\sigma(1)}$ for (2)) is placed above the threshold δ , tells that the corresponding $\sigma(1)$ -th input $d_{\sigma(1)}$ is the maximum element in (3). We say we have built an “Winner Take All” system - WTA for short.

We show that (1) and (2) evolve towards stable equilibria $\bar{x}$ and $\bar{y}$, namely, $\bar{x}$ is locally while $\bar{y}$ is globally asymptotically stable.

Let us now see the engineering relevance of (1) and (2). (1) describes an electronic circuit of n amplifiers relating their x_i input with their $g(\lambda x_i)$ output, where λ is “the gain”. The amplifiers are interconnected by n^2 elements. The circuit is of “neural Hopfield type” ([1]–[9], [14]) and can perform the WTA computational task on list (3) which is introduced by current sources. The computing time is short and this is why this type of circuits competes well with their digital counterparts. Still, they encounter the problem of a huge number of interconnections causing interference malfunctions. This is why a better solution is to replace the “local” feedback, used in the above described circuit, by a “global” one ([10], [11]) described by (2). This means that the amplifier (negative) outputs are gathered together, then amplified, and then split towards each input. Thus, the number of interconnections decreases from n^2 to $2n$. Our paper shows that the WTA task is still carried out if the parameters are properly chosen. Moreover, the stability performances are enhanced: instead of a local asymptotic one we find a global stability property.

2. PRELIMINARIES

If $\bar{x} \in \mathbb{R}^n$ is the stationary solution of (1) and $(\bar{y}, \bar{u}) \in \mathbb{R}^{n+1}$ that of (2), then we see that $\bar{x}$ and $\bar{y}$ satisfy the same equation

$$(5) \quad \begin{cases} 0 = -bz_i - p \sum_{k=1}^n g(\lambda z_k) + d_i - p(n-1), \\ i \in \overline{1, n}. \end{cases}$$

Throughout, about the function $g : \mathbb{R} \rightarrow (-1, 1)$ we suppose that

$$(H1) \quad \begin{cases} g \in C^1, \ g'(x) > 0 \text{ for all } x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} g(x) = \pm 1 \text{ where } \pm \text{ are in correspondence,} \\ \lim_{x \rightarrow \infty} xg'(x) = 0. \end{cases}$$

As in [7]–[10] we can readily see the following basic facts.

THEOREM 2.1. a) (1) and (2) have unique, globally defined, bounded, C^1 solutions, $x : [0, \infty) \rightarrow \mathbb{R}^n$, respectively, $(y, u) : [0, \infty) \rightarrow \mathbb{R}^{n+1}$.

b) There exists at least a solution $z \in \mathbb{R}^n$ of (5), i.e., (1) and (2) have at least an equilibrium.

c) For each $d \in \mathbb{R}^n \setminus E$, where E has zero measure, the number of equilibria is finite.

3. THE WTA PROPERTY

The list of inputs d_i , ordered as in (3), will be restricted by

$$(H2) \quad \begin{cases} d_i \in [0, d_m], & i \in \overline{1, n}, \\ |d_i - d_j| \geq \Delta, & i, j \in \overline{1, n}, i \neq j. \end{cases}$$

Here $\Delta = \frac{\rho d_m}{n-1}$, where $\rho \in [0, 1]$ is the “list density”. (H2) immediately yields

$$(6) \quad \underline{d}_i \leq d_i \leq \overline{d}_i, \quad i \in \overline{1, n},$$

where $\underline{d}_i = (n-1)\Delta$, $\overline{d}_i = d_m - (i-1)\Delta$. Put $d_M = p / \left(1 - \frac{\rho}{2(n-1)}\right)$, $\bar{\delta} = \frac{\Delta}{2b}$ and $\lambda_1 = \frac{1}{\delta} g^{-1} \left(1 - \frac{1}{n}\right)$. With this notation we can state

THEOREM 3.1. If (H1) and (H2) hold, then for any $d_m \in (0, d_M)$, $\delta \in (0, \bar{\delta})$ and $\lambda \geq \lambda_1$, the stationary solutions of (1) and (2) have property (4).

As it is sufficient to work on the solution z of (5), let us first observe that

$$(7) \quad z_i - z_j = (d_i - d_j) / b.$$

This means that (5) preserves the order of (3), that is,

$$(8) \quad z_{\sigma(1)} > z_{\sigma(2)} > \cdots > z_{\sigma(n)}.$$

For the proof of the above theorem we need some preparatory results. Everywhere, the hypotheses of Theorem 3.1 hold. The proofs are only sketched.

LEMMA 3.1. Suppose there exists $s \in \overline{1, n}$ such that $z_{\sigma(1)} > z_{\sigma(2)} > \cdots > z_{\sigma(s)} \geq \delta$. Then $s = 1$.

Proof. Let us set $\varepsilon = 1 - g(\lambda\delta)$. Then $\lambda \geq \lambda_1$ gives $\varepsilon < \frac{1}{n}$. From the $\sigma(s)$ -th equation in (5) we easily get $0 \leq -b\delta - p(2s-1) + pn\varepsilon + d_{\sigma(s)}$.

If we had $s > 1$ (i.e., $s \geq 2$) this would imply $d_{\sigma(s)} < d_{\sigma(2)}$ and $0 < -b\delta - 3p + pn\varepsilon + d_{\sigma(2)}$ contradicting $d_{\sigma(2)} < \overline{d}_{\sigma(2)}$ of (6) and the hypotheses. $\square$

LEMMA 3.2. If $r \in \overline{1, n}$ and $-\delta \geq z_{\sigma(r)} > z_{\sigma(r+1)} > \cdots > z_{\sigma(n)}$, then $r \geq 2$.

Proof. Suppose $r = 1$ and consider the $\sigma(1)$ -th equation in (5). We get $0 \geq b\delta + p - \varepsilon pn + d_{\sigma(1)}$, contradicting again (6) and the hypotheses. $\square$

LEMMA 3.3. *For any $i \in \overline{1, n-1}$, $z_{\sigma(i)}$ and $z_{\sigma(i+1)}$ are not together inside $(-\delta, \delta)$.*

Proof. This is because, if they were, then $z_{\sigma(i)} - z_{\sigma(i+1)} < 2\delta$ and from (7) and hypotheses we would have $\rho d_m / ((n-1)b) < 2\delta$, contradicting $\delta \leq \bar{\delta}$. $\square$

Proof of Theorem 3.1. By Lemmas 3.1, 3.2 and 3.3, there is $s \in \overline{1, n-1}$ such that

$$(9) \quad z_{\sigma(1)} > z_{\sigma(2)} > \cdots > z_{\sigma(s)} > \delta > -\delta > z_{\sigma(s+1)} > \cdots > z_{\sigma(n)},$$

or

$$(10) \quad z_{\sigma(1)} > z_{\sigma(2)} > \cdots > z_{\sigma(s)} > \delta > z_{\sigma(s+1)} > -\delta > z_{\sigma(s+2)} > \cdots > z_{\sigma(n)}.$$

Again by Lemmas 3.1 and 3.2, we get $s = 1$ in both cases. To get rid of (10), we have to show that $z_{\sigma(2)} < -\delta$. But from (5) we easily get $z_{\sigma(2)}b < -p + d_{\sigma(2)}$. It would be sufficient to have $\overline{d_{\sigma(2)}} < p - \delta b$, which holds by hypothesis. Thus, (9) is the only valid case, which ends the proof. $\square$

4. CONVERGENCE AND STABILITY

We explore here the dynamic properties of (1) and (2). Under (H1), we will denote by λ_2 a positive number such that for each $\lambda \geq \lambda_2$ and $|x| \geq \delta$ we have $\lambda g'(\lambda x) < b/(p(n-1))$.

THEOREM 4.1. *If the assumptions of Theorem (3.1) hold and, in addition, $\lambda \geq \lambda_2$ then the solution of (1) converges to an asymptotically stable stationary solution.*

Proof. Let us denote $w_i = g(\lambda x_i)$ and take $E(w) = \frac{1}{2}\langle pTw, w \rangle - \langle q, w \rangle + \frac{1}{\lambda} \sum_{i=1}^n \int_0^{w_i} g^{-1}(x) dx$, where T is the $n \times n$ matrix with all entries equal to 1, q is the vector with $q_i = d_i - p(n-1)$, and $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$. We have $\frac{\partial E}{\partial x_i} = \lambda g'(\lambda x_i) [bx_i + p \sum_{k=1}^n g(\lambda x_k) - q_i]$. Thus, along the solution of (1) we have $\frac{d}{dt}E(x(t)) = \sum_{i=1}^n \frac{\partial E}{\partial x_i} \frac{dx_i}{dt} = - \sum_{i=1}^n \frac{\lambda g'(\lambda x_i)}{a} \left(\frac{dx_i}{dt} \right)^2 < 0$. Thus, $x(t) \rightarrow \bar{x}$ (see [15] for instance).

Next, we take $J = -bI - \lambda pTG'$ the jacobian matrix of (1) computed at $\bar{x}$. Here I is the identity matrix and $G' = \text{diag}(g'(\lambda \bar{x}_i))$. As T is symmetric and G' positive definite, the eigenvalues – say μ – of $-TG'$ are real. We have $\mu \leq (n-1) \max \{g'(\lambda \bar{x}_i); i \in \overline{1, n}\}$. Therefore, the eigenvalues $-b + \lambda p\mu$ of J computed at $\bar{x}_i$, where $|\bar{x}_i| > \delta$ (see Theorem 3.1), are negative for $\lambda \geq \lambda_2$. $\square$

THEOREM 4.2. *Suppose (H1) holds. Then (2) has a unique equilibrium $(\bar{y}, \bar{u})$ which is globally asymptotically stable. In particular, any trajectory tends to it, $(y(t), u(t)) \rightarrow (\bar{y}, \bar{u})$ as $t \rightarrow \infty$.*

Proof. Let us transform system (2) in a sourceless one. Put $y(t) = \tilde{y}(t) + \bar{y}$ and $u(t) = \tilde{u}(t) + \bar{u}$, where $(\bar{y}, \bar{u})$ is the stationary state and $(\tilde{y}, \tilde{u})$ the “perturbation”. We get

$$(11) \quad \begin{cases} \alpha \frac{d\tilde{y}}{dt} = -b\tilde{y} + b\tilde{u}, \\ a \frac{d\tilde{u}}{dt} = -b\tilde{u} - p \sum_{k=1}^n G(\tilde{y}_k), \end{cases}$$

where $G(\tilde{y}_k) = g(\lambda y_k) - g(\lambda \bar{y}_k)$.

It is now sufficient to study the dynamics of $0 \in \mathbb{R}^{n+1}$, an equilibrium of (11). Let $\mathbf{x} = (x_1, \dots, x_n)$ and take $L : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ defined by $L(\mathbf{x}, x_{n+1}) = \sum_{i=1}^n \int_0^{x_i} G(x) dx + \frac{b}{2\alpha p} x_{n+1}^2$ and compute its time derivative along the solution of (11). We get $\frac{d}{dt}L(\tilde{y}(t), \tilde{u}(t)) = -\frac{b^2}{a\alpha p} \tilde{u}^2 - \frac{b}{\alpha} \sum_{i=1}^n \tilde{y}_i G(\tilde{y}_i) < 0$ by (H1). On the other hand, we can easily show that $L(x, x_{n+1}) > 0$ for all $(x, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$; $L(0, 0) = 0$ and $\lim_{x \rightarrow \pm\infty} \int_0^x G(z) dz = \pm\infty$, where $\pm$ are in correspondence. Hence $\|(x, x_{n+1})\| \rightarrow \infty$ implies $L(x, x_{n+1}) \rightarrow +\infty$, i.e., L is “radially unbounded”. By the Liapunov theory we have global asymptotic stability. $\square$

5. COMMENTS

We have considered two differential systems modeling two variants of WTA electronic circuits of neural type. The first one has n amplifiers with a (negative) local feedback of n^2 elements. The second one has $n+1$ amplifiers and $2n$ interconnections grouped as a global feedback. The reduction of interconnections from n^2 to $2n$ is a major achievement, subject to the condition that other properties are not worsened. We checked here two of them the WTA main task is still performed and stability preserved. The given parameters of the circuits $(a, b, p, g(x))$ for the first circuit and $a, \alpha, b, p, g(x)$ for the second) and the minimum density ρ of the list to be processed, are the starting quantities. Our theorems give instruments for designing:

- A computable d_M , the maximum of the list elements. In other words, our circuit must be fed by successive lists scaled to $[0, d_M]$. We emphasize that the density of list elements is not at all restricted.
- A computable lower bound of λ , the minimum gain of amplifiers; our circuits work with high gain.

- A computable δ , the threshold of WTA, useful for adjusting the moments of switching of our machine.

In addition, and very encouraging, the global feedback circuit shows an enhancement of stability from local to a global asymptotic one.

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