AXIAL COUETTE FLOW OF AN OLDROYD-B FLUID DUE TO A TIME-DEPENDENT SHEAR STRESS

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The velocity field and the shear stress corresponding to the motion of an Oldroyd-B fluid due to an infinite circular cylinder subject to a longitudinal time-dependent shear stress are established by means of Hankel transforms. The exact solutions, presented under series form, can be easy specialized to give the similar solutions for Maxwell, second grade and Newtonian fluids performing the same motion. Finally, some characteristics of the motion as well as the influence of the material parameters on the behavior of the fluid are shown by graphical illustrations.

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1. INTRODUCTION

Navier-Stokes equations can well describe the flow of Newtonian fluids. However, there are many fluids with complex microstructure, such as biological fluids, polymeric liquids, suspensions, liquid crystals in industrial processes, with non-linear viscoelastic behaviuor that cannot be described by these equations. Recently, the interest in motion problems of non-Newtonian fluids has considerably grown due to their multiple applications. Among the many models that have been used to describe their behaviour, the rate type models have received much attention. The first systematic thermodynamic study of these models is that of Rajagopal and Srinivasa [15], within which models for a variety of rate type viscoelastic fluids can be obtained. Among them the Oldroyd-B model seems to be more amenable to analysis and more importantly experimental. As a result of their wide implications, a lot of papers regarding these fluids have been recently published [2, 5–13, 17, 20].

However, it is worth pointing out that in all these papers the authors studied motions for which the velocity field is given on the boundary. The first exact solutions corresponding to motions of non-Newtonian fluids for which the shear stress is given on the boundary seem to be those of Bandelli and Rajagopal [3] for cylindrical domains. Recently, new exact solutions for similar problems have been established in [1] and [14] for Newtonian and second grade fluids and in [18] and [19] for Oldroyd-B fluids with fractional derivatives. However, the results of the last two mentioned papers have a meaningful drawback. The authors claim to study motions due to longitudinal or rotational constant shears on the boundary. Unfortunately, their boundary conditions (28) from [18] and (31) from [19] correspond to problems with time-dependent shear stresses on the boundary.

The main purpose of our paper is to determine the velocity field and the shear stress corresponding to the unsteady motion of an Olroyd-B fluid produced by an infinite circular cylinder subject to a longitudinal time-dependent shear stress. The general solutions, obtained by means of Hankel transforms and presented in series form in terms of Bessel functions $J_0(\cdot)$, $J_1(\cdot)$ and $J_2(\cdot)$, satisfy both the governing equations and all imposed initial and boundary conditions. Moreover, they can be easily specialized to give the similar solutions for Maxwell, second grade and Newtonian fluids performing the same motion.

2. GOVERNING EQUATIONS

The Cauchy stress $\mathbf{T}$ in an incompressible Oldroyd-B fluid is given [2, 5–13, 15, 17, 20] by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^{\mathrm{T}})], \tag{2.1}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ the first Rivlin Ericksen tensor, μ the dynamic viscosity of the fluid, λ and λ_r are relaxation and retardation times, the superscript T indicates the transpose operation and the superposed dot indicates the material time derivative. The model characterized by the constitutive equations (2.1) contains as special cases the upper-convected Maxwell model for $\lambda_r = 0$ and the Newtonian fluid model for $\lambda_r = \lambda = 0$. In some special flows, as those to be considered here, the governing equations for an Oldroyd-B fluid resemble those for a second grade fluid. For the problem under consideration we shall assume a velocity field and an extra-stress of the form

$$\mathbf{V} = \mathbf{V}(r,t) = v(r,t)\mathbf{e}_z, \quad \mathbf{S} = \mathbf{S}(r,t), \tag{2.2}$$

where $\mathbf{e}_z$ is the unit vector in the z-direction of the system of cylindrical coordinates r, θ and z. For such flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to time $t = 0$, then

$$\mathbf{V}(r,0) = \mathbf{0}, \quad \mathbf{S}(r,0) = \mathbf{0} \tag{2.3}$$

and Eqs. $(2.1)_2$ and (2.2) imply $S_{rr} = S_{r\theta} = S_{\theta z} = S_{\theta\theta} = 0$.

In the absence of body forces and a pressure gradient in the axial direction, the balance of linear momentum and the constitutive equation $(2.1)_2$

lead to the relevant equations

$$(1+\lambda\partial_t)\tau(r,t)=\mu(1+\lambda_r\partial_t)\partial_r v(r,t),\quad \rho\partial_t v(r,t)=\left(\partial_r+\frac{1}{r}\right)\tau(r,t),\tag{2.4}$$

where ρ is the constant density of the fluid and $\tau = S_{rz}$ is the shear stress that is different of zero.

Eliminating τ between Eqs. (2.4) we obtain the governing equation

$$\lambda\partial_t^2 v(r,t)+\partial_t v(r,t)=(\nu+\alpha\partial_t)\left(\partial_r^2+\frac{1}{r}\partial_r\right)v(r,t),\tag{2.5}$$

where $\alpha = \nu\lambda_r$ and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. The partial differential equation (2.5), with adequate initial and boundary conditions, can be solved in principle by several methods, their effectiveness strictly depending on the domain of definition. In our case the integral transforms technique appears to be a systematic, efficient and powerful tool. The Hankel transform can be used to eliminate the spatial variable Eq. (2.5).

However, the partial differential equation (2.5) is one order higher in t than the similar equation for Newtonian and second grade fluids. Thus, in order to solve a well-posed problem for Oldroyd-B or Maxwell fluids one has to require an additional initial condition, apart from the requirement that the fluid is initially at rest. As early as 1966, Srivastava [16] solved a similar problem for fluids of Maxwell type and assumed that the time derivative of velocity is zero at time $t = 0$. Recently, Hayat et al [11], Tan and Masuoka [17] and Aksel et al. [2] have solved unsteady problems for Oldroyd-B fluids using the initial conditions $v = \partial_t v = 0$ at $t = 0$, although the second condition seems to have no physical significance. It has been adopted for mathematical convenience. For all that, for the comparison of the behaviour of some flows for various fluid models the adoption of such a condition does not detract from the overall conclusions regarding the behaviour differences.

3. AXIAL COUETTE FLOW THROUGH AN INFINITE CIRCULAR CYLINDER

Let us consider an incompressible Oldroyd-B fluid at rest in an infinite circular cylinder of radius R. After the initial moment the cylinder is pulled by a time-dependent shear stress along its axis

$$\tau(R,t)=f[t-\lambda(1-\mathrm{e}^{-\frac{t}{\lambda}})],\quad t>0,\tag{3.1}$$

where f is a constant. Due to the shear, the fluid is gradually displaced, its velocity being of the form (2.2). The governing equation is Eq. (2.5) while the appropriate initial and boundary conditions are

$$v(r,0)=\partial_t v(r,0)=0\quad \text{for } r\in[0,R),\tag{3.2}$$

respectively,

$$(1+\lambda\partial_t)\tau(R,t)=\mu(1+\lambda_r\partial_t)\partial_r v(R,t)=ft,\quad t\geq 0. \tag{3.3}$$

In order to determine the velocity field, let us denote by

$$v_{nH}(t)=\int_0^R rv(r,t)J_0(rr_n)\mathrm{d}r,\quad n=1,2,3,\ldots \tag{3.4}$$

the finite Hankel transform of $v(r,t)$, where r_n are the positive roots of the transcendental equation $J_1(Rr)=0$, and use the relation [4, Eq. (13.4.31)]

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t)J_0(rr_n)\mathrm{d}r=RJ_0(Rr_n)\frac{\partial v(R,t)}{\partial r}-r_n^2 v_{nH}(t). \tag{3.5}$$

Multiplying Eq. (2.5) by $rJ_0(rr_n)$, integrating the result with respect to r from 0 to R and using the boundary condition (3.3) and identity (3.5), we find that

$$\lambda\ddot{v}_{nH}(t)+(1+\alpha r_n^2)\dot{v}_{nH}(t)+\nu r_n^2 v_{nH}(t)=\frac{f}{\rho}tRJ_0(Rr_n),\quad t>0. \tag{3.6}$$

It also follows from (3.2) that

$$v_{nH}(0)=\dot{v}_{nH}(0)=0. \tag{3.7}$$

The solution of the linear ordinary differential equation (3.6), with the initial conditions (3.7), is given by

$$v_{nH}(t)=\frac{f}{\mu}\frac{RJ_0(Rr_n)}{r_n^2}\left[t-\frac{\mathrm{e}^{q_{2n}t}-\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}-\frac{1+\alpha r_n^2}{\nu r_n^2}\left(1-\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}\right)\right], \tag{3.8}$$

where $q_{1n},q_{2n}=\frac{-(1+\alpha r_n^2)\pm\sqrt{(1+\alpha r_n^2)^2-4\nu\lambda r_n^2}}{2\lambda}$.

Finally, applying the inverse Hankel transform formula [4, Eq. (13.4.30)] and using the identity

$$r^2=4\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}, \tag{3.9}$$

we find for the velocity field $v(r,t)$ the simple expression

$$v(r,t)=\frac{fr^2}{2\mu R}(t-\lambda_r)-\frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)} \tag{3.10}$$

$$+\frac{2f}{\mu R}\sum_{n=1}^{\infty}\left[\frac{1+\alpha r_n^2}{\nu r_n^2}\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}-\frac{\mathrm{e}^{q_{2n}t}-\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}\right]\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)},$$

or equivalently

$$(3.11)\quad v(r,t)=\frac{fr^2}{2\mu R}(t-\lambda_r)-\frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\left(1-\lambda\frac{q_{1n}^2\mathrm{e}^{q_{2n}t}-q_{2n}^2\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}\right)\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}.$$

Solving Eq. $(2.4)_1$ with respect to $\tau(y,t)$ and taking into account Eq. $(2.3)_2$, we find that

$$(3.12)\qquad \tau(r,t)=\frac{\mu}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}}\int_0^{\mathrm{t}}\mathrm{e}^{\frac{\tau}{\lambda}}(1+\lambda_r\partial_\tau)\partial_r v(r,\tau)\mathrm{d}\tau.$$

Substituting (3.10) into (3.12) and using the identities

$$q_{1n}q_{2n}=\frac{\nu r_n^2}{\lambda},\quad q_{3n}q_{4n}=\frac{\nu(\lambda-\lambda_r)r_n^2}{\lambda^2},\quad q_{1n}q_{3n}=-\nu r_n^2\frac{1+\lambda_r q_{1n}}{\lambda},$$
$$q_{2n}q_{4n}=-\nu r_n^2\frac{1+\lambda_r q_{2n}}{\lambda},\quad q_{1n}q_{4n}=\frac{\nu r_n^2+q_{1n}}{\lambda},\quad q_{2n}q_{3n}=\frac{\nu r_n^2+q_{2n}}{\lambda},$$

where $\lambda q_{3n}=1+\lambda q_{1n}$ and $\lambda q_{4n}=1+\lambda q_{2n}$, after lengthy but straightforward computations, we obtain, for the shear stress the formula

$$\begin{aligned}(3.13)\qquad \tau(r,t)&=\frac{fr}{R}\left[t-\lambda\left(1-\mathrm{e}^{-\frac{t}{\lambda}}\right)\right]+\frac{2f}{\nu R}\left(1-\mathrm{e}^{-\frac{t}{\lambda}}\right)\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3J_0(Rr_n)}\\
&-\frac{2f}{\nu R}\frac{1}{\lambda-\lambda_r}\sum_{n=1}^{\infty}\left[\frac{\mathrm{e}^{q_{2n}t}-\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}-\lambda_r\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}+\lambda_r\mathrm{e}^{-\frac{t}{\lambda}}\right]\frac{(1+\alpha r_n^2)J_1(rr_n)}{r_n^3J_0(Rr_n)}\\
&-\frac{2f}{\nu R}\frac{\lambda}{\lambda-\lambda_r}\sum_{n=1}^{\infty}\left[\frac{q_{4n}\mathrm{e}^{q_{1n}t}-q_{3n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}-\mathrm{e}^{-\frac{t}{\lambda}}\right]\frac{J_1(rr_n)}{r_n^3J_0(Rr_n)}\\
&-\frac{2f}{\nu R}\frac{\lambda_r}{\lambda-\lambda_r}\sum_{n=1}^{\infty}\left[\frac{q_{4n}\mathrm{e}^{q_{1n}t}-q_{3n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}-\mathrm{e}^{-\frac{t}{\lambda}}\right]\frac{(1+\alpha r_n^2)J_1(rr_n)}{r_n^3J_0(Rr_n)}\\
&+\frac{2f}{\nu R}\frac{\lambda_r}{\lambda-\lambda_r}\sum_{n=1}^{\infty}\left[\frac{q_{2n}\mathrm{e}^{q_{2n}t}-q_{1n}\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}+\nu r_n^2\frac{\mathrm{e}^{q_{2n}t}-\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}-\mathrm{e}^{-\frac{t}{\lambda}}\right]\frac{J_1(rr_n)}{r_n^3J_0(Rr_n)}.\end{aligned}$$

Of course, Eq. (3.13) can be further processed to give the simple form

$$(3.14)\quad \tau(r,t)=\frac{fr}{R}\left[t-\lambda\left(1-\mathrm{e}^{-\frac{t}{\lambda}}\right)\right]+\frac{2f}{\nu R}\sum_{n=1}^{\infty}\left[1-\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}\right]\frac{J_1(rr_n)}{r_n^3J_0(Rr_n)}.$$

4. LIMITING CASES

1. Letting $\lambda_r \to 0$, in Eqs. (3.11) and (3.14) we get the solutions

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\left(1 - \lambda\frac{q_{5n}^2 \mathrm{e}^{q_{6n}t} - q_{6n}^2 \mathrm{e}^{q_{5n}t}}{q_{6n} - q_{5n}}\right)\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}, \tag{4.1}$$

$$\tau(r,t) = \frac{fr}{R}\left[t - \lambda\left(1 - \mathrm{e}^{-\frac{t}{\lambda}}\right)\right] + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\left[1 - \frac{q_{6n}\mathrm{e}^{q_{5n}t} - q_{5n}\mathrm{e}^{q_{6n}t}}{q_{6n} - q_{5n}}\right]\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}, \tag{4.2}$$

corresponding to a Maxwell fluid performing the same motion. In the above relations,

$$q_{5n}, q_{6n} = \frac{-1 \pm \sqrt{1 - 4\nu\lambda r_n^2}}{2\lambda}, \quad q_{7n} = 1 + \lambda q_{5n} \quad \text{and} \quad q_{8n} = 1 + \lambda q_{6n}.$$

2. Letting $\lambda \to 0$ in Eqs. (3.11) and (3.14), the similar solutions (cf. [14, Eqs. (30) and (34)])

$$v(r,t) = \frac{fr^2}{2\mu R}(t - \lambda_r) - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\left[1 - (1 + \alpha r_n^2)\exp\left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2}\right)\right]\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}, \tag{4.3}$$

$$\tau(r,t) = \frac{fr}{R}t + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\left[1 - \exp\left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2}\right)\right]\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}, \tag{4.4}$$

corresponding to a second grade fluid are recovered.

3. Finally, letting $\lambda \to 0$ in Eqs. (4.1) and (4.2) or $\lambda_r \to 0$ in (4.3) and (4.4), the solutions

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\left(1 - \mathrm{e}^{-\nu r_n^2 t}\right)\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}, \tag{4.5}$$

$$\tau(r,t) = \frac{fr}{R}t + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\left(1 - \mathrm{e}^{-\nu r_n^2 t}\right)\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}, \tag{4.6}$$

for a Newtonian fluid are recovered [1, 14]. Of course, for the last two cases (Newtonian and second grade fluids), the boundary condition obtained from (3.1) for $\lambda \to 0$ is

$$\tau(R,t) = ft. \tag{4.7}$$

5. NUMERICAL RESULTS AND CONCLUSIONS

The aim of this note is to provide exact solutions for the velocity field $v(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the unsteady flow of an Oldroyd-B fluid due to an infinite circular cylinder subject to a longitudinal time-dependent shear stress. Direct computations show that these solutions, obtained by means of Hankel transforms and presented in series form in terms of Bessel functions $J_0(\cdot)$, $J_1(\cdot)$ and $J_2(\cdot)$, satisfy both the governing equations and all imposed initial and boundary conditions. Furthermore, the similar solutions for Maxwell, second grade and Newtonian fluids performing the same motion are obtained as limiting cases from the general solutions for $\lambda_r \to 0$, $\lambda \to 0$, respectively, λ_r and $\lambda \to 0$.

Finally, in order to reveal some relevant physical aspects of the solutions obtained, diagrams of the velocity field $v(r,t)$ given by Eq. (3.11) have been drawn against r for different values of t and of the material constants. In Figure 1 the velocity profiles corresponding to an Oldroyd-B fluid are drawn for three different times. They give the behavior of the non-Newtonian fluid.

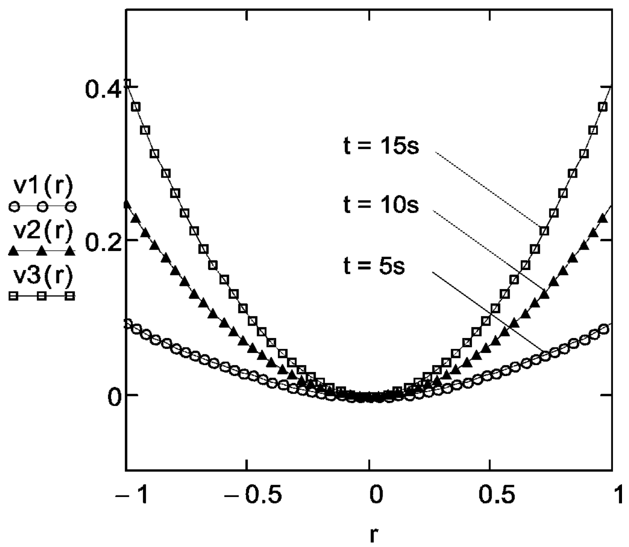

Fig. 1. Profiles of the velocity field $v(r,t)$ given by Eq. (3.11) – curves v1(r), v2(r), v3(r), for $\nu = 0.0357541$, $\mu = 32$, $R = 1$, $f = 2$, $\lambda = 5$, $\lambda_r = 2$ and different values of t.

The influence of the retardation time λ_r and the kinematic viscosity ν on the motion of the fluid can be observed from Figures 2 and 3. The two parameters, as it was to be expected, have similar effects on the motion. The velocity of the fluid decreases if λ_r or ν increases. Figure 4 spotlights the fact that for λ and $\lambda_r \to 0$ the velocity of the Oldroyd-B fluid tends to that of the Newtonian fluid, their diagrams being almost identical.

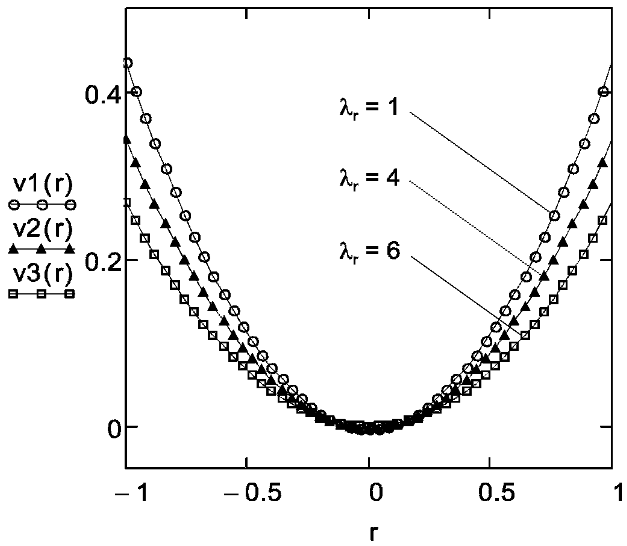

Fig. 2. Profiles of the velocity field $v(r,t)$ given by Eq. (3.11) – curves v1(r), v2(r), v3(r), for $\nu = 0.0357541$, $\mu = 32$, $R = 1$, $f = 2$, $\lambda = 7$, $t = 15\,\mathrm{s}$ and different values of λ_r.

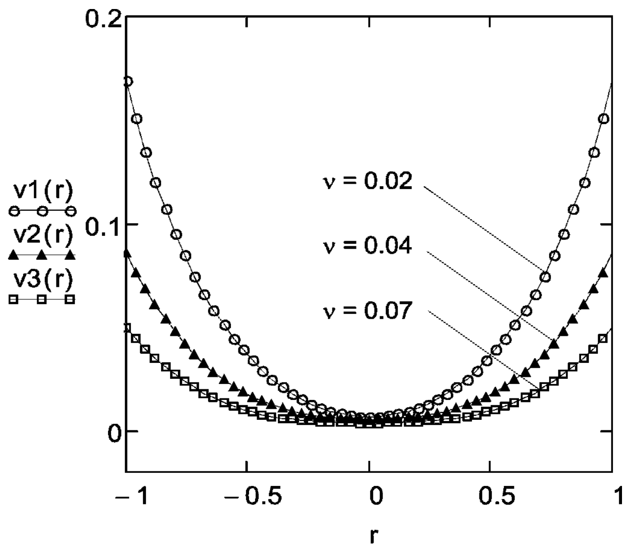

Fig. 3. Profiles of the velocity field $v(r,t)$ given by Eq. (3.11) – curves v1(r), v2(r), v3(r), for $R = 1$, $f = 2$, $\lambda = 8$, $\lambda_r = 7$, $t = 32\,\mathrm{s}$ and different values of ν.

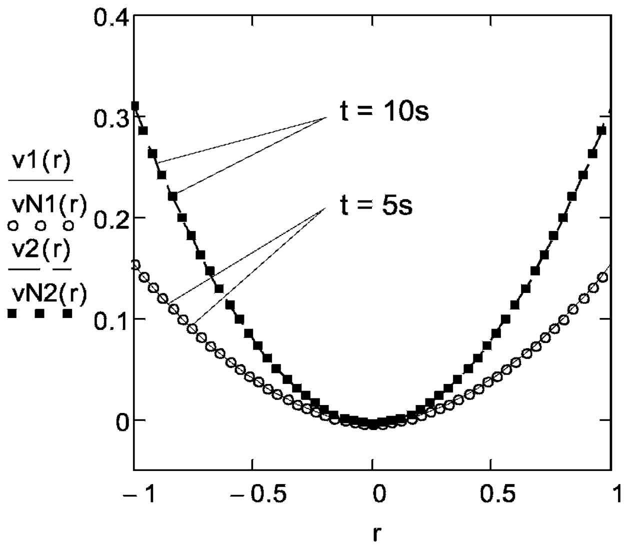

Fig. 4. Profiles of the velocity field $v(r,t)$ given by Eq. (3.11) – curves v1(r), v2(r) and Eq. (4.5) – curves vN1(r), vN2(r) for $\nu = 0.0357541$, $\mu = 32$, $R = 1$, $f = 2$, $\lambda = 0.01$, $\lambda_r = 0.01$ and different values of t.

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