STARLIKENESS AND UNIVALENCE CRITERION FOR NON-LINEAR OPERATORS

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In this paper we investigate the starlikness and univalence of certain integral operators, considering the class of univalent functions defined by the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < \lambda$, $|z| < 1$, where $f(z) = z + a_2 z^2 + \cdots$ is analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of analytic functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Denote by S the subclass of $\mathcal{A}$ consisting of functions which are univalent. Given $\alpha < 1$, a function $f \in \mathcal{A}$ is called starlike of order α, denote by $S^*(\alpha)$, if and only if $\mathrm{Re}(zf'(z)/f(z)) > \alpha$. A function $f \in \mathcal{A}$ is said convex of order α, denote by $K(\alpha)$, if and only if $zf'(z) \in S^*(\alpha)$. In particular, the classes $S^*(0) = S^*$ and $K(0) = K$ are the well-known classes of starlike and convex functions in U, respectively. In 1990 Komatu [2] introduced a integral operator

$$L_a^\lambda f(z) = \frac{a^\lambda}{\Gamma(\lambda)} \int_0^1 t^{a-2} \left(\log \frac{1}{t}\right)^{\lambda-1} f(tz)\mathrm{d}t, \tag{1.2}$$

$z \in U$, $a > 0$, $\lambda \geq 0$, $f(z) \in \mathcal{A}$. Thus, if $f(z) \in \mathcal{A}$ is of the form (1.1), it is easily seen from (1.2) that

$$L_a^\lambda f(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1}\right)^\lambda a_n z^n, \quad a > 0,\ \lambda \geq 0. \tag{1.3}$$

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We now define a function $L_a^\lambda(z)$ by

$$L_a^\lambda(z) = 1 + \sum_{n=1}^{\infty} \left(\frac{a}{a+n}\right)^\lambda z^n, \quad a > 0, \ \lambda \geq 0. \tag{1.4}$$

Using the above relation, it is easy to verify that

$$L_a^\lambda f(z) = z L_a^\lambda(z) * f(z),$$

and

$$z(L_a^{\lambda+1}(z))' = a L_a^\lambda(z) - a L_a^{\lambda+1}(z). \tag{1.5}$$

Let $V(m)$ denote the class of all functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < m, \quad z \in U, \ 0 < m \leq 1. \tag{1.6}$$

We set $V(1) = V$. We remark that from $f \in V(m)$ it follows that $\frac{f(z)}{z} \neq 0$ for $z \in U$. It is well-known that $V \subset S$ (see [4]) and so, for $0 \leq m \leq 1$ one has $V(m) \subset S$.

For proving our results we need the following lemmas.

Lemma 1.1 ([1]). *Let $h(z)$ be analytic and convex univalent in the unit disk U with $h(0) = 1$. Also let*

$$g(z) = 1 + b_1 z + b_2 z^2 + \cdots$$

be analytic in U. If

$$g(z) + \frac{z g'(z)}{c} \prec h(z), \quad z \in \mathbb{U}, \ c \neq 0, \tag{1.7}$$

then

$$g(z) \prec \psi(z) = \frac{c}{z^c} \int_0^z t^{c-1} h(t) \mathrm{d}t \prec h(z), \quad z \in U, \ \Re(c) \geq 0, \ c \neq 0. \tag{1.8}$$

and $\psi(z)$ is the best dominant of (1.7) [7].

Lemma 1.2 ([8]). *If f, g are analytic and F, G are convex functions such that $f \prec F$, $g \prec G$, then $f * g \prec F * G$.*

Lemma 1.3 ([9]). *Assume $a_1 = 1$ and $a_n \geq 0$ for $n \geq 2$, such that $\{a_n\}$ is a convex decreasing sequence, i.e.,*

$$a_n - 2a_{n+1} + a_{n+2} \geq 0 \quad \text{and} \quad a_{n+1} - a_{n+2} \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

Then

$$\operatorname{Re}\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > \frac{1}{2} \quad \text{for all } z \in U.$$

Lemma 1.4. *If $a > 0$ and $\lambda \geq 0$, then $\operatorname{Re}\{L_a^\lambda(z)\} > \frac{1}{2}$ for all $z \in U$.*

Proof. From the definition of $L_a^\lambda(z)$ we have

$$L_a^\lambda(z) = 1 + \sum_{n=2}^{\infty}\left(\frac{a}{a+n-1}\right)^{\lambda} z^{n-1} := 1 + \sum_{n=2}^{\infty} B_n z^{n-1}.$$

Since λ and a are positive, we have $B_n > 0$ for all $n \geq 2$. We can easily find that

$$B_{n+1} - B_{n+2} = \left(\frac{a}{a+n}\right)^{\lambda} - \left(\frac{a}{a+n+1}\right)^{\lambda} = a^{\lambda}\frac{(a+n+1)^{\lambda} - (a+n)^{\lambda}}{(a+n)^{\lambda}(a+n+1)^{\lambda}} \geq 0,$$

for all $n \geq 2$.

Also,

$$B_n - 2B_{n+1} + B_{n+2} =$$

$$= a^{\lambda}\frac{((a+n)(a+n+1))^{\lambda} - 2((a+n)^2-1)^{\lambda} + ((a+n)(a+n-1))^{\lambda}}{(a+n-1)^{\lambda}(a+n)^{\lambda}(a+n+1)^{\lambda}}.$$

Suppose $x = a+n$, and

$$f(x) = (x(x+1))^{\lambda} - 2(x^2-1))^{\lambda} + (x(x-1))^{\lambda}.$$

From the assumption we have $x > 1$. For proving $B_n - 2B_{n+1} + B_{n+2} \geq 0$ it is sufficient to show that $f(x) \geq 0$ for $x > 1$. But it is equivalent with

$$\left(\frac{x}{x-1}\right)^{\lambda} + \left(\frac{x}{x+1}\right)^{\lambda} \geq 2, \quad x > 1.$$

Let

$$g(x) = \left(\frac{x}{x-1}\right)^{\lambda} + \left(\frac{x}{x+1}\right)^{\lambda}.$$

Then we have

$$g'(x) = \lambda x^{\lambda-1}\left(\frac{-(x+1)^{\lambda+1} + (x-1)^{\lambda+1}}{(x-1)^{\lambda+1}(x+1)^{\lambda+1}}\right),$$

which is negative for all $x > 1$. Therefore, $g(x)$ is a decreasing function and takes its minimum value equal to 2 at infinity. Hence $g(x) \geq 2$ for all $x > 1$, and from Lemma 1.3 we get our result. □

Lemma 1.5 ([5]). *If $f \in V(m)$, $d := |f''(0)|/2 \leq 1$ and $0 \leq m \leq \frac{\sqrt{2-\ d^2}-2}{2}$, then $f \in S^*$.*

Lemma 1.6 ([7]). *If $f(z) = z + a_{n+1}z^{n+1} + \cdots (n \geq 2)$ belongs to $V(m)$ and*

$$0 \leq m \leq \frac{n-1}{\sqrt{(n-1)^2+1}},$$

then $f \in S^*$.

2. MAIN RESULTS

We now start our first result beginning with

THEOREM 2.1. *Let $a > 0$, $\lambda \geq 0$ and $f \in V(m)$ satisfy the condition*

$$\left(\frac{z}{f(z)}\right) * L_a^{\lambda+1}(z) \neq 0, \quad \forall z \in U,$$

and G be the transform defined by

$$G(z) = \frac{z}{(z/f(z)) * L_a^{\lambda+1}(z)}, \quad z \in U.$$

Further, let c be a nonnegative real number such that $c=\left|\left(\frac{a}{a+1}\right)^{\lambda+1}\frac{f''(0)}{2}\right| \leq 1.$ *Then we have the following*

(1) $G \in V\left(\frac{am}{a+2}\right)$;

(2) $G \in S^*$, *whenever* $0 < m \leq \frac{a+2}{2a}\left(\sqrt{2-c^2} - c\right)$.

Proof. From the definition of G we obtain

$$\frac{z}{G(z)} = \frac{z}{f(z)} * L_a^{\lambda+1}(z).$$

Differentiating $\frac{z}{G(z)}$ we get that

$$z\left(\frac{z}{G(z)}\right)' = \frac{z}{G(z)} - \left(\frac{z}{G(z)}\right)^2 G'(z). \tag{2.1}$$

It is easy to see that

$$z\left(\frac{z}{f(z)}\right)' * L_a^{\lambda+1}(z) = z\left(\frac{z}{f(z)} * L_a^{\lambda+1}(z)\right)'. \tag{2.2}$$

From (1.5) and (2.2) we deduce that

$$z\left(\frac{z}{f(z)} * L_a^{\lambda+1}(z)\right)' = a\frac{z}{f(z)} * L_a^{\lambda}(z) - a\frac{z}{f(z)} * L_a^{\lambda+1}(z),$$

or

$$z\left(\frac{z}{G(z)}\right)' + a\left(\frac{z}{G(z)}\right) = a\frac{z}{f(z)} * L_a^{\lambda}(z). \tag{2.3}$$

Let us define

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z) := 1 + d_2 z^2 + \cdots .$$

Then $p(z)$ is analytic in U, with $p(0) = 1$ and $p'(0) = 0$. Combining (2.1) with (2.3), one can obtain

$$p(z) = (1+a)\frac{z}{G(z)} - a\frac{z}{f(z)} * L_a^{\lambda}(z). \tag{2.4}$$

By differentiating $p(z)$ we get that

$$zp'(z) = (1+a)z\left(\frac{z}{G(z)}\right)' - az\left(\frac{z}{f(z)}\right)' * L_a^{\lambda}(z). \tag{2.5}$$

In view of (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned} ap(z) + zp'(z) &= a(1+a)\frac{z}{G(z)} - a^2\frac{z}{f(z)} * L_a^{\lambda}(z) + \\ &\quad + (1+a)z\left(\frac{z}{G(z)}\right)' - az\left(\frac{z}{f(z)}\right)' * L_a^{\lambda}(z) \\ &= a(1+a)\frac{z}{f(z)} * L_a^{\lambda}(z) - a^2\frac{z}{f(z)} * L_a^{\lambda}(z) - a\left(\frac{z}{f(z)}\right)' * L_a^{\lambda}(z) \\ &= a\frac{z}{f(z)} * L_a^{\lambda}(z) - a\left[\frac{z}{f(z)} - \left(\frac{z}{f(z)}\right)^2 f'(z)\right] * L_a^{\lambda}(z) \\ &= a\left(\frac{z}{f(z)}\right)^2 f'(z) * L_a^{\lambda}(z). \end{aligned}$$

Hence

$$p(z) + \frac{1}{a}zp'(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) * L_a^{\lambda}(z). \tag{2.6}$$

Since $\mathrm{Re}(L_a^{\lambda}(z)) \geq \frac{1}{2}$, by Herglotz theorem we can write

$$L_a^{\lambda}(z) = \int_{|x|=1} \frac{1}{1-xz}\mathrm{d}\mu(x),$$

where $\mu(x)$ is a probability measure on the unit disc $|x| = 1$, that is,

$$\int_{|x|=1} \mathrm{d}\mu(x) = 1.$$

From $f(z) \in V(m)$ we have

$$\left(\frac{z}{f(z)}\right)^2 f'(z) \prec 1 + mz^2 := h(z).$$

Since $h(z)$ is convex in U, from the well known result [3] and (2.6) we obtain

$$p(z) + \frac{1}{a}zp'(z) \prec \int_{|x|=1} h(tz)\mathrm{d}\mu(t) \prec h(z).$$

It now follows by Lemma 1.1 that

$$p(z) \prec \psi(z) = \frac{a}{z^a}\int_0^z t^{a-1}(1+mt^2)\mathrm{d}t.$$

Therefore,

$$p(z) \prec 1 + \frac{ma}{a+2}z^2.$$

Also, it is easy to see that the second part is a consequence of Lemma 1.5. □

We note that by considering more restrictions on a, λ we can improve the above result. It is well-known that for $a > 0$ and $\lambda \in \{0, 1, 2, 3, \ldots\}$ the function $L_a^\lambda(z)$ is convex. So, we can get the following result.

THEOREM 2.2. *Let $a > 0$, $\lambda \in \{0, 1, 2, 3, \ldots\}$ and $f \in V(m)$ satisfy the condition*

$$\left(\frac{z}{f(z)}\right) * L_a^{\lambda+1}(z) \neq 0, \quad \forall z \in U,$$

and G be the transform defined by

$$G(z) = \frac{z}{(z/f(z)) * L_a^{\lambda+1}(z)}, \quad z \in U.$$

Further, let c be a nonnegative real number such that $c = \left| \left(\frac{a}{a+1}\right)^{\lambda+1} \frac{f''(0)}{2} \right| \leq 1$. Then we have the following

(1) *$G \in V\left(m\left(\frac{a}{a+2}\right)^{\lambda+1}\right)$. The result is sharp especially when $|f''(0)/2| \leq 1-m$.*

(2) *$G \in S^*$, whenever $0 < m \leq \left(\frac{a+2}{a}\right)^{\lambda+1}\left(\frac{\sqrt{2-c^2}-c}{2}\right)$.*

Proof. Let us define

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z),$$

then $p(z)$ is analytic in U, with $p(0) = 1$ and $p'(0) = 0$. Using the same method as on Theorem 2.1 we get

$$p(z) + \frac{1}{a}zp'(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) * L_a^\lambda(z). \tag{2.7}$$

But $1 + mz^2$ and $L_a^\lambda(z)$ are convex, and

$$\left(\frac{z}{f(z)}\right)^2 f'(z) \prec 1 + mz^2.$$

Using Lemma 1.2, (2.7) yields

$$p(z) + \frac{1}{a}zp'(z) \prec 1 + m\left(\frac{a}{a+2}\right)^\lambda z^2.$$

It follows from Lemma 1.1 that

$$p(z) \prec 1 + m\left(\frac{a}{a+2}\right)^{\lambda+1} z^2.$$

Therefore,

$$|p(z) - 1| \leq \left(\frac{a}{a+2}\right)^{\lambda+1} |z|^2. \tag{2.8}$$

To prove the sharpness we follow the same way as in [6]. Let the function $f \in V(m)$ be defined by

$$f(z) = \frac{z}{1 - a_2 z + mz^2}, \quad z \in U,$$

where $a_2 = f''(0)/2$ and $|a_2| \leq 1 - m$, so that $1 - a_2 z + mz^2 \neq 0$ for all $z \in U$. Moreover, since $\lambda \geq 0$, $a > 0$, it follows that $(a+2)^{\lambda+1} > (a+1)^{\lambda+1} > (a)^{\lambda+1}$ and, therefore

$$1 - a_2\left(\frac{a}{a+2}\right)^{\lambda+1} z + m\left(\frac{a}{a+2}\right)^{\lambda+1} z^2| \neq 0,$$

for all $z \in U$, provided $|a_2| \leq 1-m$. Now, from the definition of $G(z)$ we obtain

$$G(z) = \frac{z}{1 - a_2\left(\frac{a}{a+2}\right)^{\lambda+1} z + m\left(\frac{a}{a+2}\right)^{\lambda+1} z^2},$$

which is analytic on U, $\frac{z}{G(z)} \neq 0$ on U and $\left(\frac{z}{G(z)}\right)^2 G'(z) - 1 - m\left(\frac{a}{a+2}\right)^{\lambda+1}$. This means that $G \in V\left(m\left(\frac{a}{a+2}\right)^{\lambda+1}\right)$. Also, the second part is a easy consequence of Lemma 1.5. Hence the proof is ended. □

Using Lemma 1.6, Theorem 2.1 can be generalized as follows.

THEOREM 2.3. *Let* $n \geq 2$, $f(z) = z + a_{n+1}z^{n+1} + \cdots \in V(m)$. *Suppose* $a > 0$, $\lambda \geq 0$ *and*

$$\left(\frac{z}{f(z)}\right) * L_a^{\lambda+1}(z) \neq 0, \quad \forall z \in U,$$

and G *be the transform defined by*

$$G(z) = \frac{z}{(z/f(z)) * L_a^{\lambda+1}(z)}, \quad z \in U.$$

Then we have the following

(1) $G \in V\left(\frac{am}{a+n}\right)$.

(2) $G \in S^*$, *whenever* $0 < m \leq \frac{(a+n)(n-1)}{a\sqrt{(n-1)^2+1}}$.

Proof. Let us define

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z) = 1 - a_{n+1}\left(\frac{a}{a+2}\right)^\lambda z^n + \cdots .$$

Then $p(z)$ is analytic in U, with $p(0) = 1$ and $p'(0) = \cdots = p^{(n-1)}(0) = 0$. Using the same method as on Theorem 2.1 we get

$$p(z) + \frac{1}{a} zp'(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) * L_a^\lambda(z). \tag{2.9}$$

Hence the result follows as Theorem 2.1. □

Finally, using the similar methods as in Theorems 2.2 and 2.3 we get the following result and we omit the details.

THEOREM 2.4. *Let $n \geq 2$, $f(z) = z + a_{n+1}z^{n+1} + \cdots \in V(m)$. Also suppose $a > 0$, $\lambda \geq 0$ and*

$$\left(\frac{z}{f(z)}\right) * L_a^{\lambda+1}(z) \neq 0, \quad \forall z \in U,$$

and G be the transform defined by

$$G(z) = \frac{z}{(z/f(z)) * L_a^{\lambda+1}(z)}, \quad z \in U.$$

Then we have the following

(1) $G \in V\left(m\left(\frac{a}{a+n}\right)^{\lambda+1}\right)$;

(2) $G \in S^*$, *whenever* $0 < m \leq \left(\frac{(a+n)}{a}\right)^{\lambda+1} \frac{(n-1)}{\sqrt{(n-1)^2+1}}$.

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