ON THE X-RANKS WITH RESPECT TO
A LOW GENUS PROJECTIVE CURVE

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Let $X \subset \mathbb{P}^{2k+1}$ be a smooth, non-degenerate and real projective curve. For each $P \in \mathbb{P}^{2k+1}(\mathbb{R})$ we study the set of all integers $\sharp(S \cap X(\mathbb{R}))$ for all possible sets $S \subset X(\mathbb{C})$ such that $P \in \langle S \rangle$ and $\sharp(S) = k + 1$ (mainly when $p_a(X) \leq 1$).

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1. INTRODUCTION

Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety defined over $\mathbb{C}$. For any $P \in \mathbb{P}^n(\mathbb{C})$ the X-rank $r_X(P)$ of $P$ is the minimal cardinality of a finite set $S \subset X(\mathbb{C})$ such that $P \in \langle S \rangle$, where $\langle \rangle$ denote the linear span. In the applications when $X$ is a Veronese embedding of $\mathbb{P}^m$ the X-rank is also called the “structured rank” or “symmetric tensor rank” (this is related to the virtual array concept considered in sensor array processing ([5], [8], [9])). On this topic there was the 2008 AIM workshop Geometry and representation theory of tensors for computer science, statistics and other areas and a summer school: “Geometry of tensors and applications”, 14–18, June 2010, at the Sophus Lie Conference Center, Nordfjordeid, Norway. In the applications quite often the base field is the real field ([9], [6] and references therein). In this case there are several possible related questions and here we introduce another one, different from the notion of real rank considered in [6].

For any algebraic scheme $Y$ defined over $\mathbb{C}$ let $Y(\mathbb{C})$ denote the set of all $\mathbb{C}$-points with the euclidean topology. We will explicitly say when we will use the Zariski topology on $Y(\mathbb{C})$. Thus $Y(\mathbb{C})$ is Hausdorff. If $Y$ is projective, then $Y(\mathbb{C})$ is compact. Now assume that $Y$ is defined over $\mathbb{R}$. Any subset of $Y(\mathbb{C})$ (and in particular the real locus $Y(\mathbb{R}) \subset Y(\mathbb{C})$) inherits a topology. The complex conjugation $\sigma$ is a homeomorphism $\sigma : Y(\mathbb{C}) \rightarrow Y(\mathbb{C})$ and $Y(\mathbb{R}) = \{ P \in Y(\mathbb{C}) : \sigma(P) = P \}$. With this topology if $Y$ is smooth and projective, then $Y(\mathbb{R})$ is a compact real manifold with pure dimension $\dim(Y)$ (sometimes empty). Thus if $Y$ is a smooth and projective curve, then $Y(\mathbb{R})$ is a disjoint union of finitely many circles.

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For any $P \in \mathbb{P}^n(\mathbb{C})$ let $S(X, P)$ denote the set of all $S \subset X(\mathbb{C})$ computing $r_X(P)$, i.e., such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Set $E_X(k) := \{P \in \mathbb{P}^n(\mathbb{C}) : r_X(P) = k\}$. For any integer $t \geq 1$ let $\sigma_t(X)$ or $\sigma_t(X)(\mathbb{C})$ denote the closure in $\mathbb{P}^n(\mathbb{C})$ of the union of all $(t - 1)$-dimensional linear subspaces spanned by $t$ points of $X(\mathbb{C})$. For any $P \in \mathbb{P}^n(\mathbb{C})$ let $b_X(P)$ denote the border $X$-rank of $P$, i.e., the first integer $s > 0$ such that $P \in \sigma_s(X)$. Let $z_X(P)$ be the minimal integer $\deg(Z)$ for some zero-dimensional scheme $Z \subset X(\mathbb{C})$ such that $P \in \langle Z \rangle$. Notice that if $Z$ computes $z_X(P)$, then $P \notin \langle Z' \rangle$ for any $Z' \subset Z$, $Z' \neq Z$. In this paper we only use the function $z_X$ when $X$ is a smooth curve of genus zero or one. In these cases $z_X = b_X$ ([3], Proposition 5.1). Set $U := E_X(\rho_X)$. Fix $X \subset \mathbb{P}^n$ defined over $\mathbb{R}$ and $P \in \mathbb{P}^n(\mathbb{R})$. Recall that $r_X(P)$ will always mean the complex $X$-rank of $X$, i.e., the rank with respect to $X(\mathbb{C})$. Since $X$ and $P$ are real, $\sigma(S(X, P)) = S(X, P)$. Set $S(X, P)^o := \{S \in S(X, P) : \sigma(S) = S\}$. Each element of $S(X, P)^o$ is a disjoint union of some points of $X(\mathbb{R})$ and some complex conjugate pairs of points of $X(\mathbb{C}) \setminus X(\mathbb{R})$. For any integer $t$ such that $t \equiv r_X(P) \pmod{2}$ let $S(X, P; \mathbb{R})_t^o$ denote the set of all $S \in S(X, P)^o$ which are unions of $t$ points of $X(\mathbb{R})$ and $(r_X(P) - t)/2$ complex conjugate pairs of points of $X(\mathbb{C})$.

The complex conjugation $\sigma$ acts on $S(X, P)$. If $\sigma : S(X, P) \to S(X, P)$ has no fixed point, then we say that the signature of $P$ with respect to the real curve $X$ is $\emptyset$ and that the extended signature is $\{\emptyset\}$. If there is at least one such fixed point, then the signature $\tau(X, \mathbb{R}, P)$ of $P$ with respect to the real curve $X$ is the pair $(r_X(P), U)$, where $U := \{t \in \mathbb{N} : S(X, P; \mathbb{R})_t^o \neq \emptyset\}$. We are interested in the possible signatures of the points of $U_X(\mathbb{R}) := U_X \cap \mathbb{P}^n(\mathbb{R})$. Notice that points of $U_X(\mathbb{R})$ with different signatures lie in different connected components of $U_X(\mathbb{R})$ for the euclidean topology. If there is $S \in S(X, P)$ such that $\sigma(S) \neq S$, then we call $\tau(X, \mathbb{R}, P)' := \tau(X, \mathbb{R}, P) \cup \{\emptyset\}$ the extended signature of $P$. If there is no such $S$ then we call $\tau(X, \mathbb{R}, P)' := \tau(X, \mathbb{R}, P)$ the extended signature of $P$. If $X(\mathbb{R})$ has several connected components, say $T_1, \ldots, T_s$, then we may refine the signature of $\tau(X, \mathbb{R}, P)$ of $P$ with respect to the real variety $X$ by prescribing how many of the real points are contained in each component $T_1, \ldots, T_s$ of $X(\mathbb{R})$.

In this note we prove the following results.

**Theorem 1.1.** Let $X \subset \mathbb{P}^n$ be a real and non-degenerate embedding of the real integral curve $X$. Assume $X_{\text{reg}}(\mathbb{R}) \neq \emptyset$. We have $\rho_X = \lfloor (n + 2)/2 \rfloor$. Each non-empty admissible pair $([\lfloor (n + 2)/2 \rfloor, t], 0 \leq t \leq \lfloor (n + 2)/2 \rfloor$, $t \equiv$
\((n + 2)/2\) (mod 2) arises as the signature of a non-empty open subset (for the euclidean topology) of \(U_X(\mathbb{R})\).

Theorem 1.2. Let \(X \subseteq \mathbb{P}^{2k+1}, k \geq 1\), be a rational normal curve defined over \(\mathbb{R}\). We have \(p_X = k + 1\), every \(P \in U_X(\mathbb{R})\) has a unique signature and each non-empty admissible pair \((k + 1, t)\), \(0 \leq t \leq k + 1\), \(t \equiv k + 1 \pmod{2}\) arises as the signature of a non-empty open subset (for the euclidean topology) of \(U_X(\mathbb{R})\). \(\{\emptyset\}\) is not in the extended signature of any \(P \in U_X(\mathbb{R})\).

The last assertion of Theorem 1.2 shows that sometimes in the statement of Theorem 1.1 we need to exclude the \(\emptyset\) signature.

Theorem 1.3. Let \(X \subseteq \mathbb{P}^{2k+1}, k \geq 1\), be a real linearly normal embedding of a smooth curve of genus 1. Assume \(X(\mathbb{R}) \neq \emptyset\). Set \(U'_X := \{P \in U_X : O_X(1) \not\cong O_X(2Z)\text{ for all } Z \in S(X, P)\}\). \(U'_X\) contains a non-empty Zariski open subset of \(U_X\).

(a) Every \(P \in U'_X(\mathbb{R})\) has at most two signatures.

(b) Fix integers \(t_1, t_2\) such that \(0 \leq t_1 \leq t_2 \leq k + 1\) and \(t_1 \equiv t_2 \equiv k + 1 \pmod{2}\). Then there exists a non-empty open subset \(A(t_1, t_2)\) of \(U'_X(\mathbb{R})\) (for the euclidean topology) such that every \(P \in A(t_1, t_2)\) has \((k + 1, \{t_1, t_2\})\) as its only extended signatures.

Notice that in part (b) of Theorem 1.3 we allow the case \(t_1 = t_2\): every \(P \in A(t_1, t_1)\) has \((k + 1, \{t_1\})\) as its only extended signature.

2. THE PROOFS

We lift from [2] the following lemma and its proof.

Lemma 2.1. Fix any \(P \in \mathbb{P}^n(\mathbb{C})\) and two zero-dimensional subschemes \(A, B\) of \(X(\mathbb{C})\) such that \(A \neq B, P \notin \langle A \rangle, P \in \langle B \rangle, P \notin \langle A' \rangle\) for any \(A' \subset A, A' \neq A, \) and \(P \notin \langle B' \rangle\) for any \(B' \subset B, B' \neq B\). Then \(h^1(\mathbb{P}^n, \mathcal{I}_{A \cup B}(1)) > 0\).

Proof. Since \(A\) and \(B\) are zero-dimensional, we have \(h^1(\mathbb{P}^n, \mathcal{I}_{A \cup B}(1)) \geq \max\{h^1(\mathbb{P}^n, \mathcal{I}_A(1)), h^1(\mathbb{P}^n, \mathcal{I}_B(1))\}\). Thus we may assume \(h^1(\mathbb{P}^n, \mathcal{I}_A(1)) = h^1(\mathbb{P}^n, \mathcal{I}_B(1)) = 0\), i.e., \(\dim(\langle A \rangle) = \deg(A) - 1\) and \(\dim(\langle B \rangle) = \deg(B) - 1\). Set \(D := \langle A \rangle \cap \langle B \rangle\) (scheme-theoretic intersection). Thus \(\deg(A \cup B) = \deg(A) + \deg(B) - \deg(D)\). Since \(D \subseteq A\) and \(A\) is linearly independent, we have \(\dim(\langle D \rangle) = \deg(D) - 1\). Since \(h^1(\mathbb{P}^n, \mathcal{I}_{A \cup B}(1)) > 0\) if and only if \(\dim(\langle A \cup B \rangle) \leq \deg(A \cup B) - 2\), we get \(h^1(\mathbb{P}^n, \mathcal{I}_{A \cup B}(1)) > 0\) if and only if \(\langle D \rangle \subset \langle A \rangle \cap \langle B \rangle\) and \(\langle D \rangle \neq \langle A \rangle \cap \langle B \rangle\). Since \(A \neq B, D \subset A, D \neq A\). Hence \(P \notin \langle D \rangle\). Since \(P \in \langle A \rangle \cap \langle B \rangle\), we are done.

Lemma 2.2. Let \(C \subset \mathbb{P}^{2k+1}, k \geq 1\), be a rational normal curve. Fix \(P \in \mathbb{P}^{2k+1}\) such that there is a zero-dimensional subscheme \(A \subset C\) such that
$P \in \langle A \rangle$, $P \notin \langle A' \rangle$ for any $A' \subset A$, $A' \neq A$. and $\deg(A) \leq k + 1$. Let $B \subset C$ be a zero-dimensional scheme such that $P \in \langle B \rangle$ and $\deg(B) \leq \deg(A)$. Then $B = A$. 

Proof. Use Lemma 2.1, the inequality $\deg(A \cup B) \leq \deg(C) + 1$ and that any zero-dimensional scheme with degree $\leq c + 1$ of a rational normal curve of $\mathbb{P}^c$ is linearly independent.

Remark 2.3. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. We have $\dim(\sigma_t(X)) = \min\{n, 2t - 1\}$ for all integer $t \geq 1$ ([1], Remark 1.6). Hence $\rho_X = [(n + 2)/2]$ and $\cup_X = E_X((n + 2)/2))$.

For any scheme $X \subset \mathbb{P}^n$ let $\hat{X} \subset \mathbb{A}^{n+1}$ denote its affine cone.

Lemma 2.4. Let $X, Y \subset \mathbb{P}^n$ be complex integral subvarieties. Fix $P \in X_{\text{reg}}(\mathbb{C})$ and $Q \in Y_{\text{reg}}(\mathbb{C})$ such that $T_P X \cap T_Q Y = \emptyset$. Then the join $J(X, Y)$ of $X$ and $Y$ has a smooth branch of dimension $\dim(X) + \dim(Y) + 1$ at a general point $E$ of the line $\langle\{P, Q\}\rangle$ and the natural addition map $\hat{X} + \hat{Y} \to J(\hat{X}, \hat{Y})$ considered in [1], §1, is a submersion at $E$.

Proof. Let $\alpha : \mathbb{A}^{n+1} \times \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ denote the addition. Set $u := \alpha|(\hat{X}_{\text{reg}}) \times (\hat{Y}_{\text{reg}})$. Since $\dim(\hat{X} \times \hat{Y}) = \dim(X) + \dim(Y) + 2$, the differential of the map $\alpha : (\hat{X}_{\text{reg}}) \times (\hat{Y}_{\text{reg}}) \to \mathbb{A}^{n+1}$ has always rank at most $\dim(X) + \dim(Y) + 2$. Our assumption means that it achieves its maximal value at $(\hat{P}, \hat{Q})$ ([1], Corollary 1.11). Hence $\text{Im}(u)$ has a smooth branch of dimension $\dim(X) + \dim(Y) + 2$ in a neighborhood of $u(\hat{P}, \hat{Q})$, proving the last assertion. Taking the projection $\mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ we get the other statements of the lemma.

Lemma 2.5. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate complex projective curve. Fix an integer $t \geq 2$ and $P_1, \ldots, P_t \in X_{\text{reg}}(\mathbb{C})$ such that $\dim(\langle T_{P_1} X \cup \cdots \cup T_{P_t} X \rangle) = 2t - 1$. Then $\dim(\langle\{P_1, \ldots, P_t\}\rangle) = t - 1$ and a general point of $\langle\{P_1, \ldots, P_t\}\rangle$ is contained in a smooth branch of $\sigma_t(X)$.

Proof. Since $\langle T_{P_1} X \cup \cdots \cup T_{P_t} X \rangle$ is generated by $\langle\{P_1, \ldots, P_t\}\rangle$ and $t$ points (one for each tangent line), we obviously have $\dim(\langle\{P_1, \ldots, P_t\}\rangle) = t - 1$. If $t = 2$, then the second assertion is Lemma 2.4 applied to $(X, Y, P, Q) := (X, X, P_1, P_2)$. Now assume $t \geq 3$. We use induction on $t$. We apply Lemma 2.4 to $(X, Y, P, Q) := (X, \sigma_t^{-1}(X), P_1, Q)$ where $Q$ is a general element of $\langle\{P_2, \ldots, P_t\}\rangle$.

Lemma 2.6. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate projective curve defined over $\mathbb{R}$. Call $T_1, \ldots, T_s$ the connected components of $X_{\text{reg}}(\mathbb{R})$. Fix $(\alpha, \beta) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ such that $2\alpha + 4\beta \leq n + 1$. If $s = 0$, then assume $\alpha = 0$. If $s > 0$, then for each $i \in \{1, \ldots, \alpha\}$ fix an integer $\gamma(i) \in \{1, \ldots, s\}$. There are $\alpha$ real points $P_1, \ldots, P_\alpha$ of $X$ with $P_i \in T_{\gamma(i)}$ for all $i$ and $\beta$
$Q'_i, Q''_i, 1 \leq i \leq \beta$, of complex conjugate points of $X(\mathbb{C}) \setminus X(\mathbb{R})$ such that 

$$\dim((\bigcup_{i=1}^{\alpha} T_{P_i}X \cup \bigcup_{i=1}^{\beta} (T_{Q'_i}X \cup T_{Q''_i}X))) = 2\alpha + 4\beta - 1.$$  

**Proof.** By Terracini’s lemma ([1], Corollary 1.11) and by [1], Remark 1.6, there is a Zariski open dense subset $U$ of $X^{\alpha+2\beta}(\mathbb{C})$ such that 

$$\dim((\bigcup_{i=1}^{\alpha+2\beta} T_{Q_i}X))) = 2\alpha + 4\beta - 1$$  

for every $(Q_1, \ldots, Q_{\alpha+2\beta}) \in U$. For any $x < \alpha + 2\beta$ and fixed $Q_1, \ldots, Q_x$ such that $\dim((\bigcup_{i=1}^{\alpha+2\beta} T_{Q_i}X))) = 2x - 1$ the set of all $(Q_{x+1}, \ldots, Q_{\alpha+2\beta})$ that we may add to $(Q_1, \ldots, Q_x)$ and have $\dim((\bigcup_{i=1}^{\alpha+2\beta} T_{Q_i}X))) = 2\alpha + 4\beta - 1$ is a non-empty open subset of $X^{\alpha+2\beta-1}$. Since each $T_i$ is Zariski dense in $X$, the previous observation reduces the proof of the lemma for the pair $(\alpha, \beta)$ to the proof of the lemma for the pair $(0, \beta)$. To prove the case $(0, \beta)$ it is sufficient to use induction on $\beta$ and the observation that the set of all pairs $(Q, \sigma(Q))$ are Zariski dense in $(X_{\text{reg}}(\mathbb{C}) \setminus X_{\text{reg}}(\mathbb{R})) \times (X_{\text{reg}}(\mathbb{C}) \setminus X_{\text{reg}}(\mathbb{R}))$.

**Proof of Theorem 1.1.** It is well-known that $\rho_X = \lfloor (n + 2)/2 \rfloor$ ([1], Remark 1.6). Fix any $P \in U_X(\mathbb{R})$. Apply Lemma 2.5 with $\alpha + 2\beta = \lfloor (n + 2)/2 \rfloor$. Then apply Lemma 2.5 and conclude the proof.

**Proof of Theorem 1.2.** Lemma 2.2 gives the uniqueness of the set $S(X, P)$. Thus $\sigma(S) = S$. Hence $P$ has a unique signature and its extended signature does not contain $\emptyset$. The remaining assertions are particular cases of Theorem 1.1, concluding the proof.

**Lemma 2.7.** Let $X$ be a real smooth curve of genus 1 such that $X(\mathbb{R}) \neq \emptyset$. Fix an integer $d > 0$ and $L \in \text{Pic}^d(X)(\mathbb{R})$. Let $[L](\mathbb{R})$ denote the set of all $\sigma$-invariant elements of the linear system $|L|$. Fix any integer $t$ such that $0 \leq t \leq d$ and $t \equiv d \pmod{2}$. Let $A(t)$ be the set of all $S \in [L](\mathbb{R})$ such that $S$ is reduced and it has exactly $t$ real points. Then $A(t) \neq \emptyset$ and $A(t)$ contains a non-empty open subset of the $(d-1)$-dimensional real projective space $|L|(\mathbb{R})$.

**Proof.** Fix any set $S \subset X(\mathbb{C})$ such that $g(S) = d - 1$ and either $S \subset X(\mathbb{R})$ (case $t = d$) or $S$ is the union of $t$ real points and $((d - t)/2 - 1)$ pairs of complex conjugate points and one point $Q \in X(\mathbb{C}) \setminus X(\mathbb{R})$ (case $t \neq d$). The line bundle $L(-S)$ has degree 1. Thus there is a unique $Q' \in X(\mathbb{C})$ such that $L(-S) \cong \mathcal{O}_X(Q')$. Since $L$ is real and $S \setminus \{Q\}$ is $\sigma$-invariant, we have $Q' = \sigma(Q)$. First assume $t < d$. Since $Q \in X(\mathbb{C}) \setminus X(\mathbb{R})$, we have $\sigma(Q) \neq Q$. Since $S \setminus \{Q\}$ is $\sigma$-invariant and reduced, we get that $S \cup \{\sigma(Q)\}$ is reduced and $\sigma$-invariant. By construction $S \cup \{\sigma(Q)\} \in A(t)$. Varying $S$ as above we get the lemma when $t \neq d$. Now assume $t = d$. In this case there is a unique effective divisor $S + Q' \in |L|$ containing $S$. Since $S$ is real, the uniqueness of this divisor implies that it is defined over $\mathbb{R}$, i.e., $Q' \in X(\mathbb{R})$. We only need to prove that $S + Q'$ is reduced for general $S$, i.e., $Q' \neq S$ for general $S$. This is
true because the set of all unreduced elements of $|L|(\mathbb{C})$ is an algebraic set of dimension $\leq d - 2$ by Bertini’s theorem.

Proof of Theorem 1.3. Since $\deg(O_X(1)) = 2k + 2$ is even, there are 4 non-isomorphic $L \in \text{Pic}^{k+1}(X)$ such that $L^{\otimes 2} \cong O_X(1)$. Call $L_i$, $1 \leq i \leq 4$, these line bundles. For each $P \in U_X \setminus U'_X$ there is $i \in \{1, 2, 3, 4\}$ and $Z \in |L_i|$ such that $P \in \langle Z \rangle$. Since $\dim(|L_i|) = k$ and $\dim(\langle Z \rangle) = k$, we see that $U_X \setminus U'_X$ is contained in a hypersurface of $\mathbb{P}^{2k+1}$. Hence $U'_X$ contains a non-empty Zariski open subset of $U_X$. Since the embedding $X \hookrightarrow \mathbb{P}^{2k+1}$ is real, $\sigma(U'_X) = U'_X$.

(i) Fix $P \in \sigma_{k+1}(X) \setminus \sigma_k(X)$ and assume the existence of 3 distinct degree $k+1$ subschemes $A_i$, $i = 1, 2, 3$, of $X(\mathbb{C})$ computing $z_X(P)$. Thus $\dim(\langle A_i \rangle) = k$ for all $i$. First assume $A_i \cap A_j = \emptyset$ for all $i \neq j$. Since $P \in \langle A_1 \rangle \cap \langle A_2 \rangle$, $A \cup B$ is contained in a hyperplane of $\mathbb{P}^n$. Since $\deg(A_1 \cup A_2) = 2k + 2$, $A_1 + A_2 \in |O_X(1)|$. Thus $A_2 \in |O_X(1)(-A_1)|$. Using $A_3$ instead of $A_2$ we get $A_3 \in |O_X(1)(-A_1)|$. Thus $O_X(A_2) \cong O_X(A_3)$. In the same way we get $O_X(1) \cong O_X(A_3)$. Thus $O_X(1) \cong O_X(2A_1)$. Since $P \in U_X \setminus U'_X$ now assume $A_i \cap A_j \neq \emptyset$ for some $i < j$, say if $(i, j) = (1, 2)$. Let $B_{1,2}$ be the maximal divisor contained both in $A_1$ and $A_2$. Hence $D := A_1 + A_2 - B_{1,2}$ is a subscheme of degree at most $2k + 1$ of the linearly normal curve $X$. By Riemann-Roch every effective divisor of $X$ with degree at most $2k + 1$ is a linearly independent subscheme of $\mathbb{P}^n$, contradicting Lemma 1.2. Hence part (a) is proved.

(ii) Fix any real line bundle $M$ on $X$ such that $\deg(M) = k + 1$ and set $N := O_X(1) \otimes M^*$. Hence $N$ is a degree $k + 1$ real line bundle. Assume the existence of $A \in |M|$ such that $\sigma(A) = A$, $A$ is reduced and $A$ has exactly $t_1$ real connected components. Assume the existence of $B \in |M|$ such that $\sigma(B) = B$, $B$ is reduced and $B$ has exactly $t_2$ real connected components. Assume $A \cap B = \emptyset$. Since any zero-dimensional subscheme of $X$ with degree at most $2k + 1$ is linearly independent, $\dim(\langle A \rangle) = \dim(\langle B \rangle) = k$. Since $A + B \in |O_X(1)|$, $A + B$ is contained in a hyperplane. Since $A \cap B = \emptyset$, we saw in step (i) that $\langle A \rangle \cap \langle B \rangle$ is a unique point, $Q_{A,B}$. Since $\langle A \rangle$ and $\langle B \rangle$ are real, $Q_{A,B} \in \mathbb{P}^n(\mathbb{R})$. Assume for the moment $z_X(Q_{A,B}) = k + 1$. Any such $Q_{A,B}$ has signatures $(k + 1; t_1)$ and $(k + 1; t_2)$ and no other extended signature, because part (a), i.e., step (i), gives the non-existence of $Z \subset X(\mathbb{C})$ such that $\deg(Z) = k + 1$, $Q_{A,B} \in \langle Z \rangle$ and $Z \notin \{A, B\}$. Since $\sigma_k(X) \cap \mathbb{P}^n(\mathbb{R})$ has Hausdorff dimension $\leq 2k$, to conclude it is sufficient to find families of sets $A, B$ as above such that the set of the associated points $Q_{A,B}$ (without imposing the condition $z_X(Q_{A,B}) = k + 1$) cover a non-empty open subset of $\mathbb{P}^n(\mathbb{R})$. Start with any real line bundle $M$ on $X$. Apply Lemma 2.7 with respect to the integer $d := k + 1$ first to the integer $t_1$ and $M$ and then with
respect to the integer $t_2$ and the real line bundle $O_X(1) \otimes M^*$. The proof of Theorem 1.3 is over.

**Remark 2.8.** Let $X \subset \mathbb{P}^4$ be a real rational normal curve. There is a non-empty open subset $\Omega$ of $U_X(\mathbb{R})$ formed by points with real rank 4 ([6], case $d = 4$ of the Main Theorem). Recall that $\rho_X = 3$. Fix any $P \in \Omega$ and take any signature $(3, a)$ of $P$. Since 3 is odd, $a > 0$. Since $P$ has not real rank 3, $a \neq 3$. Thus $(3, \{1\})$ is the only signature of $P$ with respect to $X$.

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