Elliptic divisibility sequences are generalizations of a class of integer divisibility sequences called Lucas sequences. There has been much interest in cases where the terms of Lucas sequences and also sequences of Pell numbers are squares or cubes [7, 8]. In [6], we determine in which cases a term of an elliptic divisibility sequence can be a square if one of the first six terms is zero. We also determine the cube terms of these sequences.

AMS 2010 Subject Classification: 11B37, 11B39, 11D09.

Key words: elliptic divisibility sequences, cubes.

1. INTRODUCTION

A divisibility sequence is a sequence \((h_n)\) \((n \in \mathbb{N})\) of integers with the property that \(h_m|h_n\) if \(m|n\). One of the oldest examples of a divisibility sequence is the Fibonacci sequence. There are also divisibility sequences satisfying a nonlinear recurrence relation. These are the elliptic divisibility sequences and this relation comes from the recursion formula for elliptic division polynomials associated to an elliptic curve.

An elliptic divisibility sequence (or EDS) is a sequence of integers \((h_n)\) satisfying a non-linear recurrence relation
\[
h_{m+n}h_{m-n} = h_{m+1}h_{m-1}h_n^2 - h_{n+1}h_{n-1}h_m^2
\]
and with the divisibility property that \(h_n\) divides \(h_m\) whenever \(n\) divides \(m\) for all \(m \geq n \geq 1\).

A solution of (1) is proper if \(h_0 = 0\), \(h_1 = 1\), and \(h_2h_3 \neq 0\). Such a proper solution will be an EDS if and only if \(h_2, h_3, h_4\) are integers with \(h_2|h_4\). The sequence \((h_n)\) with initial values \(h_1 = 1\), \(h_2, h_3\) and \(h_4\) is denoted by \([1 \, h_2 \, h_3 \, h_4]\).

EDSs are interesting because they were the first non-linear divisibility sequences to be studied. Morgan Ward wrote several papers detailing the arithmetic theory of EDSs [13, 14]. For the arithmetic properties of EDSs, see also [3, 4, 5, 11, 12]. To classify EDSs we need to know the following definition.

An integer \(m\) is said to be a divisor of the sequence \((h_n)\) if it divides some term with positive subscript. Let \(m\) be a divisor of \((h_n)\). If \(\rho\) is an integer such
that $m|h_\rho$ and there is no integer $j$ such that $j$ is a divisor of $\rho$ with $m|h_j$ then $\rho$ is said to be rank of apparition of $m$ in $(h_n)$. Ward proved that the multiples of $m$ are regularly spaced in $(h_n)$ in the following theorem.

**Theorem 1** [14]. Let $p$ be a prime divisor of an elliptic divisibility sequence $(h_n)$, and let $\rho$ be its smallest rank of apparition. If $h_{\rho+1} \not\equiv 0(p)$ then $h_n \equiv 0(p)$ if and only if $n \equiv 0(\rho)$.

2. CUBES IN ELLIPTIC DIVISIBILITY SEQUENCES

EDSs are generalizations of a class of integer divisibility sequences called Lucas sequences. The question of when a term of a Lucas sequence can be square has generated interest in the literature [1, 2, 9, 10]. Similar results concerning cubes were also obtained for specific sequences such as Fibonacci, Lucas and Pell numbers [7, 8]. Although authors wrote many papers on when the terms of Lucas sequences are perfect squares or cubes, they did not touch the question of in which cases a term of an EDS can be a perfect square or a cube.

In [6], we partly answer the question for square terms in an elliptic divisibility sequence and in this work we determine in which cases a term of an elliptic divisibility sequence can be a cube. In [6] we gave formulæ for the general term of EDSs whose second, third, fourth, fifth or sixth term is zero, and we determine the cube terms of such EDSs. Below we shall denote an unspecified cube by $C$.

2.1. Sequences whose second term is zero

First consider the EDSs with rank two. We know that if $h_2 = 0$ then every term of the sequence with even subscript is zero. Ward gave the general term of $(h_n)$ with second term zero. But we rearrange the general term of $(h_n)$ in [6], i.e., if $(h_n)$ is an elliptic divisibility sequence with initial values $[10h_30]$ ($h_3 \neq 0$) and $n$ is odd, then $(h_n)$ is given by

\[
h_n = h_{2k+1} = \varepsilon h_3^{k(k+1)/2},
\]

where

\[
\varepsilon = \begin{cases} 
+1 & \text{if } n \equiv 1, 3 \pmod{8}, \\
-1 & \text{if } n \equiv 5, 7 \pmod{8}.
\end{cases}
\]

**Theorem 2**. Let $(h_n)$ be an elliptic divisibility sequence with initial values $[10h_30]$
with $h_3 \neq 0$. If $n \equiv 1, 5 \pmod{6}$, then $h_n = C$.

Proof. Let $n \equiv 1$ or $5 \pmod{6}$, then $k = 3r$ or $3r + 2$ $(r, k \in \mathbb{N})$, on substituting these equations into (2), we have

$$h_n = \varepsilon h_3^{(9r^2+3r)/2} \quad \text{and} \quad h_n = \varepsilon h_3^{(9r^2+15r+6)/2},$$

respectively. Hence, $h_n = C$. □

As particular cases of the preceding results and Theorem 2 we deduce the following corollary.

Corollary 1. Let $(h_n)$ be an elliptic divisibility sequence with initial values

$$[10h_30]$$

with $h_3 \neq 0$.

1. If $h_3 = C$, then $h_n = C$, for all $n$.
2. If $h_3 \neq C$, then $h_n = C$ iff $n \equiv 1, 5 \pmod{6}$.

2.2. Sequences whose third term is zero

Now consider the EDSs with third term zero. By Theorem 1, we know that if $h_3 = 0$ then $h_{3n} = 0$ for all integers $n \neq 0$. In [6] we proved that if $(h_n)$ is an elliptic divisibility sequence with initial values $[1h_20h_4] \ (h_2, h_4 \neq 0)$ then $(h_n)$ is given by

$$(3) \quad h_n = h_{3k+a} = \varepsilon h_4^{(k+1)/2} h_2^{(k+2a-2)(k+2a-3)/2},$$

where $a \equiv n \pmod{3}$ and

$$\varepsilon = \begin{cases} 
+1 & \text{if } n \equiv 1, 2, 4, 5 \pmod{12}, \\
-1 & \text{if } n \equiv 7, 8, 10, 11 \pmod{12}.
\end{cases}$$

Theorem 3. Let $(h_n)$ be an elliptic divisibility sequence with initial values

$$[1h_20h_4]$$

with $h_2, h_4 \neq 0$.

1. If $n \equiv 1, 8 \pmod{9}$, then $h_n = C$.
2. If $n \equiv 2, 7 \pmod{9}$, then $h_n = C$ iff $h_2 = C$.
3. If $n \equiv 4, 5 \pmod{9}$, then $h_n = C$ iff $h_4 = C$.

Proof. For part 1, let $n \equiv 1$ or $8 \pmod{9}$, then $k = 3r$ or $3r + 2$ $(r, k \in \mathbb{N})$, on substituting these equations into (3), we have

$$h_n = \varepsilon h_4^{(9r^2+3r)/2} h_2^{(9r^2-3r)/2} \quad \text{and} \quad h_n = \varepsilon h_4^{(9r^2+15r+6)/2} h_2^{(9r^2+21r+12)/2}.$$
respectively, hence, \( h_n = C \).

For part 2, let \( n \equiv 2 \) or \( 7 \) (mod \( 9 \)), then \( k = 3r \) or \( 3r + 2 \) (\( r, k \in \mathbb{N} \)). Putting these equations in (3), we have

\[
h_n = \varepsilon h_4 (9r^2 + 3r + 1)/2 h_2 (9r^2 + 3r)/2 \quad \text{and} \quad h_n = \varepsilon h_4 (9r^2 + 9r + 2)/2 h_2 (9r^2 + 9r + 2)/2
\]

respectively, so \( h_n = C \) iff \( h_2 = C \).

Next for part 3, let \( n \equiv 4 \) or \( 5 \) (mod \( 9 \)), then \( k = 3r + 1 \) (\( r, k \in \mathbb{N} \)) in these two cases and so,

\[
h_n = \varepsilon h_4 (9r^2 + 3r + 1)/2 h_2 (9r^2 + 3r)/2 \quad \text{and} \quad h_n = \varepsilon h_4 (9r^2 + 9r + 2)/2 h_2 (9r^2 + 15r + 6)/2
\]

by (3). Therefore, \( h_n = C \) iff \( h_4 = C \). □

As particular cases of the preceding results and Theorem 3 we deduce the following corollary.

**Corollary 2.** Let \((h_n)\) be an elliptic divisibility sequence with initial values

\[ [1h_20h_4] \]

with \( h_2, h_4 \neq 0 \).

1. If \( h_2, h_4 = C \), then \( h_n = C \), for all \( n \).
2. If \( h_2, h_4 \neq C \), then \( h_n = C \) iff \( n \equiv 1, 8 \) (mod \( 9 \)).
3. If \( h_2 = C \), and \( h_4 \neq C \), then \( h_n = C \) iff \( n \equiv 1, 2, 7, 8 \) (mod \( 9 \)).
4. If \( h_2 \neq C \), and \( h_4 = C \), then \( h_n = C \) iff \( n \equiv 1, 4, 5, 8 \) (mod \( 9 \)).

### 2.3. Sequences whose fourth term is zero

Let \((h_n)\) be an elliptic divisibility sequence with fourth term being zero. By Theorem 1, we know that if \( h_4 = 0 \) then \( h_{4n} = 0 \) for all integers \( n \neq 0 \). In [6] we proved that if \((h_n)\) is an elliptic divisibility sequence with initial values \([1h_2h_30]\) \( (h_2, h_3 \neq 0) \) then \((h_n)\) is given by

\[
h_n = h_{4k+a} = \varepsilon h_2^\alpha h_3^\beta (2k^2 + ak + \alpha),
\]

where \( a \equiv n \) (mod \( 4 \)) and

\[
\varepsilon = \begin{cases} +1 & \text{if } n \equiv 1, 2, 3 \text{ (mod } 8), \\ -1 & \text{if } n \equiv 5, 6, 7 \text{ (mod } 8), \end{cases}
\]

\[
\alpha = \frac{1}{2} a^2 - \frac{3}{2} a + 1 \quad \text{and} \quad \beta = \begin{cases} 1 & \text{if } 2 \mid n, \\ 0 & \text{if } 2 \nmid n. \end{cases}
\]

**Theorem 4.** Let \((h_n)\) be an elliptic divisibility sequence with initial values

\[ [1h_2h_30] \]
with \( h_2, h_3 \neq 0 \).

1. If \( n \equiv 1, 5 \pmod{6} \), then \( h_n = C \).
2. If \( n \equiv 2, 4 \pmod{6} \) and \( n \neq 4k \) (\( k \in \mathbb{N} \)), then \( h_n = C \) iff \( h_2 = C \).
3. If \( n \equiv 3 \pmod{6} \), then \( h_n = C \) iff \( h_3 = C \).
4. If \( n \equiv 0 \pmod{6} \) and \( n \neq 12k \) (\( k \in \mathbb{N} \)), then \( h_n = C \) iff \( h_2h_3 = C \).

Proof. For part 1, let \( n \equiv 1 \pmod{6} \) either \( k = 3r/2 \) for some even integer \( r \) or \( k = (3r-1)/2 \) for some odd integer \( r \). On substituting this into (4), we have

\[
h_n = \varepsilon h_3^{(9r^2+3r)/2}
\]

and if \( n \equiv 5 \pmod{6} \), then \( k = (3r+2)/2 \) for some even integer \( r \) or \( k = (3r+1)/2 \) for some odd integer \( r \), and

\[
h_n = \varepsilon h_3^{(9r^2+15r+6)/2}.
\]

Hence, \( h_n = C \).

For part 2, let \( n \equiv 2 \pmod{6} \) and \( n \neq 4k \) with \( k \in \mathbb{N} \) then \( k = 3r/2 \) for some even integer \( r \). Putting this into (4), we have

\[
h_n = \varepsilon h_2h_3^{(9r^2+6r)/2},
\]

and if \( n \equiv 4 \pmod{6} \), then \( k = (3r+1)/2 \) for some odd integer \( r \), and so

\[
h_n = \varepsilon h_2h_3^{(9r^2+12r+3)/2}.
\]

Hence \( h_n = C \) iff \( h_2 = C \).

For part 3, \( n \equiv 3 \pmod{6} \), then \( k = (3r+1)/2 \) for some odd integer \( r \) or \( k = 3r/2 \) for some even integer \( r \), and so

\[
h_n = \varepsilon h_3^{(9r^2+9r+2)/2}
\]

by (4). Hence, \( h_n = C \) iff \( h_3 = C \).

For part 4, let \( n \equiv 0 \pmod{6} \) and \( n \neq 12k \) with \( k \in \mathbb{N} \) then \( k = 3r+1 \) for some odd integer \( r \). Hence

\[
h_n = \varepsilon h_2h_3^{18r^2+18r+4}
\]

by (4), that is \( h_n = C \) iff \( h_2h_3 = C \). \( \square \)

As particular cases of the preceding results and Theorem 4 we deduce the following corollary.

**Corollary 3.** Let \((h_n)\) be an elliptic divisibility sequence with initial values

\[ [1h_2h_30] \]

with \( h_2, h_3 \neq 0 \).

1. If \( h_2, h_3 = C \), then \( h_n = C \), for all \( n \).
2. If \( h_2, h_3 \neq C \), then \( h_n = C \) iff \( n \equiv 1, 5, 7, 11 \pmod{12} \).
3. If $h_2 = C$, and $h_3 \neq C$, then $h_n = C$ iff $n \equiv 1, 2, 4, 5, 7, 10, 11 \pmod{12}$.
4. If $h_2 \neq C$, and $h_3 = C$, then $h_n = C$, iff $n \equiv 1, 3, 5, 7, 9, 11 \pmod{12}$.

2.4. Sequences whose fifth term is zero

Let $(h_n)$ be an elliptic divisibility sequence with fifth term zero. By Theorem 1, we know that if $h_5 = 0$ then $h_{5n} = 0$ for all integers $n \neq 0$. In [6] we proved that if $(h_n)$ is an elliptic divisibility sequence with initial values $[1h_2h_3h_4]$ $(h_2, h_3, h_4 \neq 0)$ and with fifth term zero then $(h_n)$ is given by

\[ h_n = h_{5k+a} = \varepsilon h_3^{5k^2+2ak+a} h_2^{-(5k^2+2ak+b)} \]

where $a \equiv n \pmod{5}$ and

\[ \varepsilon = \begin{cases} +1 & \text{if } n \equiv 1, 2, 3, 4 \pmod{10}, \\ -1 & \text{if } n \equiv 6, 7, 8, 9 \pmod{10}, \end{cases} \]

\[ \alpha = \frac{1}{2}a^2 - \frac{3}{2}a + 1 \quad \text{and} \quad \beta = a^2 - 4a + 3. \]

Theorem 5. Let $(h_n)$ be an elliptic divisibility sequence with initial values

$[1h_2h_3h_4]$,

where $h_2, h_3, h_4 \neq 0$ and with fifth term zero.

1. If $n \equiv 1, 4, 11, 14 \pmod{15}$, then $h_n = C$.
2. If $n \equiv 2, 7, 8, 13 \pmod{15}$, then $h_n = C$ iff $h_2 = C$.
3. If $n \equiv 3, 12 \pmod{15}$, then $h_n = C$ iff $h_3 = C$.
4. If $n \equiv 6, 9 \pmod{15}$, then $h_n = C$ iff $h_3 = C h_2$.

Proof. We prove parts 2 and 4. Other parts can be seen in the same way.

For part 2, let $n \equiv 2 \pmod{15}$, then $k = 3r$ $(r, k \in \mathbb{N})$. Putting this in (5), we have

\[ h_n = \varepsilon h_3^{45r^2+12r} h_2^{-45r^2-12r+1} \]

and similarly if $n \equiv 7, 8, 13 \pmod{15}$, then

\[ h_n = \varepsilon h_3^{45r^2+42r+9} h_2^{-45r^2-42r-8}, \quad h_n = \varepsilon h_3^{45r^2+48r+12} h_2^{-45r^2-48r-11} \]

and

\[ h_n = \varepsilon h_3^{45r^2+78r+33} h_2^{-45r^2-78r-32}. \]

respectively. Hence, $h_n = C$ iff $h_2 = C$.

For part 4, let $n \equiv 6, 9 \pmod{15}$, then by (5) we have

\[ h_n = \varepsilon h_3^{45r^2+36r+7} h_2^{-45r^2-36r-7} \quad \text{and} \quad h_n = \varepsilon h_3^{45r^2+54r+16} h_2^{-45r^2-54r-16} \]

respectively. So, $h_n = C$ iff $h_3 = h_2 C$. □
As particular cases of the preceding results and Theorem 5 we deduce the following corollary.

**Corollary 4.** Let \((h_n)\) be an elliptic divisibility sequence with initial values
\[1h_2h_3h_4],\]
where \(h_2, h_3, h_4 \neq 0, h_2 | h_3\) and with fifth term zero.

1. If \(h_2, h_3 = C\), then \(h_n = C\), for all \(n\).
2. If \(h_2, h_3 \neq C\), then \(h_n = C\) iff
\[n \equiv 1, 4, 11, 14 \pmod{15}\]
3. If \(h_2 = C\), and \(h_3 \neq C\), then \(h_n = C\) iff \(n \equiv 1, 2, 4, 7, 8, 11, 13, 14 \pmod{15}\).
4. If \(h_2 \neq C\), and \(h_3 = C\), then \(h_n = C\) iff \(n \equiv 1, 3, 4, 11, 12, 14 \pmod{15}\).

2.5. **Sequences whose sixth term is zero**

Let \((h_n)\) be an elliptic divisibility sequence with sixth term zero. By Theorem 1, we know that if \(h_6 = 0\) then \(h_{6n} = 0\) for all integers \(n \neq 0\). In [6] we proved that if \((h_n)\) is an elliptic divisibility sequence with initial values \([1h_2h_3h_4]\) \((h_2, h_3, h_4 \neq 0)\) and with sixth term zero then \((h_n)\) is given by
\[h_n = h_{6k+a} = \varepsilon h_2^\alpha h_3^\beta c^{3k^2+ak+\gamma},\]
where \(a \equiv n \pmod{6}\),
\[\varepsilon = \begin{cases} +1 & \text{if } n \equiv 1, 2, 3, 4, 5 \pmod{12}, \\ -1 & \text{if } n \equiv 7, 8, 9, 10, 11 \pmod{12} \end{cases} \]
and
\[\alpha = \begin{cases} 1 & \text{if } 2 \mid n, \\ 0 & \text{if } 2 \nmid n, \end{cases} \quad \beta = \begin{cases} 1 & \text{if } 3 \mid n, \\ 0 & \text{if } 3 \nmid n, \end{cases} \quad \gamma = \begin{cases} 0 & \text{if } a \leq 3, \\ a-3 & \text{if } a > 3. \end{cases} \]

**Theorem 6.** Let \((h_n)\) be an elliptic divisibility sequence with initial values
\[1h_2h_3h_4],\]
where \(h_2h_3h_4 \neq 0, h_4 = ch_2\) for some \(c \in \mathbb{N}\) and with sixth term zero.

1. If \(n \equiv 1, 17 \pmod{18}\), then \(h_n = C\).
2. If \(n \equiv 5, 7, 11, 13 \pmod{18}\), then \(h_n = C\) iff \(c = C\).
3. If \(n \equiv 2, 16 \pmod{18}\), then \(h_n = C\) iff \(h_2 = C\).
4. If \(n \equiv 3, 9, 15 \pmod{18}\), then \(h_n = C\) iff \(h_3 = C\).
5. If \(n \equiv 4, 14 \pmod{18}\), then \(h_n = C\) iff \(h_4 = C\).
6. If \(n \equiv 8, 10 \pmod{18}\), then \(h_n = C\) iff \(ch_4 = C\).
Proof. We prove parts 2 and 4. Other parts can be seen in the same way.

Let \( n \equiv 5 \pmod{18} \), then \( k = 3r \ (r, k \in \mathbb{N}) \). Putting this in (6), we have

\[
h_n = \varepsilon c^{27r^2 + 15r + 2}
\]

and similarly if \( n \equiv 7, 11, 13 \pmod{18} \), then

\[
h_n = \varepsilon c^{27r^2 + 21r + 4}, \quad h_n = \varepsilon c^{27r^2 + 33r + 10}, \quad h_n = \varepsilon c^{27r^2 + 39r + 14}
\]

respectively. Hence, \( h_n = C \) iff \( c = C \).

For part 4, let \( n \equiv 3 \pmod{18} \), then \( k = 3r \ (r, k \in \mathbb{N}) \) and by (6) we have

\[
h_n = \varepsilon h_3c^{27r^2 + 9r},
\]

and similarly if \( n \equiv 9, 15 \pmod{18} \), then

\[
h_n = \varepsilon h_3c^{27r^2 + 27r + 6}, \quad h_n = \varepsilon h_3c^{27r^2 + 45r + 18}
\]

respectively. Hence, \( h_n = C \) iff \( h_3 = C \). \( \square \)

As particular cases of the preceding results and Theorem 6 we deduce the following corollary.

**Corollary 5.** Let \( (h_n) \) be an elliptic divisibility sequence with initial values

\[[h_2h_3h_4],
\]

where \( h_2h_3h_4 \neq 0, \ h_4 = ch_2 \) for some \( c \in \mathbb{N} \) and with sixth term zero.

1. If \( h_2, h_3, c = C \) then \( h_n = C \), for all \( n \).

2. If \( h_2, h_3, c \neq C \), then \( h_n = C \) iff \( n \equiv 1, 17 \pmod{18} \) if \( h_4 = C \) but \( n \equiv 4, 14 \pmod{18} \) if \( ch_4 = C \).

3. If \( h_2 = C \), and \( h_3, c \neq C \), then \( h_n = C \) iff \( n \equiv 1, 2, 16, 17 \pmod{18} \).

4. If \( h_2, h_3 = C \), and \( c \neq C \), then \( h_n = C \) iff \( n \equiv 1, 2, 3, 9, 15, 16, 17 \pmod{18} \).

5. If \( h_2, c = C \), and \( h_3 \neq C \), then \( h_n = C \) iff \( n \equiv 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17 \pmod{18} \).

6. If \( h_3 = C \), and \( h_2, c \neq C \), then \( h_n = C \) iff \( n \equiv 1, 3, 9, 15, 17 \pmod{18} \).

7. If \( h_3, c = C \), and \( h_2 \neq C \), then \( h_n = C \) iff \( n \equiv 1, 3, 5, 7, 9, 11, 13, 15, 17 \pmod{18} \).

8. If \( c = C \), and \( h_2, h_3 \neq C \), then \( h_n = C \) iff \( n \equiv 1, 5, 7, 11, 13, 17 \pmod{18} \).

**Acknowledgement.** The authors would like to thank the referee for suggestions which have improved the presentation of this paper.
REFERENCES


Received 14 April 2010

Układ University
Faculty of Science
Department of Mathematics
Görük, 16059, Bursa – Turkey
betulgezer@uludag.edu.tr
obizim@uludag.edu.tr