The conventional Hahn-Banach extension theorem based on vector space has been widely used to obtain many important and interesting results in nonlinear analysis, vector optimization and mathematical economics. In this paper, we consider the nonstandard vector space in which the concept of inverse element will not be assumed. In this case, the Hahn-Banach extension theorems based on the concepts of nonstandard vector space and nonstandard normed space can still be created.

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Key words: nonstandard vector space, nonstandard normed space, nonstandard sublinear functional, null set, Hahn-Banach theorem.

1. INTRODUCTION

It is well-known that the topic of functional analysis is based on the vector space by referring to the monographs [1–5, 9]. However, there are some interesting sets that cannot form the real vector spaces. For example, the set of all closed intervals cannot form a real vector space, which can refer to Wu [6]. Also, the set of all fuzzy elements cannot form a real vector space, which can refer to Wu [7]. The main reason is that there are no additive inverse elements for the above-mentioned spaces. In this case, the conventional Hahn-Banach extension theorem cannot hold true in the space of all closed intervals and the space of all fuzzy elements. Fortunately, under the weaker definition of vector space, which is called the nonstandard vector space, Wu [6, 7] had presented the Hahn-Banach extension theorems for the nonstandard vector spaces consisting of all closed intervals and the nonstandard vector spaces consisting of all fuzzy elements.

Of course, there are other spaces that may not have the inverse elements. For example, in set-valued analysis, the family of all closed subsets of a topological vector space cannot have the inverse elements, which means that the conventional Hahn-Banach extension theorem cannot hold true in this space. Inspired by this motivation, we are going to propose the general concept of
nonstandard vector space which will be weaker than the conventional vector space in the sense of axioms, and to derive the Hahn-Banach extension theorem based on this general space. Fortunately, this expectation has been done in Wu [8].

Since the concept of inverse elements is not adopted in the nonstandard vector space, every thing will become complicated and non-intuitive. The main contribution of this paper is to establish the different versions of Hahn-Banach extension theorems based on the concepts of nonstandard vector space and nonstandard normed space, which can strengthen the results obtained in Wu [8]. For example, the sufficient conditions in the main Hahn-Banach extension thereom in Wu ([8], Theorem 4.1) are different from the sufficient conditions in Theorem 5.4 in this paper. On the other hand, although the different versions of Hahn-Banach extension theorems based on the interval space have been derived in Wu [6], the sufficient conditions are much neater than that of the sufficient conditions adopted in this paper, since the interval space has owned nice structure which can simplify the sufficient conditions. In other words, the Hahn-Banach extension theorems generalizes the results obtained in Wu [6].

As we shall see in the context of this paper, although the sufficient conditions for the Hahn-Banach extension theorems seem complicated, it is not hard to check that if the nonstandard vector space proposed in this paper turns into the conventional vector space, then the sufficient conditions will be satisfied automatically. In other words, the Hahn-Banach extension theorems established in this paper indeed generalizes the conventional Hahn-Banach extension theorem.

In Sections 2 and 3, the concepts of nonstandard vector space and nonstandard normed space are proposed. Also many interesting and useful results are also provided in order to derive the Hahn-Banach extension theorems. In Section 4, we also introduce the concept of nonstandard sublinear functional. Of course, in Sections 5 and 6, we present many versions of Hahn-Banach extension theorems based on the concepts of nonstandard vector space and nonstandard normed space, respectively.

2. NONSTANDARD VECTOR SPACES

Let $X$ be a universal set, and let $\mathbb{F}$ denote a scalar field, where $\mathbb{F}$ can be field of real numbers or complex numbers. We assume that the universal set $X$ is endowed with the vector addition $x \oplus y$ and scalar multiplication $\alpha x$ for any $x, y \in X$ and $\alpha \in \mathbb{F}$. We also assume that $X$ is closed under the vector addition and scalar multiplication. In this case, we call $X$ as a universal set over $\mathbb{F}$. In the conventional vector space over $\mathbb{F}$, the additive inverse element of
$x$ is denoted by $-x$, and it can also be shown that $-x = -1x$. Here, we shall not consider the concept of inverse element. However, for convenience, we still adopt $-x = -1x$.

Let $Y$ be a subset of $X$. We say that $Y$ is closed under the vector addition if $y_1, y_2 \in Y$ implies $y_1 \oplus y_2 \in Y$. We say that $Y$ is closed under scalar multiplication if $\alpha y \in Y$ for any $\alpha \in \mathbb{F}$ and $y \in Y$.

For $x, y \in X$, the subtraction $x \ominus y$ is defined by $x \ominus y = x \oplus (-y)$, where $-y$ means the scalar multiplication $(-1)y$. For any $x \in X$ and $\alpha \in \mathbb{F}$, since $-\alpha x$ means $-(\alpha x)$, we have to mention that $(-\alpha)x \neq -\alpha x$ and $\alpha (-x) \neq -\alpha x$ in general, unless $\alpha (\beta x) = (\alpha \beta)x$ for any $\alpha, \beta \in \mathbb{F}$. Here, this law will not always be assumed to be true.

- We say that the commutative law for vector addition holds true in $X$ if $x \oplus y = y \oplus x$ for any $x, y \in X$.
- We say that the associative law for vector addition holds true in $X$ if $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for any $x, y, z \in X$.

Since the universal set $X$ over $\mathbb{F}$ can just own the vector addition and scalar multiplication, in general, the universal set $X$ cannot own the zero element. The set $\Omega = \{x \ominus x : x \in X\}$ is called the null set of $X$. Therefore, the null set can be regarded as kind of zero elements of $X$. Now, we are in a position to define the concept of nonstandard vector space.

**Definition 2.1.** Let $X$ be a universal set over $\mathbb{F}$. We say that $X$ is a nonstandard vector space over $\mathbb{F}$ if the following conditions are satisfied:

(i) $1x = x$ for any $x \in X$;

(ii) $x = y$ implies $x \oplus z = y \oplus z$ and $\alpha x = \alpha y$ for any $x, y, z \in X$ and $\alpha \in \mathbb{F}$.

(iii) the commutative and associative laws for vector addition hold true in $X$.

Let $X$ be a universal set over $\mathbb{F}$. More laws about the vector addition and scalar multiplication can be defined below.

- We say that the distributive law for vector addition holds true in $X$ if $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ for any $x, y \in X$ and $\alpha \in \mathbb{F}$.
- We say that the positively distributive law for vector addition holds true in $X$ if $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ for any $x, y \in X$ and $\alpha > 0$.
- We say that the associative law for scalar multiplication holds true in $X$ if $\alpha(\beta x) = (\alpha \beta)x$ for any $x \in X$ and $\alpha, \beta \in \mathbb{F}$.
- We say that the associative law for positive scalar multiplication holds true in $X$ if $\alpha(\beta x) = (\alpha \beta)x$ for any $x \in X$ and $\alpha, \beta > 0$.
- We say that the distributive law for scalar addition holds true in $X$ if $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha, \beta \in \mathbb{F}$.
- We say that the distributive law for positive scalar addition holds true in $X$ if $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha, \beta > 0$. 

We say that the distributive law for negative scalar addition holds true in $X$ if $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha, \beta < 0$.

It is easy to see that if the distributive law for positive and negative scalar addition hold true in $X$, then $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha \beta > 0$. It is not hard to see that the distributive law for scalar addition does not hold true in $I$ (the set of all closed intervals). However, the distributive law for positive and negative scalar addition hold true in $I$.

Let $X$ be a nonstandard vector space over $F$ and $Y$ be a subset of $X$. We say that $Y$ is a subspace of $X$ if $Y$ is closed under the vector addition and scalar multiplication, i.e., $x \oplus y \in Y$ and $\alpha x \in Y$ for any $x, y \in Y$ and $\alpha \in F$.

For any $x \in X$, since the distributive law $(\alpha + \beta)x = \alpha x \oplus \beta x$ does not hold true in general, it says that $\alpha_1 x \oplus \cdots \oplus \alpha_n x \neq (\alpha_1 + \cdots + \alpha_n)x$ in general. Therefore, we need to carefully interpret the concept of linear combination.

Let $X$ be a nonstandard vector space over $F$ and let $\{x_1, \cdots, x_n\}$ be a finite subset of $X$. A linear combination of $\{x_1, \cdots, x_n\}$ is an expression of the form $y_1 \oplus \cdots \oplus y_m$, where $y_i = \alpha_i x_k$ for some $k \in \{1, \cdots, n\}$ and the coefficients $\alpha_i \in F$ for $i = 1, \cdots, m$. We allow $y_i = \alpha_i x_k$ and $y_j = \alpha_j x_k$ for the same $x_k$. In this case, we see that $y_i \oplus y_j = \alpha_i x_k \oplus \alpha_j x_k \neq (\alpha_i + \alpha_j)x_k$ in general. For any nonempty subset $S$ of $X$, the set of all linear combinations of finite subsets of $S$ is called the span of $S$, which is also denoted by $\text{span}(S)$. Then we see that $S \subseteq \text{span}(S)$. Wu [8] has shown that $\text{span}(S)$ is a subspace of $X$.

3. NONSTANDARD NORMED SPACES

Let $X$ be a nonstandard vector space over $F$ with the null set $\Omega$. Many kinds of nonstandard normed spaces can be proposed below.

Definition 3.1. Let $X$ be a nonstandard vector space over $F$. For the nonnegative real-valued function $\| \cdot \| : X \to \mathbb{R}$, we consider the following conditions:

(i) $\| \alpha x \| = |\alpha| \| x \|$ for any $x \in X$ and $\alpha \in F$;

(ii') $\| \alpha x \| = |\alpha| \| x \|$ for any $x \in X$ and $\alpha \in F$ with $\alpha \neq 0$.

(ii) $\| x \oplus y \| \leq \| x \| + \| y \|$ for any $x, y \in X$.

(iii) $\| x \| = 0$ implies $x \in \Omega$.

Different kinds of nonstandard normed spaces are defined below.

• We say that $(X, \| \cdot \|)$ is a nonstandard pseudo-seminormed space if conditions (i') and (ii) are satisfied.
We say that \((X, \| \cdot \|)\) is a nonstandard seminormed space if conditions (i) and (ii) are satisfied.

We say that \((X, \| \cdot \|)\) is a nonstandard pseudo-normed space if conditions (i'), (ii) and (iii) are satisfied.

We say that \((X, \| \cdot \|)\) is a nonstandard normed space if conditions (i), (ii) and (iii) are satisfied.

We say that the norm \(\| \cdot \|\) satisfies the null condition if condition (iii) is replaced by \(\| x \| = 0\) if and only if \(x \in \Omega\). Now we consider the following conditions:

(iv) \(\| x \oplus \omega \| \geq \| x \|\) for any \(x \in X\) and \(\omega \in \Omega\).

(iv') \(\| x \oplus \omega \| = \| x \|\) for any \(x \in X\) and \(\omega \in \Omega\).

We say that the norm \(\| \cdot \|\) satisfies the null inequality (resp. null equality) if condition (iv) (resp. (iv')) is satisfied.

Remark 3.1. We have the following observations.

(i) If the norm \(\| \cdot \|\) satisfies the null inequality and null condition, then the norm \(\| \cdot \|\) also satisfies the null equality, since \(\| x \oplus \omega \| \leq \| x \| + \| \omega \| = \| x \|\) for any \(\omega \in \Omega\).

(ii) If the norm \(\| \cdot \|\) satisfies the null condition, then \(\Omega\) is closed under the scalar multiplication, since \(\| \alpha \omega \| = |\alpha| \cdot \| \omega \| = 0\) by condition (i) in Definition 3.1.

Proposition 3.1 (Wu [8]). The following statements hold true.

(i) Let \(X\) be a nonstandard normed space such that the norm \(\| \cdot \|\) satisfies the null inequality. Assume that there exists \(\bar{x} \in X\) such that \(\| \omega \oplus 0\bar{x} \| = 0\) for any \(\omega \in \Omega\). Then \(\| x \| = 0\) if and only if \(x \in \Omega\), i.e., the norm \(\| \cdot \|\) satisfies the null condition.

(ii) Let \(X\) be a nonstandard normed space such that the norm \(\| \cdot \|\) satisfies the null equality. Then the norm \(\| \cdot \|\) satisfies the null condition.

(iii) Let \((X, \| \cdot \|)\) be a nonstandard pseudo-normed space satisfies the null inequality and null condition. Then it also satisfies the null equality.

(iv) Let \((X, \| \cdot \|)\) be a nonstandard normed space such that the norm \(\| \cdot \|\) satisfies the null inequality. Then the norm \(\| \cdot \|\) satisfies the null equality if and only if the norm \(\| \cdot \|\) satisfies the null condition.

Definition 3.2. Let \(X\) and \(Y\) be nonstandard vector spaces over the same scalar field \(\mathbb{F}\) and let \(T\) be a function from \(X\) into \(Y\).

(i) We say that \(T\) is a pseudo-linear operator if

\[ T(x_1 \oplus x_2) = T(x_1) \oplus T(x_2) \text{ and } T(\alpha x) = \alpha T(x) \]

for \(x_1, x_2, x \in X\) and \(\alpha \in \mathbb{F}\) with \(\alpha \neq 0\).
(ii) We say that $T$ is a linear operator if $T$ is a pseudo-linear operator and $T(0x) = 0T(x)$ for any $x \in X$.

(iii) If $Y$ is taken as the scalar field $\mathbb{F}$, then the pseudo-linear operator and linear operator are called pseudo-linear functional and linear functional, respectively.

**Proposition 3.2 (Wu [8])**. Let $X$ and $Y$ be two nonstandard vector spaces over the same scalar field $\mathbb{F}$ and let $T$ be a pseudo-linear operator from $X$ into $Y$. The following statements hold true.

(i) We have $T(\omega) \in \Omega_Y$ for any $\omega \in \Omega_X$, where $\Omega_X$ and $\Omega_Y$ are the null sets of $X$ and $Y$, respectively.

(ii) If $f$ is a pseudo-linear functional, then $f(\omega) = 0$ for any $\omega \in \Omega_X$.

**Definition 3.3**. Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be nonstandard pseudo-seminormed spaces. The pseudo-linear operator $T : X \to Y$ is said to be bounded if there exists nonnegative real number $c$ such that

\[
\| T(x) \|_Y \leq c \| x \|_X
\]

for all $x \in X$. We assume further that the norm $\| \cdot \|_X$ satisfies the null equality. Suppose that $X \setminus \Omega_X \neq \emptyset$. Inspired by part (i) of Proposition 3.1, the norm of operator $T$ is denoted by $\| T \|$ and is defined by

\[
\| T \| = \sup_{x \in X \setminus \Omega_X} \frac{\| T(x) \|_Y}{\| x \|_X}.
\]

Throughout this paper, when we consider the nonstandard normed space $(X, \| \cdot \|_X)$, we implicitly assume that $X \setminus \Omega_X \neq \emptyset$.

**Proposition 3.3 (Wu [8])**. Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be nonstandard normed spaces such that the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ satisfy the null equality. Suppose that $T : X \to Y$ is a bounded pseudo-linear operator. Then we have

\[
\| T(x) \|_Y \leq \| T \| \cdot \| x \|_X.
\]

**Proposition 3.4 (Wu [8])**. Let $(X, \| \cdot \|_X)$ be a nonstandard normed space such that the norm $\| \cdot \|$ satisfies the null equality, and let $(Y, \| \cdot \|_Y)$ be a nonstandard pseudo-seminormed space. Suppose that $T : X \to Y$ is a bounded pseudo-linear operator. The norm of $T$ can be rewritten as

\[
\| T \| = \sup_{x \in X \setminus \Omega_X} \| T(x) \|_Y = \sup_{\| x \|_X = 1} \| T(x) \|_Y.
\]

\[
\| T \| = \sup_{x \in X \setminus \Omega_X} \| T(x) \|_Y = \sup_{\| x \|_X \leq 1} \| T(x) \|_Y.
\]
4. NONSTANDARD SUBLINEAR FUNCTIONALS

The Hahn-Banach extension theorem concerns the extension of linear functional dominated by a sublinear functional. Since the concept of non-standard vector space is considered here, we shall propose many concepts of sublinear functionals on the nonstandard vector space over $\mathbb{F}$.

Definition 4.1. Let $p$ be a real-valued function defined on a nonstandard vector space $X$ over $\mathbb{F}$.

(i) We say that $p$ is pseudo-sublinear if, for all $x, y \in X$ and $\lambda > 0$, $p(\lambda x) = \lambda p(x)$ and $p(x \oplus y) \leq p(x) + p(y)$.

(ii) We say that $p$ is sublinear if, for all $x, y \in X$ and $\lambda \geq 0$, $p(\lambda x) = \lambda p(x)$ and $p(x \oplus y) \leq p(x) + p(y)$.

(iii) We say that $p$ is nonstandard sublinear if $p$ is pseudo-sublinear and satisfies $p(\omega) = 0$ for all $\omega \in \Omega$.

We say that $p$ satisfies the null inequality if $p(x \oplus \omega) \geq p(x)$ for any $x \in X$ and $\omega \in \Omega$.

Remark 4.1. We have the following observations.

(i) Let $p$ be a sublinear or pseudo-sublinear functional defined on a nonstandard vector space $X$ over $\mathbb{F}$ satisfying $\omega \oplus \omega = 2\omega$ for any $\omega \in \Omega$.

- If $p(\omega \oplus \omega) = p(\omega)$ for any $\omega \in \Omega$, then we have $2p(\omega) = p(2\omega) = p(\omega \oplus \omega) = p(\omega)$, which shows that $p(\omega) = 0$, i.e., $p$ is a nonstandard sublinear.

- If $p(\omega \oplus \omega) \leq p(\omega)$ for any $\omega \in \Omega$, then we have $p(\omega) \leq 0$.

- If $p(\omega \oplus \omega) \geq p(\omega)$ for any $\omega \in \Omega$, then we have $p(\omega) \geq 0$.

(ii) Let $p$ be a pseudo-sublinear functional defined on a nonstandard vector space $X$ over $\mathbb{F}$ and satisfy the null inequality. Then, for any $\omega \in \Omega$, we have

$$p(\omega) \leq p(\omega \oplus \omega) \leq p(\omega) + p(\omega),$$

which implies $p(\omega) \geq 0$.

(iii) Let $p$ be a nonstandard sublinear functional defined on a nonstandard vector space $X$ over $\mathbb{F}$ and satisfy the null inequality. Then we see that $p(x \oplus \omega) = p(x)$ for any $x \in X$ and $\omega \in \Omega$, since $p(x \oplus \omega) \leq p(x) + p(\omega) = p(x)$.

5. HAHN-BANACH EXTENSION THEOREMS IN NONSTANDARD VECTOR SPACES

Let $S$ be a nonempty set which is partially ordered by a reflexive and transitive binary relation “$\preceq$” and let $T$ be a nonempty subset of $S$. 
• An element \( x^* \in S \) is called an upper bound of \( T \) if and only if \( x \leq x^* \) for all \( x \in T \).

• An element \( x^* \in S \) is called a lower bound of \( T \) if and only if \( x^* \leq x \) for all \( x \in T \).

• An element \( x^* \in S \) is called a maximal element of \( S \) if and only if \( x^* \in S \) and \( x^* \preceq x \), then \( x^* \preceq x \). We remark that if the binary relation is also antisymmetric, then an element \( x^* \in S \) is a maximal element of \( S \) if and only if \( x \in S \) and \( x^* \preceq x \), then \( x = x^* \).

• An element \( x^* \in S \) is called a minimal element of \( S \) if and only if \( x^* \in S \) and \( x \preceq x^* \), then \( x^* \preceq x \). We remark that if the binary relation is also antisymmetric, then an element \( x^* \in S \) is a minimal element of \( S \) if and only if \( x \in S \) and \( x \preceq x^* \), then \( x = x^* \).

• The set \( S \) is called inductively ordered from above (resp. from below) if and only if each totally ordered subset of \( S \) has an upper (resp. lower) bound.

**Lemma 5.1 (Zorn’s Lemma).** Let \( S \) be a nonempty set which is partially ordered by a reflexive and transitive binary relation. If \( S \) is inductively ordered from above (resp. from below), then \( S \) has at least one maximal (resp. minimal) element.

**Definition 5.1.** Let \( X \) be a nonstandard vector space over \( \mathbb{F} \) and let \( p \) be a real-valued function defined on \( X \). We say that \( p \) satisfies the Hahn-Banach conditions if the following conditions are satisfied:

(i) \( p((-\alpha)x) = p(-(\alpha x)) \) for any \( x \in X \) and \( \alpha < 0 \);

(ii) \( p((-\alpha)x + z) = p(-(\alpha x) + z) \) for any \( x, z \in X \) and \( \alpha < 0 \);

(iii) \( p(z + (\alpha + \beta)x) \leq p(z + \alpha x + \beta x) \) for any \( x, z \in X \) and \( \alpha, \beta > 0 \);

(iv) \( p \) satisfies the positively distributive law for vector addition, i.e., \( p(\alpha(x + y)) = p(\alpha x + \alpha y) \) and \( p(z + \alpha(x + y)) = p(z + \alpha x + \alpha y) \) for any \( x, y, z \in X \) and \( \alpha > 0 \).

In the sequel, we are going to derive many versions of Hahn-Banach theorems.

**Theorem 5.1 (Basic Version of Hahn-Banach Theorem).** Let \( X \) be a nonstandard vector space over \( \mathbb{R} \). For each nonstandard sublinear functional \( p \) on \( X \) satisfying the null inequality and Hahn-Banach conditions, there exists a pseudo-linear functional \( f \) on \( X \) such that \( f \) satisfies the Hahn-Banach conditions and \( f(x) \leq p(x) \) for all \( x \in X \).

**Proof.** We consider the set \( \mathcal{F} \) of all nonstandard sublinear functionals satisfying the null inequality such that each \( h \) in \( \mathcal{F} \) satisfies the Hahn-Banach
conditions and \( h(x) \leq p(x) \) for all \( x \in X \). Then we see that \( F \neq \emptyset \), since \( p \in F \).

We are going to claim that \( F \) has at least one minimal element with respect to the partial ordering \( h_i \leq h_j \) that is defined to be the pointwise ordering, i.e., \( h_i(x) \leq h_j(x) \) for all \( x \in X \), where \( h_i, h_j \in F \). Suppose that \( \{h_i\} \) is a totally ordered subset of \( F \). We define the functional \( h \) on \( X \) by

\[
h(x) = \inf_i h_i(x).
\]

We want to show that \( h \in F \). We firstly need to claim \( h(x) > -\infty \) for all \( x \in X \). Suppose that there exists \( x_0 \in X \) such that \( h(x_0) = -\infty \). Given any real number \( k \), there exists \( j \) such that \( h_j(x_0) < k \), which implies \( h_j(x_0) + h_j(y) < k + h_j(y) \). Therefore, by the subadditivity of \( h_i \), we have

\[
h(x_0 \oplus y) = \inf_i h_i(x_0 \oplus y) \leq \inf_i [h_i(x_0) + h_i(y)]
\]

\[
\leq h_j(x_0) + h_j(y) < k + h_j(y).
\]

Since \( k \) can be sufficiently small real number, we conclude that \( h(x_0 \oplus y) = -\infty \) for any \( y \in X \). Since \( x_0 \oplus x_0 = \omega \in \Omega \) and \( h_i \in F \), we have

\[
h(z) = \inf_i h_i(z) \leq \inf_i h_i(z \oplus \omega) = h(z \oplus \omega)
\]

\[
= h(z \oplus x_0 \oplus x_0) = h(x_0 \oplus (z \oplus x_0)) = -\infty,
\]

which shows \( h(z) = -\infty \) for any \( z \in X \), i.e., \( h(\omega) = -\infty \) for any \( \omega \in \Omega \). Since each \( h_i \) is also nonstandard sublinear, we have \( h_i(\omega) = 0 \), i.e., \( h(\omega) = \inf_i h_i(\omega) = 0 \) for any \( \omega \in \Omega \). A contradiction occurs. Therefore, we conclude that \( h(x) > -\infty \) for all \( x \in X \).

It is obvious that \( h(x) \leq p(x) \), \( h(\lambda x) = \lambda h(x) \), \( h(\omega) = 0 \) and \( h(x \oplus \omega) \geq h(x) \) for any \( x \in X \), \( \omega \in \Omega \) and \( \lambda > 0 \). Since each \( h_i \) satisfies the Hahn-Banach conditions, it is also obvious that \( h \) satisfies the Hahn-Banach conditions. Now, we want to show the subadditivity of \( h \). Since \( h(x) > -\infty \) and \( h(y) > -\infty \), given any \( \epsilon > 0 \), there exists \( j, k \) such that

\[
\inf_i h_i(x) + \epsilon > h_j(x) \text{ and } \inf_i h_i(y) + \epsilon > h_k(y),
\]

which implies

\[
\inf_i h_i(x) + \inf_i h_i(y) + 2\epsilon > h_j(x) + h_k(y)
\]

\[
\geq \begin{cases} 
  h_j(x) + h_j(y) & \text{if } h_j \leq h_k \\
  h_k(x) + h_k(y) & \text{if } h_k \leq h_j 
\end{cases}
\]

\[
\geq \inf_i [h_i(x) + h_i(y)] \geq \inf_i h_i(x \oplus y)
\]

by the subadditivity of \( h_i \). Since \( \epsilon \) can be any positive number, we obtain \( h(x \oplus y) \leq h(x) + h(y) \), which shows the subadditivity of \( h \). Therefore, we
have \( h \in \mathcal{F} \) and \( h \) is also a lower bound of \( \{h_i\} \). This shows that the set \( \mathcal{F} \) is inductively ordered from below with respect to the pointwise ordering. By Zorn’s lemma, the set \( \mathcal{F} \) has at least one minimal element \( f \) in \( \mathcal{F} \), i.e., \( f(x) \leq p(x) \) for all \( x \in X \). Next we are going to show that \( f \) is a pseudo-linear functional on \( X \).

For any fixed \( y \in X \), we define the functional \( g \) on \( X \) by

\[
g(x) = \inf_{\lambda > 0} [f(x + \lambda y) - \lambda f(y)]
\]

for all \( x \in X \). First of all, we need to claim that the real-valued function \( g \) is well-defined, i.e., the infimum exists for each \( x \in X \). For each \( x \in X \) and \( \lambda > 0 \), since \( x \ominus x = \omega \in \Omega \) and \( f \in \mathcal{F} \), we have

\[
\lambda f(y) = f(\lambda y) \leq f(\lambda y + \omega) = f(\lambda y + x \ominus x) \leq f(x + \lambda y) + f(-x).
\]

This says that \(-\infty < -f(-x) \leq f(x + \lambda y) - \lambda f(y)\). By taking infimum for \( \lambda > 0 \), we obtain \(-\infty < -f(-x) \leq g(x) \) for all \( x \in X \). This shows that the real-valued function \( g \) is well-defined.

Next, we want to claim that \( g \) is a nonstandard sublinear functional satisfying the null inequality. For all \( x \in X \) and all \( \mu > 0 \), since \( f \) satisfies the positively distributive law for vector addition, we have

\[
\mu g(x) = \inf_{\lambda > 0} [\mu f(x + \lambda y) - \mu \lambda f(y)] = \inf_{\lambda > 0} [f(\mu x + \lambda y)) - \mu \lambda f(y)]
\]

\[
= \inf_{\lambda > 0} [f(\mu x + \mu \lambda y) - \mu \lambda f(y)] = \inf_{\lambda > 0} \left[ f(\mu x + \lambda y) - \lambda f(y) \right] = g(\mu x).
\]

From Remark 4.1, we also have

\[
g(\omega) = \inf_{\lambda > 0} [f(\omega + \lambda y) - \lambda f(y)] = \inf_{\lambda > 0} [f(\lambda y) - \lambda f(y)]
\]

\[
= \inf_{\lambda > 0} \lambda [f(y) - f(y)] = 0.
\]

Now, we have

\[
g(x \oplus z) = \inf_{\lambda > 0} [f(x \oplus z + \lambda y) - \lambda f(y)]
\]

\[
\leq f(x \oplus z + (\lambda + \mu)y) - (\lambda + \mu)f(y)
\]

\[
\leq f(x \oplus \lambda y + z \ominus \mu y) - \lambda f(y) - \mu f(y)
\]

(by referring to the Hahn-Banach conditions)

\[
\leq (f(x \oplus \lambda y) - \lambda f(y)) + (f(z \ominus \mu y) - \mu f(y))
\]

(by the sublinearity of \( f \)).
By taking infimum on both sides, we have
\[
g(x \oplus z) \leq \inf_{\lambda > 0} \inf_{\mu > 0} [(f(x \oplus \lambda y) - \lambda f(y)) + (f(z \oplus \mu y) - \mu f(y))]
\]
\[
= \inf_{\lambda > 0} (f(x \oplus \lambda y) - \lambda f(y)) + \inf_{\mu > 0} (f(z \oplus \mu y) - \mu f(y)),
\]
which concludes \(g(x \oplus z) \leq g(x) + g(z)\). Since \(f \in \mathcal{F}\), we have
\[
g(x \oplus \omega) = \inf_{\lambda > 0} [f(x \oplus \omega \oplus \lambda y) - \lambda f(y)] \geq \inf_{\lambda > 0} [f(x \oplus \lambda y) - \lambda f(y)] = g(x).
\]
This shows that \(g\) is a nonstandard sublinear functional satisfying the null inequality.

Next, we want to show that \(g\) satisfies the Hahn-Banach conditions. Since \(f\) satisfies the Hahn-Banach conditions, we also have
\[
g((-\alpha)x) = \inf_{\lambda > 0} [f((-\alpha)x \oplus \lambda y) - \lambda f(y)]
\]
\[
= \inf_{\lambda > 0} [f((-\alpha)x) \oplus \lambda y) - \lambda f(y)] = g((-\alpha)x) \quad \text{(where } \alpha < 0)\]
\[
g((-\alpha)x \oplus z) = \inf_{\lambda > 0} [f((-\alpha)x \oplus z \oplus \lambda y) - \lambda f(y)]
\]
\[
= \inf_{\lambda > 0} [f((-\alpha)x) \oplus z \oplus \lambda y) - \lambda f(y)] = g((-\alpha)x \oplus z) \quad \text{(where } \alpha < 0)\]
\[
g(z \oplus (\alpha + \beta)x) = \inf_{\lambda > 0} [f(z \oplus (\alpha + \beta)x \oplus \lambda y) - \lambda f(y)]
\]
\[
\leq \inf_{\lambda > 0} [f(z \oplus \alpha x \oplus \beta x \oplus \lambda y) - \lambda f(y)] = g(z \oplus \alpha x \oplus \beta x) \quad \text{(where } \alpha, \beta > 0)\]
\[
g(\alpha(x \oplus y)) = \inf_{\lambda > 0} [f(\alpha(x \oplus y) \oplus \lambda y) - \lambda f(y)]
\]
\[
= \inf_{\lambda > 0} [f(\alpha x \oplus \alpha y \oplus \lambda y) - \lambda f(y)] = g(\alpha x \oplus \alpha y) \quad \text{(where } \alpha > 0)\]
\[
g(z \oplus \alpha(x \oplus y)) = \inf_{\lambda > 0} [f(z \oplus \alpha(x \oplus y) \oplus \lambda y) - \lambda f(y)]
\]
\[
= \inf_{\lambda > 0} [f(z \oplus \alpha x \oplus \alpha y \oplus \lambda y) - \lambda f(y)] = g(z \oplus \alpha x \oplus \alpha y) \quad \text{(where } \alpha > 0)\).
\]
This shows that \(g\) satisfies the Hahn-Banach conditions. Since
\[
f(x \oplus \lambda y) - \lambda f(y) \leq f(x) + \lambda f(y) - \lambda f(y) = f(x) \leq p(x),
\]
i.e., \(g(x) \leq p(x)\) for all \(x \in X\) (by taking infimum on both sides), we conclude that \(g \in \mathcal{F}\).

From (5), we also have \(g(x) \leq f(x)\) for all \(x \in X\) by taking infimum on both sides. Since \(f\) is the minimal element of \(\mathcal{F}\) and \(g \in \mathcal{F}\), we have \(f(x) \leq g(x)\) for all \(x \in X\). Therefore, we obtain \(g = f\). Furthermore, we have
\[
f(x) = g(x) = \inf_{\lambda > 0} [f(x \oplus \lambda y) - \lambda f(y)] \leq f(x \oplus y) - f(y),
\]
which implies \(f(x) + f(y) \leq f(x \oplus y)\). Since \(f\) is also subadditive, we conclude that \(f\) is additive. Since \(g = f\), i.e., \(f(\lambda x) = \lambda f(x)\) and \(f(\omega) = 0\) for \(\lambda > 0\)
and $\omega \in \Omega$. It remains to show that $f(\lambda x) = \lambda f(x)$ for $\lambda < 0$. For any $x \in X$, we let $\omega = x \ominus x \in \Omega$. Then, by the additivity of $f$, we have

$$0 = f(\omega) = f(x \ominus x) = f(x \oplus (-x)) = f(x) + f(-x),$$

which says that $f(-x) = -f(x)$. Therefore, for $\lambda < 0$, since $f$ satisfies the Hahn-Banach conditions, we have

$$-\lambda f(x) = f((-\lambda)x) = f(-\lambda x) = -f(\lambda x),$$

which implies $f(\lambda x) = \lambda f(x)$. This shows that $f$ is a pseudo-linear functional. \hfill \square

**Corollary 5.1 (Basic Version of Hahn-Banach Theorem).** Let $X$ be a nonstandard vector space over $\mathbb{R}$ such that $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ and $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x, y \in X$ and $\alpha, \beta > 0$, and $(-\lambda)x = -(\lambda x)$ for any $\lambda < 0$. For each nonstandard sublinear functional $p$ on $X$ satisfying the null inequality, there exists a pseudo-linear functional $f$ on $X$ such that $f$ satisfies the Hahn-Banach conditions and $f(x) \leq p(x)$ for all $x \in X$.

**Proof.** Under the assumptions, the Hahn-Banach conditions will be satisfied automatically. Therefore, the results follows from Theorem 5.1 immediately. \hfill \square

It is not hard to see that if $X$ is taken as the set $I$ of all closed intervals, then we automatically have $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ and $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x, y \in X$ and $\alpha, \beta > 0$, and $(-\lambda)x = -(\lambda x)$ for any $\lambda < 0$. Therefore, Corollary 5.1 is applicable for the space $I$.

**Definition 5.2.** Let $X$ be a nonstandard vector space over $\mathbb{R}$ and let $S$ be a subset of $X$. We say that $S$ is nonstandard convex if, for any $a, b \in S$, $\lambda a \oplus (1 - \lambda)b \in S$ for $\lambda \in (0, 1)$. Let $f : S \to \mathbb{R}$ be a real-valued function defined on a nonstandard convex subset $S$. We say that $f$ is nonstandard concave if $f(\lambda x \oplus (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$ for any $x, y \in X$ and $\lambda \in (0, 1)$.

For the nonstandard convexity, it is not necessary to consider the cases of $\lambda = 0$ and $\lambda = 1$.

**Definition 5.3.** Let $X$ be a nonstandard vector space over $\mathbb{F}$ and let $p$ be a real-valued function defined on $X$.

(i) We say that $p$ satisfies the positively distributive law for vector addition if $p(\alpha(x \oplus y)) = p(\alpha x \oplus \alpha y)$ and $p(z \oplus \alpha(x \oplus y)) = p(z \oplus \alpha x \oplus \alpha y)$ for any $x, y, z \in X$ and $\alpha > 0$.

(ii) We say that $p$ satisfies the associative law for positive scalar multiplication if $p(\alpha(\beta x)) = p((\alpha \beta)x)$ and $p(z \oplus \alpha(\beta x)) = p(z \oplus (\alpha \beta)x)$ for any $x, z \in X$ and $\alpha, \beta > 0$. 
(iii) We say that $p$ satisfies the positively sub-distributive law for vector addition if $p(\alpha(x \oplus y)) \leq p(\alpha x \oplus \alpha y)$ and $p(z \oplus \alpha(x \oplus y)) \leq p(z \oplus \alpha x \oplus \alpha y)$ for any $x, y, z \in X$ and $\alpha > 0$.

(iv) We say that $p$ satisfies the sub-associative law for positive scalar multiplication if $p(\alpha(\beta x)) \leq p((\alpha \beta)x)$ and $p(z \oplus \alpha(\beta x)) \leq p(z \oplus (\alpha \beta)x)$ for any $x, z \in X$ and $\alpha, \beta > 0$.

Remark 5.1. Let $X$ be a nonstandard vector space over $\mathbb{F}$ and let $p$ be a real-valued function defined on $X$. We have the following observations.

- If $p$ is a pseudo-sublinear functional, then, for any $\alpha, \beta > 0$, we have $p(\alpha(\beta x)) = \alpha \cdot p(\beta x) = \alpha \beta \cdot p(x) = p((\alpha \beta)x)$.

- If the positively distributive law for vector addition and associative law for positive scalar multiplication hold true in $X$, then the laws in Definition 5.3 are automatically satisfied. We also remark that the positively distributive law for vector addition and associative law for positive scalar multiplication hold true automatically in the space $\mathcal{I}$ of all closed intervals.

Lemma 5.2. Let $S$ be a nonempty subset of a nonstandard vector space $X$ over $\mathbb{R}$. Let $p$ be a pseudo-sublinear functional defined on $X$ satisfying the null inequality, and let $h$ be a given real-valued function defined on $S$ with $h(x) \leq p(x)$ for all $x \in S$. Let $f$ be a real-valued function defined on $X$ by

$$f(x) = \inf_{y \in S, \lambda > 0} [p(x \oplus \lambda y) - \lambda h(y)].$$

Then the following statements hold true.

(i) The real-valued function $f$ satisfies the inequality $f(x) \leq p(x)$ for all $x \in X$.

(ii) If $p(\omega) \leq 0$ for any $\omega \in \Omega$, then, for a pseudo-linear functional $l$ defined on $X$, we have $l(x) \leq p(x)$ for all $x \in X$ and $h(x) \leq l(x)$ for all $x \in S$ if and only if $l(x) \leq f(x)$ for all $x \in X$.

(iii) Suppose that $p$ satisfies the positively distributive law for vector addition and sub-associative law for positive scalar multiplication. If the set $S$ is nonstandard convex and $h$ is nonstandard concave, then $f$ is a pseudo-sublinear functional satisfying the null inequality. If we assume further that $p$ is a nonstandard sublinear functional satisfying the null inequality, then $f$ is also a nonstandard sublinear functional satisfying the null inequality.

Proof. First of all, we need to claim that the real-valued function $f$ is well-defined, i.e., the infimum exists for each $x \in X$. For each $x \in X$, $y \in S$
and \( \lambda > 0 \), we have \( x \oplus x = \omega \in \Omega \) and
\[
\lambda h(y) \leq \lambda p(y) = p(\lambda y) \leq p(\lambda y \oplus \omega) = p(\lambda y \oplus x \oplus x) \leq p(x \oplus \lambda y) + p(-x).
\]

This says that \(-\infty < -p(-x) \leq p(x \oplus \lambda y) - \lambda h(y)\), i.e., the infimum exists. In other words, the real-valued function \( f \) is well-defined.

To prove part (i), by the sublinearity of \( p \), we have
\[
p(x \oplus \lambda y) - \lambda h(y) \leq p(x) + \lambda p(y) - \lambda h(y) = p(x) + \lambda (p(y) - h(y))
\]
for all \( x \in X \), \( y \in S \) and \( \lambda > 0 \). By taking infimum on both sides, we have
\[
\inf_{y \in S, \lambda > 0} [p(x \oplus \lambda y) - \lambda h(y)] \leq p(x) + \inf_{y \in S, \lambda > 0} \lambda (p(y) - h(y)) = p(x),
\]
since \( p(y) - h(y) \geq 0 \) for all \( y \in S \) and \( \lambda > 0 \), which shows that \( f(x) \leq p(x) \) for all \( x \in X \).

To prove part (ii), let \( l \) be a pseudo-linear functional defined on \( X \) such that \( l(x) \leq p(x) \) for all \( x \in X \) and \( h(x) \leq l(x) \) for all \( x \in S \). For \( x \in X \), \( y \in S \) and \( \lambda > 0 \), we let \( \omega = \lambda y \oplus \lambda y \in \Omega \). Then, by Proposition 3.2, we have
\[
l(x) = l(x) + l(\omega) = l(x \oplus \omega) = l(x \oplus \lambda y \oplus \lambda y) = l(x \oplus \lambda y) - \lambda l(y) \leq p(x \oplus \lambda y) - \lambda h(y).
\]

By taking infimum on both sides, we obtain \( l(x) \leq f(x) \) for all \( x \in X \). Conversely, using (i), we immediately have \( l(x) \leq p(x) \) for all \( x \in X \). On the other hand, for some \( \lambda_0 > 0 \) and for all \( x \in S \), we have
\[
-\lambda_0 l(x) = l(-\lambda_0 x) \leq f(-\lambda_0 x) = \inf_{y \in S, \lambda > 0} [p(-\lambda_0 x \oplus \lambda y) - \lambda h(y)]
\]
\[
\leq p(-\lambda_0 x \oplus \lambda_0 x) - \lambda_0 h(x) = p(\omega) - \lambda_0 h(x) \leq -\lambda_0 h(x),
\]
since \( p(\omega) \leq 0 \), which says that \( h(x) \leq l(x) \).

To prove part (iii), for all \( x \in X \) and all \( \mu > 0 \), since \( p \) satisfies the positively distributive law for vector addition, we have
\[
\mu f(x) = \inf_{y \in S, \lambda > 0} [\mu p(x \oplus \lambda y) - \mu \lambda h(y)] = \inf_{y \in S, \lambda > 0} [p(\mu(x \oplus \lambda y)) - \mu \lambda h(y)]
\]
\[
= \inf_{y \in S, \lambda > 0} [p(\mu x \oplus \mu \lambda y) - \mu \lambda h(y)]
\]
\[
= \inf_{y \in S, \lambda > 0} [p(\mu x \oplus \lambda y) - \lambda h(y)] = f(\mu x).
\]

Now, we want to show that \( f \) is subadditive. For any \( u, v \in S \) and \( \lambda, \mu > 0 \), since \( S \) is nonstandard convex, we have
\[
w = \frac{\lambda}{\lambda + \mu} u + \frac{\mu}{\lambda + \mu} v \in S.
\]
Since $p$ satisfies the positively distributive law for vector addition and sub-associative law for positive scalar multiplication, we obtain
\[
f(x \oplus y) = \inf_{z \in S, \eta > 0} \left[ p((x \oplus y) \oplus \eta z) - \eta h(z) \right]
\leq p(x \oplus y \oplus (\lambda + \mu) w) - (\lambda + \mu) h(w)
\quad \text{(by taking $z = w$ and $\eta = \lambda + \mu$)}
\leq p(x \oplus \lambda u \oplus y \oplus \mu v) - \lambda h(u) - \mu h(v)
\quad \text{(since $h$ is nonstandard concave)}
\leq (p(x \oplus \lambda u) - \lambda h(u)) + (p(y \oplus \mu v) - \mu h(v)).
\]

By taking infimum on both sides, we have
\[
f(x \oplus y) \leq \inf_{u \in S, \lambda > 0} \inf_{v \in S, \mu > 0} \left[ (p(x \oplus \lambda u) - \lambda h(u)) + (p(y \oplus \mu v) - \mu h(v)) \right]
= \inf_{u \in S, \lambda > 0} \left[ (p(x \oplus \lambda u) - \lambda h(u)) + \inf_{v \in S, \mu > 0} (p(y \oplus \mu v) - \mu h(v)) \right]
= \inf_{u \in S, \lambda > 0} (p(x \oplus \lambda u) - \lambda h(u)) + \inf_{v \in S, \mu > 0} (p(y \oplus \mu v) - \mu h(v)),
\]
which concludes $f(x \oplus y) \leq f(x) + f(y)$. Finally, we have
\[
f(x \oplus \omega) = \inf_{y \in S, \lambda > 0} \left[ p(x \oplus \omega \oplus \lambda y) - \lambda h(y) \right]
\geq \inf_{y \in S, \lambda > 0} \left[ p(x \oplus \lambda y) - \lambda h(y) \right] = f(x).
\]

Now, we assume that $p$ is a nonstandard sublinear functional satisfying the null inequality. From Remark 4.1, we also have
\[
f(\omega) = \inf_{y \in S, \lambda > 0} \left[ p(\omega \oplus \lambda y) - \lambda h(y) \right] = \inf_{y \in S, \lambda > 0} \left[ p(\lambda y) - \lambda h(y) \right]
= \inf_{y \in S, \lambda > 0} \lambda [p(y) - h(y)] = 0,
\]
since $p(y) - h(y) \geq 0$ for all $y \in S$. This shows that $f$ is also a nonstandard sublinear functional satisfying the null inequality. We complete the proof. \qed

**Theorem 5.2 (Sandwich Version of Hahn-Banach Theorem).** Let $X$ be a nonstandard vector space over $\mathbb{R}$. Let $p$ be a nonstandard sublinear functional on $X$ satisfying the null inequality, Hahn-Banach conditions and sub-associative law for positive scalar multiplication. If $S$ is a nonstandard convex subset of $X$ and $h$ is a nonstandard concave functional on $S$ with $h(x) \leq p(x)$ for all $x \in S$, then there exists a pseudo-linear functional $f$ on $X$ such that $f$ satisfies the Hahn-Banach conditions with $f(x) \leq p(x)$ for all $x \in X$ and $h(x) \leq f(x)$ for all $x \in S$. 
Proof. Let \( q \) be a functional defined on \( X \) by
\[
q(x) = \inf_{y \in S, \lambda > 0} [p(x \oplus \lambda y) - \lambda h(y)].
\] (6)

Then Proposition 5.2 (iii) shows that \( q \) is a nonstandard sublinear functional satisfying the null inequality. Since \( p \) satisfies the Hahn-Banach conditions, it is not hard to see that \( q \) also satisfies the Hahn-Banach conditions. Now applying Theorem 5.1, there exists a pseudo-linear functional \( f \) on \( X \) such that \( f \) satisfies the Hahn-Banach conditions and \( f(x) \leq q(x) \) for all \( x \in X \). Finally, since \( p(\omega) = 0 \) for any \( \omega \in \Omega \), using Proposition 5.2 (ii), we obtain the desired sandwich result. We complete the proof. \( \square \)

**Corollary 5.2 (Sandwich Version of Hahn-Banach Theorem).** Let \( X \) be a nonstandard vector space over \( \mathbb{R} \) such that \( \alpha(x \oplus y) = \alpha x \oplus \alpha y \) and \( (\alpha + \beta)x = \alpha x \oplus \beta x \) for any \( x, y \in X \) and \( \alpha, \beta > 0 \), and \( (-\lambda)x = -(\lambda x) \) for any \( \lambda < 0 \). Let \( p \) be a nonstandard sublinear functional on \( X \) satisfying the null inequality. If \( S \) is a nonstandard convex subset of \( X \) and \( h \) is a nonstandard concave functional on \( S \) with \( h(x) \leq p(x) \) for all \( x \in S \), then there exists a pseudo-linear functional \( f \) on \( X \) such that \( f \) satisfies the Hahn-Banach conditions with \( f(x) \leq p(x) \) for all \( x \in X \) and \( h(x) \leq f(x) \) for all \( x \in S \).

Proof. Under the assumptions, the Hahn-Banach conditions will be satisfied automatically. Therefore, the result follows from Theorem 5.2 immediately. \( \square \)

**Theorem 5.3 (Convex Version of Hahn-Banach Theorem).** Let \( X \) be a nonstandard vector space over \( \mathbb{R} \). Let \( p \) be a nonstandard sublinear functional on \( X \) satisfying the null inequality, Hahn-Banach conditions and sub-associative law for positive scalar multiplication. If \( S \) is a nonstandard convex subset of \( X \) and \( k = \inf_{x \in S} p(x) \), then there exists a pseudo-linear functional \( f \) on \( X \) such that \( f \) satisfies the Hahn-Banach conditions with \( f(x) \leq p(x) \) for all \( x \in X \) and \( \inf_{x \in S} f(x) = \inf_{x \in S} p(x) = k \).

Proof. Suppose that \( k = -\infty \). Theorem 5.1 says that there exists a pseudo-linear functional \( f \) on \( X \) such that \( f \) satisfies the Hahn-Banach conditions and \( f(x) \leq p(x) \) for all \( x \in X \). This shows that
\[
\inf_{x \in S} f(x) \leq \inf_{x \in S} p(x) = -\infty; \text{ that is, } \inf_{x \in S} f(x) = \inf_{x \in S} p(x) = -\infty.
\]

Now, we assume that \( k > -\infty \). Then we define the functional \( h \) on \( S \) by \( h(x) = k \) for all \( x \in S \). Then \( h(x) \leq p(x) \) for all \( x \in S \) and \( h \) is a nonstandard concave functional on \( S \), since \( \alpha k + (1 - \alpha)k = k \) for any \( \alpha \in (0, 1) \). From
Theorem 5.2, there exists a pseudo-linear functional \( f \) on \( X \) such that \( f \) satisfies the Hahn-Banach conditions and \( f(x) \leq p(x) \) for all \( x \in X \) and \( h(x) \leq f(x) \) for all \( x \in S \). Therefore, for \( x \in S \), we have \( k = h(x) \leq f(x) \leq p(x) \). By taking infimum on the inequalities, we obtain

\[
 k \leq \inf_{x \in S} f(x) \leq \inf_{x \in S} p(x) = k.
\]

We complete the proof. \( \square \)

Corollary 5.3 (Convex Version of Hahn-Banach Theorem). Let \( X \) be a nonstandard vector space over \( \mathbb{R} \) such that \( \alpha(x \oplus y) = \alpha x \oplus \alpha y \) and \( (\alpha + \beta)x = \alpha x \oplus \beta x \) for any \( x, y \in X \) and \( \alpha, \beta > 0 \), and \( (-\lambda)x = -(\lambda x) \) for any \( \lambda < 0 \). Let \( p \) be a nonstandard sublinear functional on \( X \) satisfying the null inequality. If \( S \) is a nonstandard convex subset of \( X \) and let \( k = \inf_{x \in S} p(x) \), then there exists a pseudo-linear functional \( f \) on \( X \) such that \( f \) satisfies the Hahn-Banach conditions with \( f(x) \leq p(x) \) for all \( x \in X \) and

\[
 \inf_{x \in S} f(x) = \inf_{x \in S} p(x) = k.
\]

Proof. Under the assumptions, the Hahn-Banach conditions will be satisfied automatically. Therefore, the result follows from Theorem 5.3 immediately. \( \square \)

Theorem 5.4 (Hahn-Banach Extension Theorem). Let \( X \) be a nonstandard vector space over \( \mathbb{R} \). Let \( p \) be a nonstandard sublinear functional on \( X \) satisfying the null inequality, Hahn-Banach conditions and sub-associative law for positive scalar multiplication. Let \( Z \) be a subspace of \( X \). If \( f \) is a pseudo-linear functional defined on \( Z \) with \( f(x) \leq p(x) \) for all \( x \in Z \), then there exists a pseudo-linear functional \( \hat{f} \) on \( X \) such that \( \hat{f} \) satisfies the Hahn-Banach conditions with \( \hat{f}(x) \leq p(x) \) for all \( x \in X \) and \( \hat{f}(x) = f(x) \) for all \( x \in Z \).

Proof. Since \( Z \) is a subspace of \( X \), it says that \( Z \) is also a nonstandard convex subset of \( X \). Since \( f \) is a pseudo-linear functional defined on \( Z \), we also see that \( f \) is a nonstandard concave functional defined on \( Z \). Theorem 5.2 says that there exists a pseudo-linear functional \( \hat{f} \) on \( X \) such that \( \hat{f} \) satisfies the Hahn-Banach conditions and \( \hat{f}(x) \leq p(x) \) for all \( x \in X \) and \( f(x) \leq \hat{f}(x) \) for all \( x \in Z \). For any \( x \in Z \), since \( -x \in Z \), we have

\[
 f(x) \leq \hat{f}(x) = -\hat{f}(-x) \leq -f(-x) = -(f(x)) = f(x)
\]

for all \( x \in Z \). This shows that \( \hat{f}(x) = f(x) \) for all \( x \in Z \). We complete the proof. \( \square \)

We also remark that if \( X \) is a real vector space, then all the assumptions of Theorem 5.4 will be satisfied (it is not hard to check). In this case, Theorem 5.4 is reduced to the conventional Hahn-Banach theorem.
Corollary 5.4 (Hahn-Banach Extension Theorem). Let $X$ be a non-standard vector space over $\mathbb{R}$ such that $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ and $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x, y \in X$ and $\alpha, \beta > 0$, and $(-\lambda)x = -(\lambda x)$ for any $\lambda < 0$. Let $p$ be a nonstandard sublinear functional on $X$ satisfying the null inequality. Let $Z$ be a subspace of $X$. If $f$ is a pseudo-linear functional defined on $Z$ with $f(x) \leq p(x)$ for all $x \in Z$, then there exists a pseudo-linear functional $\hat{f}$ on $X$ such that $\hat{f}$ satisfies the Hahn-Banach conditions with $\hat{f}(x) \leq p(x)$ for all $x \in X$ and $\hat{f}(x) = f(x)$ for all $x \in Z$.

Proof. Under the assumptions, the Hahn-Banach conditions will be satisfied automatically. Therefore, the results follows from Theorem 5.4 immediately. \qed

The following result is useful for investigating the generalized Hahn-Banach extension theorem.

Lemma 5.3. Let $X$ be a nonstandard vector space over $\mathbb{F}$ and let $p$ be a real-valued function defined on $X$ satisfying the following conditions:

(i) $p(x \oplus y) \leq p(x) + p(y)$ for any $x, y \in X$;
(ii) $p(\alpha x) = |\alpha|p(x)$ for any $x \in X$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
(iii) $p(x \oplus \omega) \geq p(x)$ for any $x \in X$ and $\omega \in \Omega$.

Then $p(x) \geq 0$ for all $x \in X$.

Proof. For any $\omega \in \Omega$, by conditions (i) and (iii), we have

$$2p(\omega) = p(\omega) + p(\omega) \geq p(\omega \oplus \omega) \geq p(\omega),$$

which implies $p(\omega) \geq 0$. By conditions (i) and (ii), we also have

$$0 \leq p(\omega) = p(x \ominus x) = p(x \oplus (-x)) \leq p(x) + p(-x) = 2p(x),$$

which implies $p(x) \geq 0$ for any $x \in X$. We complete the proof. \qed

Theorem 5.5 (Generalized Hahn-Banach Extension Theorem). Let $X$ be a nonstandard vector space over $\mathbb{F}$ such that the following condition is satisfied:

(i) for any $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$, $(\alpha + i\beta)x = \alpha x \oplus i\beta x$, $i(x \oplus y) = ix \oplus iy$ and $i(\alpha x) = (i\alpha)x = \alpha(ix)$; if $\mathbb{F}$ is taken as $\mathbb{R}$, then this condition is not needed;

Let $p$ be a real-valued function defined on $X$ such that the following conditions are satisfied:

(ii) $p$ satisfies the sub-associative law for positive scalar multiplication;
(iii) $p$ satisfies the Hahn-Banach conditions;
(iv) $p(x \oplus y) \leq p(x) + p(y)$ for any $x, y \in X$;
(v) $p(\alpha x) = |\alpha|p(x)$ for any $x \in X$ and $\alpha \in \mathbb{F}$ with $\alpha \neq 0$;
(vi) $p(x \oplus \omega) \geq p(x)$ for any $x \in X$ and $\omega \in \Omega$;
Suppose that $f$ is a pseudo-linear functional defined on a subspace $Z$ of $X$ such that
\begin{equation}
|f(z)| \leq p(z) \text{ for all } z \in Z.
\end{equation}

Then there exists a pseudo-linear functional $\hat{f}$ on $X$ such that $\hat{f}$ satisfies the Hahn-Banach conditions with $|\hat{f}(x)| \leq p(x)$ for all $x \in X$ and $\hat{f}(z) = f(z)$ for all $z \in Z$.

**Proof.** Suppose that $F$ is taken as $\mathbb{R}$. From (7), we have
\begin{equation}
\forall z \in Z, \quad f(z) \leq |f(z)| \leq p(z).
\end{equation}

By Theorem 5.4, there is a linear extension $\hat{f}$ from $Z$ to $X$ such that $\hat{f}$ satisfies the Hahn-Banach conditions and
\begin{equation}
\forall x \in X, \quad \hat{f}(x) \leq p(x),
\end{equation}
where $\hat{f}$ is a pseudo-linear functional. Using (8) and condition (v), we obtain
\[ -\hat{f}(x) = \hat{f}(-x) \leq p(-x) = |-1|p(x) = p(x), \]
that is, $\hat{f}(x) \geq -p(x)$. Combining it with (8), we obtain $|\hat{f}(x)| \leq p(x)$ for all $x \in X$.

Suppose that $F$ is taken as $\mathbb{C}$. The functional $f$ is a complex-valued function, which can be written as $f(x) = f_1(x) + if_2(x)$, where $f_1(x) = \text{Re } f(x)$ and $f_2(x) = \text{Im } f(x)$ are real-valued functions. Now we define
\[ \tilde{X} = \{ix : x \in X\}, \quad \tilde{Z} = \{iz : z \in Z\}, \quad \tilde{\Omega} = \{i\omega : \omega \in \Omega\} \]
and consider the new sets
\[ \hat{X} = X \cup \tilde{X} \quad \text{and} \quad \hat{Z} = Z \cup \tilde{Z}. \]

Then, under condition (i), we see that the nonstandard vector space $X$ over $\mathbb{C}$ is equivalent to the nonstandard vector space $\hat{X}$ over $\mathbb{R}$ in the sense that each element in $X$ over $\mathbb{C}$ can be rewritten as an element in $\hat{X}$ over $\mathbb{R}$ and vice versa. By definition, the null set $\hat{\Omega}$ of $\hat{X}$ is given by
\[ \hat{\Omega} = \{\hat{x} \ominus \hat{x} : \hat{x} \in \hat{X}\} = \{x \ominus x : x \in X\} \cup \{ix \ominus ix : x \in X\}. \]

By condition (i), we also have
\[ \hat{\Omega} = \{x \ominus x : x \in X\} \cup \{i(x \ominus x) : x \in X\} = \Omega \cup \tilde{\Omega}. \]

Since $Z$ is a complex-valued subspace of $X$, we see that $\hat{Z}$ is a subspace of $\hat{X}$. Furthermore, the complex-valued subspace $Z$ is equivalent to the real-valued subspace $\hat{Z}$ in the sense that each element in $Z$ over $\mathbb{C}$ can be rewritten as an element in $\hat{Z}$ over $\mathbb{R}$ and vice versa. Under these settings, conditions (ii)–(vii) of this theorem can be regarded as the corresponding conditions based on
the nonstandard vector space $\hat{X}$ over $\mathbb{R}$. Since $f$ is a pseudo-linear functional on the complex-valued subspace $Z$, we are going to show that $f_1$ and $f_2$ are also pseudo-linear functionals on the real-valued subspace $\hat{Z}$. Indeed, for any $x, y \in \hat{Z} = Z \cup \bar{Z}$, we have

$$[f_1(x) + f_1(y)] + i[f_2(x) + f_2(y)] = f_1(x) + if_2(x) + f_1(y) + if_2(y)$$

$$\quad = f(x) + f(y) = f(x \oplus y) = f_1(x \oplus y) + if_2(x \oplus y)$$

and, for $\alpha \in \mathbb{R}$ with $\alpha \neq 0$ and $x \in \hat{Z}$, we have

$$\alpha f_1(x) + i\alpha f_2(x) = \alpha f(x) = f(\alpha x) = f_1(\alpha x) + if_2(\alpha x).$$

By comparing the real and imaginary parts, we see that $f_1$ and $f_2$ are the pseudo-linear functionals on the real-valued subspace $\hat{Z}$. Since the real part of a complex number cannot exceed the absolute value, from (7), we have $f_1(x) \leq |f(x)| \leq p(x)$ for all $x \in \hat{Z}$. By Theorem 5.4, there is a linear extension $\hat{f}_1$ of $f_1$ from $\hat{Z}$ to $\hat{X}$ such that $\hat{f}_1$ satisfies the Hahn-Banach conditions and $\hat{f}_1(x) \leq p(x)$ for all $x \in \hat{X}$, where $\hat{f}_1$ is also a pseudo-linear functional on $\hat{X}$. Now, for every $x \in Z$, we have

$$i[f_1(x) + if_2(x)] = if(x) = f(ix) = f_1(ix) + if_2(ix).$$

Comparing the real parts on both sides, we obtain

$$f_2(x) = -f_1(ix) \text{ for all } x \in Z.$$  \hfill (9)

For $x \in X$, we set

$$\hat{f}(x) = \hat{f}_1(x) - if_1(ix).$$  \hfill (10)

Therefore, if $x \in Z \subseteq \hat{Z}$, i.e., $ix \in \bar{Z} \subseteq \hat{Z}$, then $\hat{f}_1(x) = f_1(x)$ and $\hat{f}_1(ix) = f_1(ix)$, since $\hat{f}_1$ is a linear extension of $f_1$ from $\hat{Z}$ to $\hat{X}$. Therefore, from (9) and (10), we see that

$$\hat{f}(x) = f_1(x) + if_2(x) = f(x)$$

for $x \in Z$, i.e., $\hat{f}$ is an extension of $f$ from $Z$ to $X$. Now we are going to show that $\hat{f}$ is a pseudo-linear functional on the nonstandard vector space $X$ over $\mathbb{C}$. From (10), we have

$$\hat{f}(x \oplus y) = \hat{f}_1(x \oplus y) - if_1(i(x \oplus y))$$

$$\quad = \hat{f}_1(x \oplus y) - if_1(ix \oplus iy) \quad \text{(by condition (i))}$$

$$\quad = \hat{f}_1(x) + \hat{f}_1(y) - if_1(ix) - if_1(iy) \quad \text{(by the linearity of } \hat{f}_1 \text{ on } \hat{X})$$

$$\quad = \hat{f}(x) + \hat{f}(y)$$
and, for any complex number $\alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$, we have
\[
\hat{f}((\alpha + i\beta)x) = \hat{f}(\alpha x \oplus i\beta x) \quad \text{(by condition (i))}
\]
\[
= \hat{f}_1(\alpha x \oplus i\beta x) - i\hat{f}_1(\alpha(ix) \oplus (-\beta)x) \quad \text{(by condition (i))}
\]
\[
= \alpha \hat{f}_1(x) + \beta \hat{f}_1(ix) - i[\alpha \hat{f}_1(ix) - \beta \hat{f}_1(x)]
\]
(by the linearity of $\hat{f}_1$ on $\hat{X}$)
\[
= (\alpha + i\beta)[\hat{f}_1(x) - i\hat{f}_1(ix)]
\]
\[
= (\alpha + i\beta)\hat{f}(x).
\]

This shows that $\hat{f}$ is indeed a pseudo-linear functional on the nonstandard vector space $X$ over $\mathbb{C}$. We remain to prove $|\hat{f}(x)| \leq p(x)$ for all $x \in X$. Now, for $x \in X$ with $\hat{f}(x) = 0$, it is easy to see that $|\hat{f}(x)| = 0 \leq p(x)$ by Lemma 5.3. Therefore, we consider $x \in X$ with $\hat{f}(x) \neq 0$. Then we can write
\[
\hat{f}(x) = |\hat{f}(x)|e^{i\gamma}, \ i.e.,
\]
\[
(11) \quad |\hat{f}(x)| = \hat{f}(x)e^{-i\gamma} = \hat{f}(e^{-i\gamma}x).
\]

Since $|\hat{f}(x)|$ is real, using (11), we obtain $\hat{f}(e^{-i\gamma}x) = \text{Re} \hat{f}(e^{-i\gamma}x)$. Therefore, using (10) and condition (v), we have
\[
|\hat{f}(x)| = \hat{f}(e^{-i\gamma}x) = \text{Re} \hat{f}(e^{-i\gamma}x) = \hat{f}_1(e^{-i\gamma}x) \leq p(e^{-i\gamma}x) = |e^{-i\gamma}|p(x) = p(x).
\]

Therefore, we obtain the desired result. Finally, since $\hat{f}_1$ satisfies the Hahn-Banach conditions, using (10) and condition (i), we see that $\hat{f}$ also satisfies the Hahn-Banach conditions. We complete the proof. \[\square\]

**Corollary 5.5 (Generalized Hahn-Banach Extension Theorem).** Let $X$ be a nonstandard vector space over $\mathbb{F}$ such that the following conditions are satisfied:

(i) for any $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$, $(\alpha + i\beta)x = \alpha x \oplus i\beta x$, $i(x \oplus y) = ix \oplus iy$ and $i(\alpha x) = (i\alpha)x = \alpha(ix)$; if $\mathbb{F}$ is taken as $\mathbb{R}$, then this condition is not needed;

(ii) $(-\lambda)x = -(\lambda x)$ for any $x \in X$ and $\lambda < 0$;

(iii) $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ and $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x, y \in X$ and $\alpha, \beta > 0$.

Let $p$ be a real-valued function defined on $X$ satisfying the following conditions:

(iv) $p(x \oplus y) \leq p(x) + p(y)$ for any $x, y \in X$;

(v) $p(\alpha x) = |\alpha|p(x)$ for any $x \in X$ and $\alpha \in \mathbb{F}$ with $\alpha \neq 0$.

(vi) $p(x \oplus \omega) \geq p(x)$ for any $x \in X$ and $\omega \in \Omega$.

(vii) $p(\omega) = 0$ for any $\omega \in \Omega$. 


Suppose that $f$ is a pseudo-linear functional defined on a subspace $Z$ of $X$ such that $|f(z)| \leq p(z)$ for all $z \in Z$. Then there exists a pseudo-linear functional $\hat{f}$ on $X$ such that $\hat{f}$ satisfies the Hahn-Banach conditions with $|\hat{f}(x)| \leq p(x)$ for all $x \in X$ and $\hat{f}(z) = f(z)$ for all $z \in Z$.

Proof. Under conditions (ii) and (iii), we see that $p$ satisfies the Hahn-Banach conditions. Therefore, the conditions in Theorem 5.5 are all satisfied. □

We also remark that conditions (ii) and (iii) in Corollary 5.5 are automatically satisfies in the space $\mathcal{I}$ of all closed intervals.

6. HAHN-BANACH EXTENSION THEOREMS IN NONSTANDARD NORMED SPACES

In the sequel, we are going to derive the Hahn-Banach theorems in non-standard normed spaces. In order for the convenient discussion, we introduce many terminologies.

**Definition 6.1.** Let $(X, \| \cdot \|)$ be a nonstandard pseudo-seminormed space.

(i) We say that the norm $\| \cdot \|$ satisfies the **positively distributive law for vector addition** if $\| \alpha(x \oplus y) \| = \| \alpha x \oplus \alpha y \|$ and $\| z \oplus \alpha(x \oplus y) \| = \| z \oplus \alpha x \oplus \alpha y \|$ for any $x, y, z \in X$ and $\alpha > 0$.

(ii) We say that the norm $\| \cdot \|$ satisfies the **distributive law for positive scalar addition** if $\| (\alpha + \beta)x \| = \| \alpha x \oplus \beta x \|$ and $\| z \oplus (\alpha + \beta)x \| = \| z \oplus \alpha x \oplus \beta x \|$ for any $x, z \in X$ and $\alpha, \beta > 0$.

(iii) We say that the norm $\| \cdot \|$ satisfies the **associative law for positive scalar multiplication** if $\| \alpha(\beta x) \| = \| (\alpha\beta)x \|$ and $\| z \oplus \alpha(\beta x) \| = \| z \oplus (\alpha\beta)x \|$ for any $x, z \in X$ and $\alpha, \beta > 0$.

(iv) We say that the norm $\| \cdot \|$ satisfies the **positively sub-distributive law for vector addition** if $\| \alpha(x \oplus y) \| \leq \| \alpha x \oplus \alpha y \|$ and $\| z \oplus \alpha(x \oplus y) \| \leq \| z \oplus \alpha x \oplus \alpha y \|$ for any $x, y, z \in X$ and $\alpha > 0$.

(v) We say that the norm $\| \cdot \|$ satisfies the **sub-associative law for positive scalar multiplication** if $\| \alpha(\beta x) \| \leq \| (\alpha\beta)x \|$ and $\| z \oplus \alpha(\beta x) \| \leq \| z \oplus (\alpha\beta)x \|$ for any $x, z \in X$ and $\alpha, \beta > 0$.

Let $(X, \| \cdot \|)$ be a nonstandard pseudo-seminormed space. If $\| \cdot \|$ satisfies the positively distributive law for vector addition, then, by induction on $n$, we have $\| \alpha(x_1 \oplus \cdots \oplus x_n) \| = \| \alpha x_1 \oplus \cdots \oplus \alpha x_n \|$ and $\| z \oplus \alpha(x_1 \oplus \cdots \oplus x_n) \| = \| z \oplus \alpha x_1 \oplus \cdots \oplus \alpha x_n \|$ for any $\alpha \in \mathbb{F}$ and $z, x_1, \cdots, x_n \in X$. However, if the equality $\| z \oplus \alpha(x \oplus y) \| = \| z \oplus \alpha x \oplus \alpha y \|$ is omitted in the definition, then we cannot have the equality $\| \alpha(x_1 \oplus \cdots \oplus x_n) \| = \| \alpha x_1 \oplus \cdots \oplus \alpha x_n \|$ by
induction on \( n \). The similar situation also applies to the case of distributive law for positive scalar addition.

**Definition 6.2.** Let \((X, \| \cdot \|)\) be a nonstandard pseudo-seminormed space. We say that \( \| \cdot \| \) satisfies the **Hahn-Banach conditions** if the following conditions are satisfied:

(i) \( \| (-\alpha)x \| = \| -(\alpha x) \| \) for any \( x \in X \) and \( \alpha < 0 \);

(ii) \( \| (-\alpha)x \oplus z \| = \| -(\alpha x) \oplus z \| \) for any \( x, z \in X \) and \( \alpha < 0 \);

(iii) \( \| z \oplus (\alpha + \beta)x \| \leq \| z \oplus \alpha x \oplus \beta x \| \) for any \( x, z \in X \) and \( \alpha, \beta > 0 \).

(iv) \( \| \cdot \| \) satisfies the positively distributive law for vector addition.

**Theorem 6.1 (Hahn-Banach Extension Theorem in Nonstandard Normed Space – I).** Let \((X, \| \cdot \|)\) be a nonstandard pseudo-normed space. Suppose that the following conditions are satisfied:

(i) for any \( \alpha, \beta \in \mathbb{R} \) and \( x, y \in X \), \((\alpha + i\beta)x = \alpha x \oplus i\beta x \), \( i(x \oplus y) = ix \oplus iy \) and \( i(\alpha x) = (i\alpha)x = \alpha(ix) \); if \( \mathbb{F} \) is taken as the field of real numbers, then this condition is not needed;

(ii) the norm \( \| \cdot \| \) satisfies the sub-associative law for positive scalar multiplication;

(iii) the norm \( \| \cdot \| \) satisfies the Hahn-Banach conditions.

(iv) the norm \( \| \cdot \| \) satisfies the null condition;

(v) the norm \( \| \cdot \| \) satisfies the null inequality.

Suppose that \( f \) is a pseudo-linear functional defined on a subspace \( Z \) of \( X \) such that \( |f(z)| \leq \| z \| \) for all \( z \in Z \). Then there exists a pseudo-linear functional \( \hat{f} \) on \( X \) such that \( |\hat{f}(x)| \leq \| x \| \) for all \( x \in X \) and \( \hat{f}(z) = f(z) \) for all \( z \in Z \).

**Proof.** We define the function \( p(x) = \| x \| \geq 0 \) for all \( x \in X \). Then \( p \) satisfies conditions (ii)–(vii) of Theorem 5.5. Therefore, the result follows from Theorem 5.5 immediately. \( \Box \)

**Corollary 6.1 (Hahn-Banach Extension Theorem in Nonstandard Normed Space – II).** Let \((X, \| \cdot \|)\) be a nonstandard pseudo-normed space. Suppose that the following conditions are satisfied:

(i) for any \( \alpha, \beta \in \mathbb{R} \) and \( x, y \in X \), \((\alpha + i\beta)x = \alpha x \oplus i\beta x \), \( i(x \oplus y) = ix \oplus iy \) and \( i(\alpha x) = (i\alpha)x = \alpha(ix) \); if \( \mathbb{F} \) is taken as the field of real numbers, then this condition is not needed;

(ii) \( (-\lambda)x = -(\lambda x) \) for any \( x \in X \) and \( \lambda < 0 \);

(iii) \( \alpha(x \oplus y) = \alpha x \oplus \alpha y \) and \( (\alpha + \beta)x = \alpha x \oplus \beta x \) for any \( x, y \in X \) and \( \alpha, \beta > 0 \).

(iv) the norm \( \| \cdot \| \) satisfies the null condition;
(v) the norm \( \| \cdot \| \) satisfies the null inequality.

Suppose that \( f \) is a pseudo-linear functional defined on a subspace \( Z \) of \( X \) such that \( |f(z)| \leq \|z\| \) for all \( z \in Z \). Then there exists a pseudo-linear functional \( \hat{f} \) on \( X \) satisfying \( |\hat{f}(x)| \leq \|x\| \) for all \( x \in X \) and \( \hat{f}(z) = f(z) \) for all \( z \in Z \).

Proof. We define the function \( p(x) = \|x\| \geq 0 \) for all \( x \in X \). Then the result follows from Corollary 5.5 immediately. \( \square \)

**THEOREM 6.2 (Hahn-Banach Extension Theorem in Nonstandard Normed Space – III).** Let \( (X, \| \cdot \|) \) be a nonstandard normed space such that the following conditions are satisfied:

(i) for any \( \alpha, \beta \in \mathbb{R} \) and \( x, y \in X \), \( (\alpha + i\beta)x = \alpha x \oplus i\beta x \), \( i(x \oplus y) = ix \oplus iy \) and \( i(\alpha x) = (i\alpha)x = \alpha(ix) \); if \( \mathbb{F} \) is taken as the field of real numbers, then this condition is not needed;

(ii) the norm \( \| \cdot \| \) satisfies the null condition;

(iii) the norm \( \| \cdot \| \) satisfies the null inequality;

(iv) the norm \( \| \cdot \| \) satisfies the sub-associative law for positive scalar multiplication;

(v) the norm \( \| \cdot \| \) satisfies the Hahn-Banach conditions.

Suppose that \( f \) is a bounded pseudo-linear functional defined on a subspace \( Z \) of \( X \), where \( Z \neq \Omega \). Then there exists a bounded pseudo-linear functional \( \hat{f} \) on \( X \) such that \( \hat{f}(z) = f(z) \) for all \( z \in Z \) and \( \| \hat{f} \|_X = \| f \|_Z \), where

\[
\| \hat{f} \|_X = \sup_{x \in X \setminus \Omega, \| x \| = 1} |\hat{f}(x)| \quad \text{and} \quad \| f \|_Z = \sup_{x \in Z \setminus \Omega, \| x \| = 1} |f(x)|.
\]

If \( Z = \Omega \) then \( f(z) = 0 \) for all \( z \in Z \) and its linear extension is \( \hat{f}(x) = 0 \) for all \( x \in X \). In this case, we have \( \| f \|_Z = 0 = \| \hat{f} \|_X \).

Proof. By Proposition 3.1, we see that the norm \( \| \cdot \| \) also satisfies the null equality. For \( Z \neq \Omega \), from Proposition 3.3, we have \( |f(x)| \leq \| f \|_Z \cdot \| x \| \) for all \( x \in Z \). Let \( p(x) = \| f \|_Z \cdot \| x \| \). Then we see that \( p \) can be defined on all of \( X \). Since we are going to apply Theorem 5.5, we need to check that \( p(x) \) satisfies conditions (ii)–(vii). Conditions (ii) and (iii) in Theorem 5.5 are obviously satisfied by conditions (iv) and (v) of this theorem.

For condition (iv), we have

\[
p(x \oplus y) = \| f \|_Z \cdot \| x \oplus y \| \leq \| f \|_Z (\| x \| + \| y \|) = p(x) + p(y).
\]

For condition (v), we have, \( \alpha \in \mathbb{F} \) with \( \alpha \neq 0 \),

\[
p(\alpha x) = \| f \|_Z \cdot \| \alpha x \| = |\alpha| \cdot \| f \|_Z \cdot \| x \| = |\alpha| p(x).
\]
For condition (vi), we have

\[ p(x \oplus \omega) = \| f \|_Z \cdot \| x \oplus \omega \| \geq \| f \|_Z \cdot \| x \| = p(x). \]

For condition (vii), we have \( p(\omega) = \| f \|_Z \cdot \| \omega \| = 0 \) for any \( \omega \in \Omega \) by Proposition 3.1 (ii). Therefore, Theorem 5.5 says that there exists a pseudo-linear functional \( \hat{f} \) on \( X \) such that \( \hat{f}(z) = f(z) \) for all \( z \in Z \) and \( |\hat{f}(x)| \leq p(x) = \| f \|_Z \cdot \| x \| \) for any \( x \in X \). Applying Proposition 3.4 and taking the supremum over all \( x \in X \) with norm 1, we obtain the inequality

\[ (12) \quad \| \hat{f} \|_X = \sup_{x \in X \setminus \Omega} |\hat{f}(x)| \leq \| f \|_Z. \]

Since \( \hat{f} \) is an extension of \( f \), the norm should increase, i.e., \( \| \hat{f} \|_X \geq \| f \|_Z \). Combining it with (12), we obtain \( \| \hat{f} \|_X = \| f \|_Z \). Finally, for \( Z = \Omega \), since the norm \( \| \cdot \| \) satisfies the null condition, we have \( |f(z)| \leq \| z \| = 0 \), which shows that \( f(z) = 0 \) for all \( z \in Z \). Therefore, the extension is \( \hat{f} = 0 \). We complete the proof. \( \square \)

**Corollary 6.2 (Hahn-Banach Extension Theorem in Nonstandard Normed Space – IV).** Let \((X, \| \cdot \|)\) be a nonstandard normed space such that the following conditions are satisfied:

(i) for any \( \alpha, \beta \in \mathbb{R} \) and \( x, y \in X \), \((\alpha + i\beta)x = \alpha x \oplus i\beta x\), \(i(x \oplus y) = ix \oplus iy\) and \(i(\alpha x) = (i\alpha)x = \alpha(ix)\); if \( \mathbb{F} \) is taken as the field of real numbers, then this condition is not needed;

(ii) the norm \( \| \cdot \| \) satisfies the and null condition;

(iii) the norm \( \| \cdot \| \) satisfies the null inequality;

(iv) \((-\lambda)x = -(\lambda x)\) for any \( x \in X \) and \( \lambda < 0 \);

(v) \(\alpha(x \oplus y) = \alpha x \oplus \alpha y\) and \((\alpha + \beta)x = \alpha x \oplus \beta x\) for any \( x, y \in X \) and \( \alpha, \beta > 0 \).

Suppose that \( f \) is a bounded pseudo-linear functional defined on a subspace \( Z \) of \( X \), where \( Z \neq \Omega \). Then there exists a bounded pseudo-linear functional \( \hat{f} \) on \( X \) such that \( \hat{f}(z) = f(z) \) for all \( z \in Z \) and \( \| \hat{f} \|_X = \| f \|_Z \). If \( Z = \Omega \) then \( f(z) = 0 \) for all \( z \in Z \) and its linear extension is \( \hat{f}(x) = 0 \) for all \( x \in X \). In this case, we have \( \| f \|_Z = 0 = \| \hat{f} \|_X \).

**Proof.** Under the assumptions, the conditions in Theorem 6.2 will be satisfied automatically. \( \square \)

Next, we also provide an interesting application of using Hahn-Banach theorems. As a matter of fact, the following proposition is also very useful in the future study.
Proposition 6.1. Let $(X, \| \cdot \|)$ be a nonstandard normed space such that the following conditions are satisfied:

(i) for any $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$, $(\alpha + i\beta)x = \alpha x \oplus i\beta x$, $i(x \oplus y) = ix \oplus iy$ and $i(\alpha x) = (i\alpha)x = \alpha(ix)$; if $\mathbb{F}$ is taken as the field of real numbers, then this condition is not needed;

(ii) The null set $\Omega$ is closed under the vector addition;

(iii) the norm $\| \cdot \|$ satisfies the null condition;

(iv) the norm $\| \cdot \|$ satisfies the null inequality;

(v) the distributive law for vector addition and associative law for scalar multiplication hold true;

(vi) $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha, \beta > 0$.

(vii) Let $\hat{x} \in X \setminus \Omega$ be given such that $\| \alpha_1\hat{x} \oplus \cdots \oplus \alpha_n\hat{x} \| = |\sum_{i=1}^{n} \alpha_i| \cdot \| \hat{x} \|$ for any finite sequence $\{\alpha_1, \cdots, \alpha_n\}$ in $\mathbb{F}$.

Then there exists a bounded pseudo-linear functional $\hat{f}$ on $X$ such that $\| \hat{f} \| = 1$ and $\hat{f}(\hat{x}) = \| \hat{x} \|$.

Proof. We consider the subset $Z$ of $X$ defined by $Z = \text{span} (\{\hat{x}\} \cup \Omega)$. Then $Z$ is a subspace of $X$. By conditions (ii) and (iii) and Remark 3.1, we also see that $\Omega$ is a subspace of $X$. Therefore, for any $z \in Z$, $z$ has the form of $\omega \oplus \alpha_1\hat{x} \oplus \cdots \oplus \alpha_n\hat{x}$ or $\alpha_1\hat{x} \oplus \cdots \oplus \alpha_n\hat{x}$ for some finite sequence $\{\alpha_1, \cdots, \alpha_n\}$ in $\mathbb{F}$ and some $\omega \in \Omega$.

We define a real-valued function $f$ on $Z$ by

$$f(z) = \begin{cases} 
(\sum_{i=1}^{n} \alpha_i) \| \hat{x} \| & \text{if } z \in Z \setminus \Omega \text{ with } z = \omega \oplus \alpha_1\hat{x} \oplus \cdots \oplus \alpha_n\hat{x} \\
0 & \text{if } z \in \Omega.
\end{cases}$$

The function $f$ is well-defined, which can be realized after proving the pseudo-linearity.

We first show that $f$ is a pseudo-linear functional on $Z$. For $z_1 = \omega_1 \oplus \alpha_1\hat{x} \oplus \cdots \oplus \alpha_n\hat{x}$ and $z_2 = \omega_2 \oplus \hat{\alpha}_1\hat{x} \oplus \cdots \oplus \hat{\alpha}_m\hat{x}$, we have

$$f(z_1 \oplus z_2) = \left( \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \hat{\alpha}_j \right) \| \hat{x} \| = f(z_1) + f(z_2)$$

For $z_1 = \alpha_1\hat{x} \oplus \cdots \oplus \alpha_n\hat{x}$ and $z_2 = \hat{\alpha}_1\hat{x} \oplus \cdots \oplus \hat{\alpha}_m\hat{x}$, we similarly have $f(z_1 \oplus z_2) = f(z_1) + f(z_2)$. For $z_1 \in \Omega$ and $z_2 = \omega \oplus \hat{\alpha}_1\hat{x} \oplus \cdots \oplus \hat{\alpha}_m\hat{x}$ or $z_2 = \hat{\alpha}_1\hat{x} \oplus \cdots \oplus \hat{\alpha}_m\hat{x}$, we similarly have $f(z_1 \oplus z_2) = f(z_1) + f(z_2)$. For $\alpha \neq 0$, we have

$$f(\alpha z_1) = f((\alpha \omega_1 \oplus (\alpha \alpha_1)\hat{x} \oplus \cdots \oplus (\alpha \alpha_n)\hat{x})$$

$$= \alpha \left( \sum_{i=1}^{n} \alpha_i \right) \| \hat{x} \| = \alpha f(z_1)$$
by condition (v). We similarly have \( f(\alpha z_1) = \alpha f(z_1) \) for \( z_1 = \alpha_1 \hat{x} \oplus \cdots \oplus \alpha_n \hat{x} \).

We also need to claim that if \( z_1 = z_2 \in Z \), then \( f(z_1) = f(z_2) \). Since \( z_1 \oplus z_2 \in \Omega \), we have

\[
0 = f(z_1 \oplus z_2) = f(z_1 \oplus (-z_2)) = f(z_1) + f(-z_2) = f(z_1) - f(z_2),
\]

which implies \( f(z_1) = f(z_2) \). Therefore, we conclude that \( f \) is a pseudo-linear functional on \( Z \). From Proposition 3.1, we also see that the norm \( \| \cdot \| \) satisfies the null equality. Now, for \( z = \omega \oplus \alpha_1 \hat{x} \oplus \cdots \oplus \alpha_n \hat{x} \), by condition (vii), we have

\[
\| z \| = \| \omega \oplus \alpha_1 \hat{x} \oplus \cdots \oplus \alpha_n \hat{x} \| = \| \alpha_1 \hat{x} \oplus \cdots \oplus \alpha_n \hat{x} \| = \sum_{i=1}^{n} \alpha_i \cdot \| \hat{x} \|.
\]

Therefore, we obtain

\[
|f(z)| = \begin{cases} 
\| z \| & \text{if } z \in Z \setminus \Omega \\
0 & \text{if } z \in \Omega
\end{cases}
\]

This shows that \( f \) is a bounded pseudo-linear functional with norm \( f \|_{Z} = 1 \). Corollary 6.2 says that there exists a bounded pseudo-linear functional \( \hat{f} \) on \( X \) such that \( \| \hat{f} \|_X = \| f \|_{Z} = 1 \) and \( \hat{f}(z) = f(z) \) for \( z \in Z \). In particular, \( \hat{f}(\hat{x}) = f(\hat{x}) = \| \hat{x} \| \). We complete the proof. \( \Box \)

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