ON A CLASS OF $N(k)$-CONTACT METRIC MANIFOLDS

UDAY CHAND DE, AHMET YILDIZ and SUJIT GHOSH

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The object of the present paper is to study ξ-concircularly flat and ϕ-concircularly flat $N(k)$-contact metric manifolds. Beside these, we also study $N(k)$-contact metric manifolds satisfying $Z(\xi, X).S = 0$. Finally, we construct an example to verify some results.

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1. INTRODUCTION

A transformation of a $(2n + 1)$-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [16, 21]. A concircular transformation is always a conformal transformation [21]. Here, geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus, the geometry of concircular transformations, *i.e.*, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [6]). An interesing invariant of a concircular transformation is the concircular curvature tensor Z. It is defined by [16, 17]

$$(1.1)\quad Z(X,Y)W = R(X,Y)W - \frac{r}{2n(2n+1)}[g(Y,W)X - g(X,W)Y],$$

where $X, Y, W \in TM$ and r is the scalar curvature. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). At each point $p \in M$, decompose the tangent space T_pM into direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the

1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$. Thus, the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM) \oplus \{\xi_p\}, \qquad p \in M.$$

It may be natural to consider the following particular cases:

(1) $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow \{\xi_p\}$, *i.e.*, the projection of the image of C in $\phi(T_p(M))$ is zero.

(2) $C : T_p(M) \times T_p(M) \times T_p(M) \longrightarrow \phi(T_p(M))$, *i.e.*, the projection of the image of C in $\{\xi_p\}$ is zero. This condition is equivalent to

$$C(X,Y)\xi = 0. \tag{1.2}$$

(3) $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \longrightarrow \{\xi_p\}$, *i.e.*, when C is restricted to $\phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0. \tag{1.3}$$

An almost contact metric manifold satisfying (1.2) and (1.3) are called ξ-conformally flat and ϕ-conformally flat, respectively. Almost contact metric manifolds satisfying the cases (1), (2) and (3) are considered in [11], [12] and [13], respectively .

In [12], it is proved that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstein Sasakian manifold. In [20], the authors studied ξ-conformally flat $N(k)$-contact metric manifold. A compact ϕ-conformally flat K-contact manifold with regular contact vector field has been studied in [13]. Moreover, in [15] the author studied ϕ-conformally flat (k,μ)-contact metric manifolds. Motivated by the above studies, in this paper we study ξ-concircularly flat and ϕ-concircularly flat $N(k)$-contact metric manifolds. Analogous to the considerations of conformal curvature tensor here we define the following:

Definition 1.1. A $(2n+1)$-dimensional $N(k)$-contact metric manifold is said to be ξ-concircularly flat if

$$Z(X,Y)\xi = 0, \qquad \text{for} \quad X, Y \in TM. \tag{1.4}$$

Definition 1.2. A $(2n+1)$-dimensional $N(k)$-contact metric manifold is said to be ϕ-concircularly flat if

$$g(Z(\phi X, \phi Y)\phi W, \phi U) = 0 \qquad \text{for} \quad X, Y, W, U \in TM. \tag{1.5}$$

In [7], D.E. Blair *et al.* started a study of concircular curvature tensor of contact metric manifolds. A $(2n+1)$-dimensional $N(k)$-contact metric manifold satisfying $Z(\xi,X).Z = 0$, $Z(\xi,X).R = 0$ and $R(\xi,X).Z = 0$ have been considered in [7]. B.J. Papantoniou [1] and D. Perrone [9] included the studies of contact metric manifolds satisfying $R(\xi, X).S = 0$, where S is the Ricci tensor. Motivated by these studies, we continue the study of the paper [7] and classify $N(k)$-contact manifolds with concircular curvature tensor Z satisfying $Z(\xi, X).S = 0$.

The present paper is organized as follows. After preliminaries in Section 3, we study ξ-concircularly flat $N(k)$-contact metric manifolds and prove that a $(2n+1)$-dimensional, $n > 1$, ξ-concircularly flat$N(k)$-contact metric manifold is locally isometric to Example 2.1. Section 4 deals with the study of ϕ-concircularly flat $N(k)$-contact metric manifolds. In this section, we prove that a ϕ-concircularly flat $(2n+1)$-dimensional, $n \geq 1$, $N(k)$-contact metric manifold is a Sasakian manifold. Section 5 is devoted to study a $(2n+1)$-dimensional, $n \geq 2$, $N(k)$-contact metric manifold satisfying $Z(\xi, X).S = 0$ and prove that the manifold satisfies $Z(\xi.X).S = 0$ if and only if it is an Einstein-Sasakian manifold. Finally, in Section 6, we construct an example of a 3-dimensional $N(k)$-contact metric manifold which verifies some results of Section 3.

2. PRELIMINARIES

A $(2n+1)$-dimensional differentiable manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying [3, 4]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0 \quad \text{and} \quad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$. Let g be the compatible Riemannian metric with almost contact structure (ϕ, ξ, η), *i.e.*,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1) it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for any vector fields X, Y. An almost contact metric structure becomes a contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$, for all vector fields X, Y.

A contact metric manifold is said to be Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant and η-Einstein if $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$, where α and β are smooth functions.

A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.4}$$

$X, Y \in TM$, where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ for which ξ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However, a 3-dimensional K-contact manifold is Sasakian [14].

It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure stisfying $R(X,Y)\xi = 0$ [5]. Again on a Sasakian manifold [18] we have

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case: D.E. Blair, T. Koufogiorgos and B.J. Papantoniou [8] introduced the (k,μ)- nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ [8] of a contact metric manifold M is defined by

$$\begin{aligned} N(k,\mu) \quad : \quad & p \longrightarrow N_p(k,\mu) = \{W \in T_pM : R(X,Y)W \\ & = (kI + \mu h)(g(Y,W)X - g(X,W)Y)\}, \end{aligned}$$

for all $X, Y \in TM$, where $(k,\mu) \in \mathbb{R}^2$. A contact metric manifold M with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold. If $\mu = 0$, the (k,μ)-nullity distribution reduces to k-nullity distribution [19]. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [19]

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [7]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [5].

However, for a $N(k)$-contact metric manifold M of dimension $(2n+1)$, we have [7]

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.5}$$

where $h = \frac{1}{2}\pounds_\xi \phi$.

$$h^2 = (k-1)\phi^2. \tag{2.6}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y]. \tag{2.7}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X]. \tag{2.8}$$

$$\begin{aligned} S(X,Y) \;=\; & 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ & +[2nk-2(n-1)]\eta(X)\eta(Y), \qquad n \geq 1. \end{aligned} \tag{2.9}$$

$$S(Y,\xi) = 2nk\eta(X). \tag{2.10}$$

$$r = 2n(2n-2+k). \tag{2.11}$$

Given a non-Sasakian (k,μ)-contact manifold M, E. Boeckx [10] introduced an invariant

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-k}}$$

and showed that for two non-Sasakian (k,μ)-manifolds M_1 and M_2, we have $I_{M_1} = I_{M_2}$ if and only if up to a D-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds.

Thus, we see that from all non-Sasakian (k,μ)-manifolds of dimension $(2n+1)$ and for every possible value of the invariant I_M, one (k,μ)-manifold M can be obtained. For $I_M > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I_M = \frac{1+c}{|1-c|}$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I_M < -1$.

Example 2.1. Using this invariant, D.E. Blair, J-S. Kim and M.M. Tripathi [7] constructed an example of a $(2n+1)$-dimensional $N(1-\frac{1}{n})$-contact metric manifold, $n > 1$. The example is given in the following:

Since the Boeckx invariant for a $(1-\frac{1}{n},0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting D-homothetic deformation will be a $(1-\frac{1}{n},0)$-manifold. That is, for $k = c(2-c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{k+a^2-1}{a^2}, \qquad 0 = \frac{\mu+2a-2}{a}$$

for a and c. The result is

$$c = \frac{\sqrt{n}\pm 1}{n-1}, \qquad a = 1+c$$

and taking c and a to be these values we obtain $N(1-\frac{1}{n})$-contact metric manifold.

The above example will be used in Theorem 3.1.

Here, we state some Lemmas which will be neccessary to prove our main results.

Lemma 2.1 ([5]). *A contact metric manifold M^{2n+1} satisfying the condition $R(X,Y)\xi = 0$ for all X, Y is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

Lemma 2.2 ([2]). *Let M be an η-Einstein manifold of dimension $(2n+1)$ $(n \geq 1)$. If ξ belongs to the k-nullity distribution, then $k = 1$ and the structure is Sasakian.*

Lemma 2.3 ([19]). *Let M be an Einstein manifold of dimension $(2n+1)$ $(n \geq 2)$. If ξ belongs to the k-nullity distribution, then $k = 1$ and the structure is Sasakian.*

3. ξ-CONCIRCULARLY FLAT $N(k)$-CONTACT METRIC MANIFOLDS

In this section, we study ξ-concirculrly flat $N(k)$-contact metric manifolds. Let M be a $(2n+1)$-dimensional, $n \geq 1$, ξ-concirculrly flat $N(k)$-contact metric manifold. Putting $W = \xi$ in (1.1) and applying (1.4) and $g(X,\xi) = \eta(X)$, we have

$$R(X,Y)\xi = \frac{r}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \tag{3.1}$$

Using (2.7) in (3.1), we obtain

$$(k - \frac{r}{2n(2n+1)})[\eta(Y)X - \eta(X)Y] = 0. \tag{3.2}$$

Now $[\eta(Y)X - \eta(X)Y] \neq 0$ in a contact metric manifold, in general. Therefore, (3.2) gives

$$k = \frac{r}{2n(2n+1)}. \tag{3.3}$$

Using (2.11) in (3.3) yields

$$k = 1 - \frac{1}{n}. \tag{3.4}$$

Hence, we can state the following:

Theorem 3.1. *If a $(2n+1)$-dimensional $(n > 1)$ $N(k)$-contact metric manifold is ξ-concircularly flat, then it is locally isometric to Example 2.1.*

Let us consider a 3-dimensional ξ-concircularly flat $N(k)$-contact metric manifold. Then $n = 1$ and in that case we have $k = 0$. Hence, in view of Lemma 2.1, we can state the following:

COROLLARY 3.1. *A 3-dimensional $N(k)$-contact metric manifold is ξ-concircularly flat if and only if the manifold is flat.*

4. ϕ-CONCIRCULARLY FLAT $N(k)$-CONTACT METRIC MANIFOLDS

This section is devoted to study ϕ-concircularly flat $N(k)$-contact metric manifolds. Let M be a $(2n+1)$-dimensional ϕ-concircularly flat $N(k)$-contact metric manifold.

Using (1.5) in (1.1) we obtain

$$g(R(\phi X, \phi Y)\phi W, \phi V) = \frac{r}{2n(2n+1)}[g(\phi Y, \phi W)g(\phi X, \phi V) - g(\phi X, \phi W)g(\phi Y, \phi V)]. \tag{4.1}$$

Let $\{e_1, e_2, ..., e_n, \phi e_1, \phi e_2, ..., \phi e_n, \xi\}$ be an orthonormal ϕ-basis of the tangent space. Putting $X = V = e_i$ in (4.1) and taking summation over $i = 1$ to $2n$ and using (2.7), we obtain

$$S(\phi Y, \phi W) = [\frac{r(2n-1)}{2n(2n+1)} + k]g(\phi Y, \phi W). \tag{4.2}$$

Replacing Y and W by ϕY and ϕW in (4.2) and using (2.1), (2.2), (2.10) and (2.11) yields

$$S(Y, W) = Ag(Y, W) + B\eta(Y)\eta(W), \tag{4.3}$$

where A and B are given by the following relations:

$$A = \frac{(2n-1)(2n-2) + 4nk}{2n+1} \quad \text{and} \quad B = \frac{2(2n-1)\{n(k-1)-1\}}{2n+1}.$$

In view of the equation (4.3) we state the following:

PROPOSITION 4.1. *A $(2n+1)$-dimensional ϕ-concircularly flat $N(k)$-contact metric manifold is an η-Einstein manifold.*

Using the Lemma 2.2 we have the following:

THEOREM 4.1. *A $(2n+1)$-dimensional ϕ-concircularly flat $N(k)$-contact metric manifold is a Sasakian manifold.*

5. $N(k)$-CONTACT METRIC MANIFOLD SATISFYING $Z(\xi, X).S = 0$

This section deals with a $(2n+1)$-dimensional, $n \geq 2$, $N(k)$-contact metric manifold satisfying $Z(\xi, X).S = 0$. Now, the relation $Z(\xi, X).S = 0$ implies

$$S(Z(\xi, X)Y, W) + S(Y, Z(\xi, X)W) = 0. \tag{5.1}$$

Using (1.1) in (5.1), we have

$$\begin{aligned} &S(R(\xi, X)Y, W) + S(Y, R(\xi, X)W) \\ &-\frac{r}{2n(2n+1)}[g(X, Y)S(\xi, W) \\ &-\eta(Y)S(X, W) + g(X, W)S(Y, \xi) - \eta(W)S(X, Y)] = 0. \end{aligned} \tag{5.2}$$

Using (2.8) and (2.10) in (5.2) yields

$$\begin{aligned} &2nk[g(X, Y)\eta(W) + g(X, W)\eta(Y)] - [S(X, W)\eta(Y) \\ &+S(X, Y)\eta(W)] - \frac{r}{2n(2n+1)}[2nk\{g(X, Y)\eta(W) \\ &+g(X, W)\eta(Y)\} - \{S(X, W)\eta(Y) + S(X, Y)\eta(W)\}] = 0. \end{aligned} \tag{5.3}$$

Putting $W = \xi$ in (5.3) and using $g(X, \xi) = \eta(X)$ and (2.10), we obtain

$$\{\frac{r}{2n(2n+1)} - 1\}[S(X, Y) - 2nkg(X, Y)] = 0. \tag{5.4}$$

Putting the value of r from (2.11) in $r = 2n(2n+1)$, we get $k = 3$. We know that $k \leq 1$ in a $N(k)$-contact metric manifold. Therefore $r \neq 2n(2n+1)$ and hence, from (5.4) we have

$$S(X, Y) = 2nkg(X, Y). \tag{5.5}$$

Therefore, a $N(k)$-contact metric manifold satisfying $Z(\xi, X).S = 0$ is an Einstein manifold. Therefore, in view of Lemma 2.3, we have the manifold is a Sasakian manifold.

Conversely, let the manifold be an Einstein-Sasakian manifold. Then we have $k = 1$ and $S(X, Y) = 2ng(X, Y)$. Therefore, we have

$$\begin{aligned} (Z(\xi, X).S)(Y, W) &= S(Z(\xi, X)Y, W) + S(Y, Z(\xi, X)W) \\ &= 2n[g((Z(\xi, X)Y, W) + g(Y, Z(\xi, X)W)]. \end{aligned} \tag{5.6}$$

Using (1.1), (2.8) and (2.11) in (5.6) we easily obtain

$$(Z(\xi, X).S)(Y, W) = 0. \tag{5.7}$$

In view of the above discussions, we state the following:

Theorem 5.1. *A $(2n+1)$-dimensional, $n \geq 2$, $N(k)$-contact metric manifold satisfies $Z(\xi, X).S = 0$ if and only if the manifold is an Einstein-Sasakian manifold.*

6. EXAMPLE

In this section, we construct an example of a $N(k)$-contact metric manifold which is ξ-concircularly flat. We consider 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard coordinate in R^3. Let e_1, e_2, e_3 are three vector fields in R^3 which satisfies

$$[e_1,e_2] = (1+\lambda)e_3, \quad [e_2,e_3] = 2e_1 \quad \text{and} \quad [e_3,e_1] = (1-\lambda)e_2,$$

where λ is a real number.

Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, \quad g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by

$$\eta(U) = g(U,e_1)$$

for any $U \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = 0, \;\; \phi e_2 = e_3, \;\; \phi e_3 = -e_2.$$

Using the linearity of ϕ and g we have

$$\eta(e_1) = 1,$$

$$\phi^2(U) = -U + \eta(U)e_1$$

and

$$g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Moreover,

$$he_1 = 0, \quad he_2 = \lambda e_2 \;\; \textit{and} \;\; he_3 = -\lambda e_3.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \\ &\nabla_{e_2} e_1 = -(1+\lambda)e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = (1+\lambda)e_1, \\ &\nabla_{e_3} e_1 = (1-\lambda)e_2, \quad \nabla_{e_3} e_2 = -(1-\lambda)e_1, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

In view of the above relations we have

$$\nabla_X \xi = -\phi X - \phi h X, \quad \text{for} \quad e_1 = \xi$$

Therefore, the manifold is a contact metric manifold with the contact structure (ϕ, ξ, η, g).

Now, we find the curvature tensors as follows:

$$R(e_1, e_2)e_2 = (1-\lambda^2)e_1, \quad R(e_3, e_2)e_2 = -(1-\lambda^2)e_3,$$
$$R(e_1, e_3)e_3 = (1-\lambda^2)e_1, \quad R(e_2, e_3)e_3 = -(1-\lambda^2)e_2,$$
$$R(e_2, e_3)e_1 = 0, \quad R(e_1, e_2)e_1 = -(1-\lambda^2)e_2, \quad R(e_3, e_1)e_1 = (1-\lambda^2)e_3.$$

In view of the expressions of the curvature tensors we conclude that the manifold is a $N(1-\lambda^2)$-contact metric manifold.

Using the expressions of the curvature tensor we find the values of the Ricci tensors as follows:

$$S(e_1, e_1) = 2(1-\lambda^2), \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = 0.$$

Hence, $r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = 2(1-\lambda^2)$.

Let X and Y are any two vector fields given by

$$X = a_1e_1 + a_2e_2 + a_3e_3 \quad \text{and} \quad Y = b_1e_1 + b_2e_2 + b_3e_3.$$

Using (1.1) we get

$$Z(X, Y)e_1 = \frac{2(1-\lambda^2)}{3}[(a_2b_1 - a_1b_2)e_2 + (a_3b_1 - a_1b_3)e_3].$$

Therefore, the manifold will be ξ-concircularly flat if $\lambda = 1$ and in that case the manifold will be a $N(0)$-contact metric manifold with $r = 0$ which verifies the result of Section 3.

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REFERENCES

[1] B.J. Papantoniou, *Contact Riemannian manifolds satisfying $R(\xi, X).R = 0$ and $\xi \in (k, \mu)$-nullity distribution.* Yokohama Math. J. **40** (1993), 149–161.

[2] C. Baikoussis and T. Koufogiorgos, *On a type of contact manifolds.* J. Geometry **46** (1993), 1–9.

[3] D.E. Blair, *Contact manifolds in Riemannian geometry.* Lecture Notes in Math. **509**, Springer-Verlag, 1976.

[4] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds.* Prog. Math. **203**, Birkhauser Boston, Inc., Boston, MA, 2002.

[5] D.E. Blair, *Two remarks on contact metric structures.* Tohoku Math. J. **29** (1977), 319–324.

[6] D.E. Blair, *Inversion theory and conformal mapping.* Stud. Math. Libr. **9**, Amer. Math. Soc., 2000.

[7] D.E. Blair, J-S. Kim and M.M. Tripathi, *On the concircular curvature tensor of a contact metric manifold.* J. Korean Math. Soc. **42** (2005), 883–892.

[8] D.E. Blair, T. Koufogiorgos and B.J. Papantoniou, *Contact metric manifolds satisfying a nullity condition.* Israel J. Math. **91** (1995), 189–214.
[9] D. Perrone, *Contact Riemannian manifolds satisfying* $R(X,\xi).R = 0$. Yokohama Math. J. **39** (1992), 141–149.
[10] E. Boeckx, *A full classification of contact metric* (k,μ)*-spaces.* Illinois J. Math. **44** (2000), *1*, 212–219.
[11] G. Zhen, *On conformal symmetric* K*-contact manifolds.* Chinese Quart. J. Math. **7** (1992), 5–10.
[12] G. Zhen, J.L. Cabrerizo, L.M. Fernández and M. Fernández, *On* ξ*-conformally flat* K*-contact metric manifolds.* Indian J. Pure Appl. Math. **28** (1997), 725–734.
[13] J.L. Cabrerizo, L.M. Fernández and M. Fernández, *The structure of a class of* K*-contact manifolds.* Acta Math. Hungar. **82** (1999), 331–340.
[14] J-B. Jun and U.K. Kim, *On 3-dimensional almost contact metric manifolds.* Kyungpook Math. J. **34** (1994), *2*, 293–301.
[15] K. Arslan, C. Murathan and C. Özgür, *On* ϕ*-conformally flat contact metric manifolds.* Balkan J. Geom. Appl., **5** (2000), 1–7.
[16] K. Yano, *Concircular geometry I. concircular transformations.* Proc. Imp. Acad. Tokyo **16** (1940), 195–200.
[17] K. Yano and S. Bochner, *Curvature and Betti numbers.* Ann. Math. Stud. **32**, Princeton Univ. Press, 1953.
[18] S. Sasaki, *Lecture notes on almost contact manifolds.* Tohoku University I, 1965.
[19] S. Tanno, *Ricci curvature of contact Riemannian manifolds.* Tohoku Math. J. **40** (1988), 441–448.
[20] U.C. De and S. Biswas, *A note on* ξ*-conformally flat contact manifolds.* Bull. Malays. Math. Soc. **29** (2006), 51–57.
[21] W. Kuhnel, *Conformal transformations between Einstein spaces.* Conform. Geometry. Aspects Math. **12** (1988), Vieweg, Braunschweig, 105–146.

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Calcutta University,
Department of Pure Mathematics,
35, Ballygaunge Circular Road,
Kolkata 700019,
West Bengal, India
uc_de@yahoo.com

Inonu University,
Education Faculty,
44280, Malatya
Turkey
ayildiz44@yahoo.com
a.yildiz@inonu.edu.tr

Madanpur K. A. Vidyalaya (H.S.),
Vill+P.O. Madanpur,
Dist. Nadia, Pin. 741245, *W. B.,*
India
ghosh.sujit6@gmail.com