VECTOR INTEGRALS AND A POINTWISE MEAN ERGODIC THEOREM FOR TRANSITION FUNCTIONS

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Our goal is to discuss the use of a certain vector integral in order to extend the weak* and pointwise mean ergodic theorems for equicontinuous Markov-Feller pairs to equicontinuous Feller transition functions.

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PROLOGUE

The main ideas of the paper can be told in the language of fairy tales: We consider Cinderella (in Romanian we call her “Cenuşăreasa”) and a handsome prince (in Romanian, “Făt-Frumos”). Cinderella (the pointwise integral that will be defined in the paper) is a modest girl without education (knows almost no measure theory, let alone ergodic theory), but she is easygoing and exists under almost any conditions. By contrast, the prince (the Bochner integral) has a very solid measure theoretical foundation and (thanks to Dunford and Schwartz [3]) is familiar with ergodic theory; however, the prince is spoiled, and exists under rather sophisticated conditions. Together, Cinderella and the prince can help us deal with two ergodic theorems. And then, of course, they will live happily ever after (again, in Romanian we say “Am încălecat pe-o șea și v-am spus povestea așa.”).

1. INTRODUCTION

In [6] and [8] we have obtained various results concerning transition probabilities: an extension of an ergodic decomposition that goes back to Kryloff, Bogoliouboff, Beboutoff and Yosida, results on supports of various types of invariant probabilities, a weak* mean ergodic theorem, and a pointwise mean

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ergodic theorem. In recent years we have extended all the above-mentioned results to transition functions, and currently we are writing the final form of a small monograph on transition functions that will include these extensions. As expected, the proofs of the extensions are more sophisticated than the proofs of the corresponding results for transition probabilities as one encounters various difficulties due to the transition function setting. Our goal in this note is to show how to overcome one such difficulty when extending the pointwise and the weak* mean ergodic theorems. In doing so, we will discuss a vector integral; it seems that the integral has never been mentioned explicitly in the literature, but, in all likelihood, it has been used implicitly earlier.

Let \((X, d)\) be a locally compact separable metric space and let \(\mathcal{B}(X)\) be the \(\sigma\)-algebra of all Borel subsets of \(X\).

As usual, we say that a map \(P : X \times \mathcal{B}(X) \to \mathbb{R}\) is a transition probability if the following two conditions are satisfied.

(i) For every \(x \in X\), the map \(\mu_x : \mathcal{B}(X) \to \mathbb{R}\) defined by \(\mu_x(A) = P(x, A)\) for every \(A \in \mathcal{B}(X)\) is a probability measure.

(ii) For every \(A \in \mathcal{B}(X)\), the function \(g_A : X \to \mathbb{R}\) defined by \(g_A(x) = P(x, A)\) for every \(x \in X\) is Borel measurable.

Let \(\mathcal{B}_b(X)\) be the Banach lattice (under the sup (uniform) norm and the pointwise order) of all real-valued bounded Borel measurable functions defined on \(X\), and let \(\mathcal{M}(X)\) be the Banach lattice of all real-valued signed Borel measures on \(X\), the norm on \(\mathcal{M}(X)\) being the total variation norm, and the order on \(\mathcal{M}(X)\) being the usual one (for the definition and a comprehensive treatment of Banach lattices, see the classical monograph of Schaefer [5]).

Given a transition probability \(P\) on \((X, d)\), we can define two operators \(S : \mathcal{B}_b(X) \to \mathcal{B}_b(X)\) and \(T : \mathcal{M}(X) \to \mathcal{M}(X)\) by

\[
(1.1) \quad Sf(x) = \int_X f(y) P(x, dy)
\]

for every \(f \in \mathcal{B}_b(X)\) and \(x \in X\), where \(P(x, dy)\) stands for \(d\mu_x(y)\), \(\mu_x\) being the probability measure defined by \(P\) and \(x \in X\) that appears at (i) in the definition of a transition probability, and by

\[
(1.2) \quad T\mu(A) = \int_X P(x, A) d\mu(x)
\]

for every \(\mu \in \mathcal{M}(X)\) and \(A \in \mathcal{B}(X)\).

It is easy to see that \(S\) and \(T\) are well-defined in the sense that the function \(Sf\) obtained using (1.1) belongs to \(\mathcal{B}_b(X)\) whenever \(f \in \mathcal{B}_b(X)\), and that if \(T\mu\) is given by (1.2), then \(T\mu\) belongs to \(\mathcal{M}(X)\) whenever \(\mu \in \mathcal{M}(X)\). Also easy to see is the fact that \(S\) and \(T\) are linear positive contractions.
of \( B_b(X) \) and \( \mathcal{M}(X) \), respectively. We call the pair \((S, T)\) the **Markov pair defined by** \( P \). For details on Markov pairs, see [8].

Let \( C_b(X) \) be the Banach sublattice of \( B_b(X) \) of all real-valued continuous bounded functions defined on \( X \).

A pair \((S, T)\) is called a **Markov-Feller pair** if \( Sf \) belongs to \( C_b(X) \) whenever \( f \in C_b(X) \). If \((S, T)\) is a Markov-Feller pair, the corresponding transition probability \( P \) is called a **Feller transition probability**. For details on Markov-Feller pairs, see [6] and [8].

A family \((P_t)_{t \in [0, +\infty)}\) of transition probabilities defined on \((X, d)\) is called a **transition function** if it has the property that

\[
P_{s+t}(x, A) = \int_X P_s(y, A)P_t(x, dy)
\]

for every \( s \in [0, +\infty) \), \( t \in [0, +\infty) \), \( A \in \mathcal{B}(X) \) and \( x \in X \). Note that (1.3) is the familiar *Chapman-Kolmogorov equation*.

Let \((P_t)_{t \in [0, +\infty)}\) be a transition function. For each \( t \in [0, +\infty) \) we can construct the Markov pair \((S_t, T_t)\) defined by \( P_t \). We call the family \(((S_t, T_t))_{t \in [0, +\infty)}\) constructed in this way the **family of Markov pairs defined (or generated) by** \((P_t)_{t \in [0, +\infty)}\). It can be shown that the families \((S_t)_{t \in [0, +\infty)}\) and \((T_t)_{t \in [0, +\infty)}\) are semigroups of operators (on \( B_b(X) \) and \( \mathcal{M}(X) \), respectively).

We say that \((P_t)_{t \in [0, +\infty)}\) is a **Feller transition function** if \((S_t, T_t)\) is a Markov-Feller pair for every \( t \in [0, +\infty) \). In this case, we also say that \((S_t)_{t \in [0, +\infty)}\) is a **Feller semigroup of operators**.

We say that \((P_t)_{t \in [0, +\infty)}\) satisfies the **standard measurability assumption (s.m.a.)** if for every \( A \in \mathcal{B}(X) \), the map \((t, s) \mapsto P_t(x, A), (t, x) \in [0, +\infty) \times X\), is jointly measurable with respect to \( t \) and \( x \) (that is, the map is measurable with respect to the product \( \sigma \)-algebra \( \mathcal{L}([0, +\infty)) \otimes \mathcal{B}(X) \), where \( \mathcal{L}([0, +\infty)) \) is the \( \sigma \)-algebra of all Lebesgue measurable subsets of \([0, +\infty))\).

Let \( C_0(X) \) be the Banach sublattice of \( C_b(X) \) (and, of course, of \( B_b(X) \)) of all real-valued continuous functions on \( X \) that vanish at infinity.

It can be shown that a Feller transition function which is \( C_0(X) \)-strongly continuous satisfies the s.m.a.

We say that \((P_t)_{t \in [0, +\infty)}\) is **\( C_0(X) \)-strongly continuous** if for every \( f \in C_0(X) \), the map \( t \mapsto S_tf, t \in [0, +\infty) \), is continuous with respect to the topology of uniform convergence on \( B_b(X) \) and the usual standard topology on \([0, +\infty)\).

We say that \((P_t)_{t \in [0, +\infty)}\) is **\( C_0(X) \)-equicontinuous** if for every \( f \in C_0(X) \), for every convergent sequence \((x_n)_{n \in \mathbb{N}}\) of elements of \( X \), and for every \( \varepsilon \in \mathbb{R}, \varepsilon > 0 \), there exists \( n_\varepsilon \in \mathbb{N} \) such that \( |S_tf(x_n) - S_tf(x)| < \varepsilon \) for every \( t \in [0, +\infty) \) and for every \( n \geq n_\varepsilon \), where \( x = \lim_{n \to +\infty} x_n \).
Now, assume that \((P_t)_{t \in [0, \infty)}\) satisfies the s.m.a. It can be shown that for every \(f \in B_b(X)\) and every \(x \in X\), the map \(t \mapsto S_t f(x), t \in [0, \infty)\) is measurable. Since the map is also bounded, the integral \(\int_0^s S_t f(x) \, dt\) exists and is a real number for every \(s \in [0, \infty)\). Hence, it makes sense to try to single out a class of transition functions \((P_t)_{t \in [0, \infty)}\) that satisfy the s.m.a. and have the property that \(\lim_{s \to \infty} \frac{1}{s} \int_0^s S_t f(x) \, dt\) exists for every \(f \in C_0(X)\) and \(x \in X\), thus extending the pointwise ergodic theorem for Feller transition probabilities that we proved in [6], Section 4.3, Corollary 4.3.2. However, at least for the time being, the only manner in which we can obtain such an extension is to obtain first an extension of the weak* mean ergodic theorem for transition probabilities discussed in Theorem 4.3.1 of [6] to transition functions. In order to obtain such an extension, we have to assign a meaning to \(\int_0^s T_t \mu \, dt\) for every \(\mu \in M(X)\) and \(s \in [0, +\infty)\) in order to be able to obtain conditions that ensure the existence of \(w^*\)-lim \(\lim_{s \to +\infty} \frac{1}{s} \int_0^s T_t \mu \, dt\) for every \(\mu \in M(X)\), where \(w^*\)-lim stands for the limit in the weak* topology of \(M(X)\). Ideally, we would like the integrals \(\int_0^s T_t \mu \, dt, s \geq 0\), to be Bochner integrals (for details on the Bochner integral, see Dinculeanu’s comprehensive monograph [2] on vector and stochastic integrals and Appendix E of the excellent textbook in measure theory of Cohn [1]) because in such a case we could try to use results from the first volume of Dunford and Schwartz’s monograph [3] to study conditions for the existence of \(w^*\)-lim \(\lim_{s \to +\infty} \frac{1}{s} \int_0^s T_t \mu \, dt\), \(\mu \in M(X)\). However, in general (actually, in most cases of interest), given \(\mu \in M(X)\), the map \(t \mapsto T_t \mu, t \geq 0\), is not strongly measurable because the range of the map fails to be separable (for the terminology used here, see Appendix E of Cohn [1]). Fortunately, we can consider a very simple vector integral that we call the pointwise (vector) integral, and that can be used to state and prove a weak* mean ergodic theorem which in turn allows us to prove the following result.

**Theorem 1.1 (Pointwise Mean Ergodic Theorem for Transition Functions).** Let \((P_t)_{t \in [0, +\infty)}\) be a Feller transition function that is \(C_0(X)\)-strongly continuous and \(C_0(X)\)-equicontinuous (note that, as pointed out earlier, such a transition function satisfies the s.m.a., as well). Then \(\lim_{s \to +\infty} \frac{1}{s} \int_0^s S_t f(x) \, dt\) exists and is a real number whenever \(f \in C_0(X)\) and \(x \in X\).

**Remark.** It is easy to find transition functions that satisfy the conditions of the above theorem, and, therefore, for which the conclusion of the theorem holds true. A particularly large and interesting family of examples can be obtained as follows (for details on unexplained terminology, see [7]).

Assume that \((X, d)\) is a locally compact separable metric semigroup with identity \(u\). We use * to denote the convolution operation on \(M(X)\). If \(\mu \in M(X)\), then we use the notation \(\mu^0 = \delta_u\) (\(\delta_u\) is the Dirac measure concentrated
at \( u \); throughout the paper, \( \delta_x \) is the Dirac measure concentrated at \( x \in X \), \( \mu^1 = \mu \), and \( \mu^n = \mu * \mu * \cdots * \mu \) for \( n \in \mathbb{N}, n \geq 2 \). Let \( \mu \in \mathcal{M}(X) \) be a probability measure. It can be shown that the series \( \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \mu^k \) converges in the norm topology of \( \mathcal{M}(X) \) to a positive element \( \nu_\alpha \) of \( \mathcal{M}(X) \) and that \( \|\nu_\alpha\| = e^\alpha \) (where, of course, \( e = \sum_{k=0}^{\infty} \frac{1}{k!} \)) whenever \( \alpha \in \mathbb{R}, \alpha > 0 \). Set \( \mu_0 = \delta_u \), and \( \mu_\alpha = e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \mu^k \) for every \( \alpha \in \mathbb{R}, \alpha > 0 \). Then \( \mu_\alpha, \alpha \in [0, +\infty) \) are probability measures, and it can be shown that \( \mu_{\alpha + \beta} = \mu_\alpha * \mu_\beta \) for every \( \alpha \in [0, +\infty) \) and \( \beta \in [0, +\infty) \) (the family \( \mu_\alpha \), \( \alpha \in [0, +\infty) \)) is an example of a one-parameter convolution semigroup; for a comprehensive discussion of such semigroups when \( X \) is a group, see Chapter III and Chapter IV of Heyer [4], a standard reference to many topics including one-parameter semigroups). For every \( \alpha \in [0, +\infty) \), let \( P_\alpha \) be the transition probability defined by \( \mu_\alpha \) (see [7] for details). It can be shown that \( (P_\alpha)_{\alpha \in [0, +\infty)} \) is a Feller transition function. If \( \mu \) is an equicontinuous probability measure, then it can be shown that \( (P_\alpha)_{\alpha \in [0, +\infty)} \) satisfies all the conditions of Theorem 1.1. Thus, if \( \mu \) is equicontinuous, the conclusion of the theorem holds true for \( (P_\alpha)_{\alpha \in [0, +\infty)} \). In particular, the conclusion of Theorem 1.1 holds true for any transition function \( (P_\alpha)_{\alpha \in [0, +\infty)} \) defined by any probability measure \( \mu \) whenever \( (X, d) \) is a compact metric semigroup. \( \square \)

Our goal in the next section is to define the above-mentioned vector integral and to discuss several of its properties that are useful for proving Theorem 1.1. In order to keep the length of this paper within reasonable limits, we will not prove Theorem 1.1 here. The proof of Theorem 1.1 will be included in the monograph that we mentioned earlier that we are currently writing.

## 2. THE POINTWISE VECTOR INTEGRAL

Let \( (\Omega, \Sigma, \nu) \) be a measure space, let \( Y \) be a nonempty set, and let \( \mathcal{F} \) be a collection of real-valued functions defined on \( Y \) (usually \( \mathcal{F} \) is a Banach lattice of such functions). We say that a function \( \varphi : \Omega \to \mathcal{F} \) is pointwise measurable if the condition below is satisfied.

(PM) For every \( y \in Y \), the function \( \varphi_y : \Omega \to \mathbb{R} \) defined by \( \varphi_y(t) = \varphi(t)(y) \) for every \( t \in \Omega \) is \( \Sigma \)-measurable.

A pointwise measurable function \( \varphi : \Omega \to \mathcal{F} \) is said to be pointwise integrable if the two conditions below are satisfied.

(PI1) For every \( y \in Y \), the function \( \varphi_y \), defined at (PM), is integrable.
The function $I : Y \to \mathbb{R}$ defined by $I(y) = \int_\Omega \varphi(t)(y) \, d\nu(t)$ for every $y \in Y$ belongs to $\mathcal{F}$.

If $\varphi$ is pointwise integrable, then the function $I$ defined at (PI2) is called the pointwise integral of $\varphi$ and is denoted by $\mathbb{P}\int_\Omega \varphi(t) \, d\nu(t)$. For $y \in Y$, we will use the notation $\mathbb{P}\int_\Omega \varphi(t)(y) \, d\nu(t)$ for $I(y)$, rather than $(\mathbb{P}\int_\Omega \varphi(t) \, d\nu(t))(y)$, which is the formally correct notation, but it is cumbersome.

Remark A. We mentioned earlier that we believe strongly that the pointwise integral has appeared often in implicit form in the literature earlier. In support of the above assertion, let us discuss briefly how one can state a restricted form of Fubini’s theorem in terms of pointwise integrals. Let $(Y, \mathcal{Y}, \mu_1)$ and $(Z, \mathcal{Z}, \mu_2)$ be two finite measure spaces, let $\mathcal{Y} \otimes \mathcal{Z}$ be the product $\sigma$-algebra of $\mathcal{Y}$ and $\mathcal{Z}$ on $Y \times Z$, and let $\mu_1 \otimes \mu_2$ be the product measure of $\mu_1$ and $\mu_2$ on the measurable space $(Y \times Z, \mathcal{Y} \otimes \mathcal{Z})$. Now, let $g : Y \times Z \to \mathbb{R}$ be a bounded $\mathcal{Y} \otimes \mathcal{Z}$-measurable function. For every $y \in Y$, let $g_y : Z \to \mathbb{R}$ be defined by $g_y(z) = g(y, z)$ for every $z \in Z$ and, for every $z \in Z$, let $g^{(z)} : Y \to \mathbb{R}$ be defined by $g^{(z)}(y) = g(y, z)$ for every $y \in Y$. Finally, let $B_0(Y)$, $B_0(Z)$, and $B_0(Y \times Z)$ be the Banach spaces of all real-valued bounded measurable functions defined on $Y$, $Z$, and $Y \times Z$, respectively. Now let $\varphi : Y \to B_0(Z)$ and $\psi : Z \to B_0(Y)$ be defined by $\varphi(y) = g_y$ and $\psi(z) = g^{(z)}$ for every $y \in Y$ and $z \in Z$, respectively. It is easy to see that both $\varphi$ and $\psi$ are pointwise measurable, and that both $\varphi$ and $\psi$ satisfy condition (PI1). By Fubini’s theorem, the pointwise integrals of $\varphi$ and $\psi$ exist and they are: $\theta_{\varphi} : Z \to \mathbb{R}$, $\theta_{\varphi}(z) = \int_Y \varphi(y)(z) \, d\mu_1(y)$ for every $z \in Z$ and $\theta_{\psi} : Y \to \mathbb{R}$, $\theta_{\psi}(y) = \int_Z \psi(z)(y) \, d\mu_2(z)$ for every $y \in Y$, respectively. Moreover, using again Fubini’s theorem, we obtain that both $\theta_{\varphi}$ and $\theta_{\psi}$ are $\mu_2$- and $\mu_1$-integrable, respectively, and

$$\int_Z \theta_{\varphi}(z) \, d\mu_2(z) = \int_Y \theta_{\psi}(y) \, d\mu_1(y) = \int_{Y \times Z} g(y, z) \, d(\mu_1 \otimes \mu_2)(y, z). \quad \Box$$

Note that the definitions of $\varphi$ and $\theta_{\varphi}$, and the measurability of $\theta_{\varphi}$ as a real-valued function defined on the measurable space $(Z, \mathcal{Z})$ do not depend on $\mu_2$. Therefore, by Remark A, the following result is obviously true.

Proposition 2.1. Let $(Y, \mathcal{Y}, \mu)$ be a finite measure space, let $(Z, \mathcal{Z})$ be a measurable space, let $B_0(Z)$ be the Banach lattice of all real-valued bounded $\mathcal{Z}$-measurable functions defined on $Z$, and let $g : Y \times Z \to \mathbb{R}$ be a bounded function measurable with respect to the product $\sigma$-algebra $\mathcal{Y} \otimes \mathcal{Z}$. Then the function $\varphi : Y \to B_0(Z)$, $\varphi(y) = g_y$ for every $y \in Y$, where $g_y : Z \to \mathbb{R}$, $g_y(z) = g(y, z)$ for every $z \in Z$, is well-defined (in the sense that $g_y \in B_0(Z)$ for every $y \in Y$) and is pointwise integrable. The pointwise integral
\[ \text{P-} \int_Y \varphi(y) \, d\mu(y) \] (which is a real-valued \(Z\)-measurable bounded function on \(Z\)) is defined by

\[ \text{P-} \int_Y \varphi(y)(z) \, d\mu(y) = \int_Y g_y(z) \, d\mu(y) \]

for every \(z \in Z\), where (as mentioned earlier) \(\text{P-} \int_Y \varphi(y)(z) \, d\mu(y)\) stands for \((\text{P-} \int_Y \varphi(y)) \, d\mu(y)) (z)\).

We will call the pointwise integral \(\text{P-} \int_Y \varphi(y) \, d\mu(y)\) that appears in Proposition 2.1 a \(B_b(Z)\)-pointwise integral. The next result shows that such pointwise integrals appear quite naturally when dealing with transition functions.

**Proposition 2.2.** Let \((P_t)_{t \in [0,+\infty)}\) be a transition function defined on a locally compact separable metric space \((X,d)\), suppose that \((P_t)_{t \in [0,+\infty)}\) satisfies the s.m.a., and let \((\langle S_t, T_t \rangle)_{t \in [0,+\infty)}\) be the family of Markov pairs defined by \((P_t)_{t \in [0,+\infty)}\). Then, for every \(f \in B_b(X)\) and every Lebesgue measurable subset \(L\) of \([0, +\infty)\) of finite Lebesgue measure, the map \(\varphi_f^{(L)} : L \to B_b(X)\) defined by \(\varphi_f^{(L)}(t) = S_t f\) for every \(t \in L\) is pointwise integrable.

**Proof.** We apply repeatedly Proposition 2.1 in the case where \(Y = L\), \(\mu\) is the Lebesgue measure on \((Y, Z = X, \mathcal{B}(X), \mathbb{R})\), and \(g_y^{(f)} : L \times X \to \mathbb{R}\) is defined by \(g_y^{(f)}(t, x) = S_t f(x)\) for every \((t, x) \in L \times X\).

First, assume that \(f = 1_A\) for some \(A \in \mathcal{B}(X)\). Then, using (1.1), we obtain that \(S_t 1_A(x) = P_t(x, A)\) for every \(t \in L\) and \(x \in X\). Since \((P_t)_{t \in [0,+\infty)}\) satisfies the s.m.a., the map \((t, x) \mapsto P_t(x, A)\), \((t, x) \in L \times X\) is jointly measurable with respect to \(t\) and \(x\); as the map is also bounded, we can apply Proposition 2.1 in order to infer that the pointwise integral \(\text{P-} \int_L S_t f(x) \, dt\) exists in this case.

If \(f \in B_b(X)\) is a simple function; that is, if there exist \(n \in \mathbb{N}\), \(n\) measurable subsets \(A_1, A_2, \ldots, A_n\) of \(X\), and \(n\) real numbers \(a_1, a_2, \ldots, a_n\) such that \(f = \sum_{i=1}^{n} a_i 1_{A_i}\), then the map \((t, x) \mapsto S_t f(x), (t, x) \in L \times X\), is jointly measurable with respect to \(t\) and \(x\) because \(S_t f(x) = \sum_{i=1}^{n} a_i P_t(x, A_i)\) for every \((t, x) \in L \times X\), and the maps \((t, x) \mapsto P_t(x, A_i), \ (t, x) \in L \times X, i = 1, 2, \ldots, n\) are jointly measurable with respect to \(t\) and \(x\). Thus, applying Proposition 2.1 again, we obtain that the pointwise integral \(\text{P-} \int_L S_t f(x) \, dt\) exists in this case, as well.

Now, if \(f \in B_b(X), f \geq 0\), then there exists a sequence \((f_n)_{n \in \mathbb{N}}\) of simple measurable functions that converges uniformly to \(f\), and such that \(0 \leq f_n \leq f\) for every \(n \in \mathbb{N}\). Since \(S_t, t \geq 0\) are (positive) contractions of \(B_b(X)\), the sequence of maps \((t, x) \mapsto S_t f_n(x), (t, x) \in L \times X, n \in \mathbb{N}\)
Let \( \varphi_t \) and assume that the range of \( n \) \( \varphi_t \) is bounded; hence, the map \( (f, t) \rightarrow S_t f(x) \) is Bochner integrable in the sense of Appendix E of Cohn [1], and has the conditions under which a function \( \varphi \), \( n \), \( x \), the pointwise and Bochner integrals are equal. Finally, if \( f \in B_b(X) \) is not necessarily a positive element of \( B_b(X) \), then \( f = f^+ - f^- \), where \( f^+ = f \vee 0 \in B_b(X) \), \( f^- = (-f) \vee 0 \in B_b(X) \), \( f^+ \geq 0 \), \( f^- \geq 0 \). Using the above discussion, we obtain that both pointwise integrals \( \int L S_t f(x) \, dt \) and \( \int L S_t f^+(x) \, dt \), \( \int L S_t f^-(x) \, dt \) exist, and the maps \( (t, x) \rightarrow S_t f^+(x) \) and \( (t, x) \rightarrow S_t f^-(x) \) are both jointly measurable with respect to \( t \) and \( x \), and bounded; hence, the map \( (t, x) \rightarrow S_t f(x) = S_t f^+(x) - S_t f^-(x) \) is also bounded and jointly measurable with respect to \( t \) and \( x \). Using Proposition 2.1, we obtain that the map \( \varphi_f^{(L)} : L \rightarrow B_b(X) \) defined by \( \varphi_f^{(L)}(t) = S_t f \) for every \( t \in L \) is pointwise integrable. \( \square \)

Let \( J \subseteq \mathbb{R} \) be a finite union of bounded intervals of \( \mathbb{R} \), and think of \( J \) as the measure space defined by the \( \sigma \)-algebra of the Lebesgue measurable subsets of \( J \) and the Lebesgue measure on \( J \). Our goal now is to obtain sufficient conditions under which a function \( \varphi : J \rightarrow B_b(X) \) is pointwise integrable, is Bochner integrable in the sense of Appendix E of Cohn [1], and has the property that the pointwise and Bochner integrals are equal.

If \( \varphi \) is Bochner integrable in the sense of Cohn [1], we will use the notation \( B_f J \varphi(t) \, dt \) for the Bochner integral of \( \varphi \) on \( J \).

**Theorem 2.3.** Let \( \varphi : J \rightarrow B_b(X) \) be a continuous bounded function, and assume that the range of \( \varphi \) is included in \( C_b(X) \). Then the following assertions are true:

(a) \( \varphi \) is pointwise integrable;

(b) \( \varphi \) is Bochner integrable in the sense of Appendix E of Cohn [1];

(c) the pointwise and the Bochner integrals of \( \varphi \) are equal.

**Proof.** (a) Let \( g : J \times X \rightarrow \mathbb{R} \) be defined by \( g(t, x) = \varphi(t)(x) \) for every \( (t, x) \in J \times X \).

We first note that \( g \) is jointly continuous with respect to \( t \) and \( x \). Indeed, let \( (t_n, x_n) \) \( \in \mathbb{N} \) be a convergent sequence of elements of \( J \times X \), set \( (t, x) = \lim (t_n, x_n) \), and let \( \varepsilon \in \mathbb{R} \), \( \varepsilon > 0 \). Since \( \varphi \) is continuous and \( (t_n)_{n \in \mathbb{N}} \) converges to \( t \), there exists \( m_\varepsilon \in \mathbb{N} \) such that \( \| \varphi(t_n) - \varphi(t) \| < \frac{\varepsilon}{2} \) for every \( n \geq m_\varepsilon \). Since \( \varphi(t) \in C_0(X) \) and \( (x_n)_{n \in \mathbb{N}} \) converges to \( x \), there exists \( n_\varepsilon \in \mathbb{N} \) such that \( |g(t, x_n) - g(t, x)| < \frac{\varepsilon}{2} \) for every \( n \geq n_\varepsilon \). Clearly, we may choose \( n_\varepsilon \) such that \( n_\varepsilon \geq m_\varepsilon \). Then, for every \( n \in \mathbb{N} \), \( n \geq n_\varepsilon \), we have

\[
|g(t, x_n) - g(t, x)| \leq |g(t, x_n) - g(t, x_n)| + |g(t, x_n) - g(t, x)| \leq \| \varphi(t_n) - \varphi(t) \| + |g(t, x_n) - g(t, x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Since \( g \) is jointly continuous with respect to \( t \) and \( x \), it also is jointly measurable with respect to \( t \) and \( x \). Clearly, \( g \) is bounded (because of the boundedness of \( \varphi \)); since \( J \) has finite Lebesgue measure, we can apply Proposition 2.1 in order to conclude that \( \varphi \) is pointwise integrable.

(b) Since \( \varphi \) is continuous, \( \varphi(J) \) is a separable subset of \( B_b(\mathbb{X}) \) (that is, \( \varphi \) has a separable range) and \( \varphi \) is measurable (i.e., \( \varphi^{-1}(E) \) is a Lebesgue measurable subset of \( J \) whenever \( E \) is a Borel subset of \( B_b(\mathbb{X}) \)). Accordingly, \( \varphi \) is strongly measurable. Now, the map \( \langle \varphi \rangle \mapsto \|\varphi(t)\|, \ t \in J \) is continuous (because \( \|\varphi(t)\| - \|\varphi(s)\| \leq \|\varphi(t) - \varphi(s)\| \) for every \( s \in J \) and \( t \in J \), and because \( \varphi \) is continuous) and bounded (because \( \varphi \) is bounded). Since \( J \) has finite Lebesgue measure, the map \( t \mapsto \|\varphi(t)\|, \ t \in J \) is integrable. Accordingly, \( \varphi \) is Bochner integrable in the sense of Appendix E of Cohn [1].

(c) Note that if \( \psi : J \to B_b(\mathbb{X}) \) is of the form \( \psi(t) = \sum_{i=1}^{n} 1_{A_i}(t)f_i, \ t \in J, \) for some \( n \in \mathbb{N} \), \( n \) Lebesgue measurable subsets \( A_1, A_2, \ldots, A_n \) of \( J \) and \( n \) elements (real-valued bounded measurable functions defined on \( \mathbb{X} \)) \( f_1, f_2, \ldots, f_n \) of \( B_b(\mathbb{X}) \), then \( \psi \) is obviously strongly measurable (such a function \( \psi \) is called a strongly measurable simple function) and Bochner integrable in the sense of Cohn (because \( J \) has finite Lebesgue measure), and, by definition, the Bochner integral in the sense of Cohn [1] of \( \psi \) is \( B\int_{J} \psi(t) \, dt = \sum_{i=1}^{n} \lambda(A_i)f_i, \) where \( \lambda \) is the Lebesgue measure (on \( J \)). Also, \( \psi \) is pointwise integrable because for every \( x \in \mathbb{X} \) the function \( t \mapsto \sum_{i=1}^{n} 1_{A_i}(t)f_i(x), \ t \in J, \) is Lebesgue integrable since \( t \mapsto \sum_{i=1}^{n} 1_{A_i}(t)f_i(x), \ t \in J, \) is a simple measurable real-valued map, and \( J \) has finite Lebesgue measure; moreover, the pointwise integral of \( \psi \) is \( (P\int_{J} \psi(t) \, dt)(x) = \sum_{i=1}^{n} \lambda(A_i)f_i(x) \) for every \( x \in X; \) hence, \( P\int_{J} \psi(t) \, dt = B\int_{J} \psi(t) \, dt. \)

Since \( \varphi \) is Bochner integrable in the sense of Cohn [1] (as we proved at (b)), using Proposition E2, p. 351 of Cohn [1], we obtain that there exists a sequence \( (\psi_k)_{k \in \mathbb{N}} \) of strongly measurable simple functions, \( \psi_k : J \to B_b(\mathbb{X}) \) for every \( k \in \mathbb{N} \), such that \( (\psi_k(t))_{k \in \mathbb{N}} \) converges uniformly on \( \mathbb{X} \) (that is, in the norm topology of \( B_b(\mathbb{X}) \)) to \( \varphi(t) \) for every \( t \in J \), such that \( \|\psi_k(t)\| \leq \|\varphi(t)\| \) for every \( k \in \mathbb{N} \) and \( t \in J \), and such that the sequence of Bochner integrals (which do exist as pointed out above) \( (B\int_{J} \psi_k(t) \, dt)_{k \in \mathbb{N}} \) converges uniformly on \( \mathbb{X} \) to \( B\int_{J} \varphi(t) \, dt. \)

Now, for every \( x \in X \), the sequence \( (u_k^{(x)})_{k \in \mathbb{N}}, u_k^{(x)} : J \to \mathbb{R}, u_k^{(x)}(t) = \psi_k(t)(x) \) for every \( t \in J \) and \( k \in \mathbb{N} \), converges pointwise to \( u^{(x)} : J \to \mathbb{R}, \)
$u^{(x)}(t) = \varphi(t)(x)$ for every $t \in J$ (because $(\psi_k(t))_{k \in \mathbb{N}}$ converges uniformly on $X$ to $\varphi(t)$ for every $t \in J$), so we can apply the dominated convergence theorem to the sequence $(u_k^{(x)})_{k \in \mathbb{N}}$ in order to infer that the sequence $(\left( \mathbb{P} \int_J \psi_k(t) \, dt \right)(x))_{k \in \mathbb{N}}$ converges to $(\mathbb{P} \int_J \varphi(t) \, dt)(x)$. Thus, the sequence $(\mathbb{P} \int_J \psi_k(t) \, dt)_{k \in \mathbb{N}}$ converges pointwise on $X$ to $\mathbb{P} \int_J \varphi(t) \, dt$.

Since (as shown in the first part of the proof) $\mathbb{B} \int_J \psi_k(t) \, dt = \mathbb{P} \int_J \psi_k(t) \, dt$ for every $k \in \mathbb{N}$, and since $(\mathbb{B} \int_J \psi_k(t) \, dt)_{k \in \mathbb{N}}$ converges uniformly on $X$ to $\mathbb{B} \int_J \varphi(t) \, dt$, we conclude that $\mathbb{B} \int_J \varphi(t) \, dt = \mathbb{P} \int_J \varphi(t) \, dt$. □

It can be shown that the integral discussed in Dunford and Schwartz [3] is an extension of the classical Bochner integral as presented in Appendix E of Cohn [1]; that is, if $\zeta : J \to B_b(X)$ is Bochner integrable in the sense of Cohn [1], then $\zeta$ is integrable in the sense of Dunford and Schwartz [3] and the two integrals are equal. If we apply Theorem 2.3 to the integrals that appear in Theorem 1.1 and Proposition 2.2, we obtain the following result.

**Corollary 2.4.** Let $(P_t)_{t \in [0, +\infty)}$ be a Feller transition function defined on a locally compact separable metric space $(X,d)$, let $((S_t, T_t))_{t \in [0, +\infty)}$ be the family of Markov-Feller pairs defined by $(P_t)_{t \in [0, +\infty)}$: and assume that $(P_t)_{t \in [0, +\infty)}$ is $C_0(X)$-strongly continuous. If $J$ is a finite union of finite length subintervals of $[0, +\infty)$ and if $f \in C_0(X)$, then the map $\varphi^{(J)}_f : J \to B_b(X)$ defined by $\varphi^{(J)}_f(t) = S_t f$ for every $t \in J$ is pointwise integrable, and Bochner integrable in the sense of Cohn [1] and of Dunford and Schwartz [3]. Moreover, the pointwise integral, and the two Bochner integrals in the sense of Cohn [1] and in the sense of Dunford and Schwartz [3] are equal.

The proof of the corollary is obvious since the map $\varphi^{(J)}_f$ that appears in the corollary satisfies the conditions of Theorem 2.3.

The corollary is useful because it allows us to think of integrals of the form $\int_J S_t f \, dt$, $f \in C_0(X)$, as Bochner integrals in the sense of Dunford and Schwartz; hence, we can use various results of [3] to study these integrals.

We now turn our attention to another type of pointwise integral that we call the $\mathcal{M}(X)$-pointwise integral.

Let $(\Omega, \Sigma, \nu)$ be a measure space, let $\varphi : \Omega \to \mathcal{M}(X)$ be a function, and let us think of $\mathcal{M}(X)$ as a Banach lattice of real-valued functions defined on $\mathcal{B}(X)$. Then it makes sense to consider the definitions stated at the beginning of this section in the present setting. Thus, we say that $\varphi$ is pointwise measurable, or $\mathcal{M}(X)$-pointwise measurable, if, for every $A \in \mathcal{B}(X)$, the function $\psi_A^{(\varphi)} : \Omega \to \mathbb{R}$ defined by $\psi_A^{(\varphi)}(t) = \varphi(t)(A)$ for every $t \in \Omega$ is measurable, and we say that $\varphi$ is pointwise integrable, or $\mathcal{M}(X)$-pointwise integrable if, for every
A \in \mathcal{B}(X), the function \( \psi_A^{(\varphi)} \) is integrable and the map \( \mathbf{I} : \mathcal{B}(X) \to \mathbb{R} \) defined by \( \mathbf{I}(A) = \int_\Omega \psi_A^{(\varphi)}(t) \, d\nu(t) \) for every \( A \in \mathcal{B}(X) \) belongs to \( \mathcal{M}(X) \). If \( \varphi \) is \( \mathcal{M}(X) \)-pointwise integrable, we call \( \mathbf{I} \) the \( \mathcal{M}(X) \)-pointwise integral of \( \varphi \) and we use the notation \( \mathbf{I} \) rather than \( \mathbf{I} \). Also, it is convenient to use the notation \( \varphi_t \) for \( \varphi(t), t \in \Omega \), and, for \( A \in \mathcal{B}(X) \), to use \( \int_\Omega \varphi_t(A) \, d\nu(t) \) for \( \mathbf{I}(A) \) rather than \( \left( \int_\Omega \varphi_t \, d\nu(t) \right)(A) \), which is formally correct, but is cumbersome and less intuitive.

Naturally, the first concern when dealing with an integral is to find conditions for integrability. In the next result, we discuss such conditions. The conditions are general enough for our purposes, and probably they are good enough for most purposes.

**Proposition 2.5.** Let \( \varphi : \Omega \to \mathcal{M}(X) \) be an \( \mathcal{M}(X) \)-pointwise measurable function. If there exists an integrable function \( \rho : \Omega \to \mathbb{R} \) such that \( \sup_{A \in \mathcal{B}(X)} \| \varphi_t(A) \| \leq \rho(t) \) for every \( t \in \Omega \), then \( \varphi \) is \( \mathcal{M}(X) \)-pointwise integrable. In particular, \( \varphi \) is \( \mathcal{M}(X) \)-pointwise integrable whenever the measure \( \nu \) is finite and there exists \( M \in \mathbb{R} \) such that \( \| \varphi_t(A) \| \leq M \) for every \( t \in \Omega \) and \( A \in \mathcal{B}(X) \).

**Proof.** Let \( \varphi : \Omega \to \mathcal{M}(X) \) and \( \rho : \Omega \to \mathbb{R} \) be as in the statement.

Let \( A \in \mathcal{B}(X) \), and note that the function \( \psi_A^{(\varphi)} : \Omega \to \mathbb{R} \) defined by \( \psi_A^{(\varphi)}(t) = \varphi_t(A) \) for every \( t \in \Omega \) is measurable since \( \varphi \) is \( \mathcal{M}(X) \)-pointwise measurable. Since \( \| \psi_A^{(\varphi)}(t) \| \leq \rho(t) \) for every \( t \in \Omega \) and \( \rho \) is integrable, \( \psi_A^{(\varphi)} \) is integrable for every \( A \in \mathcal{B}(X) \). Thus, the map \( \mathbf{I} : \mathcal{B}(X) \to \mathbb{R} \) defined by \( \mathbf{I}(A) = \int_\Omega \psi_A^{(\varphi)}(t) \, d\nu(t) \) for every \( A \in \mathcal{B}(X) \) is well-defined; accordingly, in order to prove that \( \varphi \) is \( \mathcal{M}(X) \)-pointwise integrable, we have to show that \( \mathbf{I} \) belongs to \( \mathcal{M}(X) \). Since \( \mathbf{I}(\emptyset) = 0 \), in order to prove that \( \mathbf{I} \in \mathcal{M}(X) \), we only have to prove that \( \mathbf{I} \) is \( \sigma \)-additive.

To this end, let \( (A_n)_{n \in \mathbb{N}} \) be a sequence of mutually disjoint measurable subsets of \( X \). We have to prove that

\[
\mathbf{I} \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mathbf{I}(A_n);
\]

that is, that the series \( \sum_{n=1}^{\infty} \mathbf{I}(A_n) \) is convergent and the above equality holds true.

Since \( \varphi_t \) is a signed measure for every \( t \in \Omega \), we have

\[
\mathbf{I} \left( \bigcup_{i=1}^{n} A_i \right) = \int_\Omega \varphi_t \left( \bigcup_{i=1}^{n} A_i \right) \, d\nu(t) = \sum_{i=1}^{n} \int_\Omega \varphi_t(A_i) \, d\nu(t) = \sum_{i=1}^{n} \mathbf{I}(A_i)
\]
for every \( n \in \mathbb{N} \); thus, if \( \lim_{n \to \infty} I(\bigcup_{i=1}^{n} A_i) \) exists, then the series \( \sum_{n=1}^{\infty} I(A_n) \) is convergent, and \( \sum_{n=1}^{\infty} I(A_n) = \lim_{n \to \infty} I(\bigcup_{i=1}^{n} A_i) \). This means that in order to prove (2.1) it is enough to prove that \( \lim_{n \to \infty} I(\bigcup_{i=1}^{n} A_i) \) exists and

\[
\lim_{n \to \infty} I(\bigcup_{i=1}^{n} A_i) = I(\bigcup_{n=1}^{\infty} A_n). \tag{2.2}
\]

Now, note that \( \left( \psi_{\bigcup_{i=1}^{n} A_i} \right)_{n \in \mathbb{N}} \) is a sequence of integrable functions that converges everywhere on \( \Omega \) to \( \psi_{\bigcup_{i=1}^{\infty} A_i} \), \( \left| \psi_{\bigcup_{i=1}^{n} A_i}(t) \right| \leq \rho(t) \) for every \( n \in \mathbb{N} \) and \( t \in \Omega \), and \( \left| \psi_{\bigcup_{i=1}^{\infty} A_i}(t) \right| \leq \rho(t) \) for every \( t \in \Omega \). Thus, we can apply the Lebesgue dominated convergence theorem to the sequence \( \left( \psi_{\bigcup_{i=1}^{n} A_i} \right)_{n \in \mathbb{N}} \) and conclude that \( \lim_{n \to \infty} \int_{\Omega} \psi_{\bigcup_{i=1}^{n} A_i}(t) \, d\nu(t) = \int_{\Omega} \psi_{\bigcup_{i=1}^{\infty} A_i}(t) \, d\nu(t) \).

Thus, (2.2) is true since \( I(\bigcup_{i=1}^{n} A_i) = \int_{\Omega} \psi_{\bigcup_{i=1}^{n} A_i}(t) \, d\nu(t) \) for every \( n \in \mathbb{N} \) and \( I(\bigcup_{n=1}^{\infty} A_n) = \int_{\Omega} \psi_{\bigcup_{i=1}^{\infty} A_i}(t) \, d\nu(t) \).

We have therefore proved that \( I \in \mathcal{M}(X) \), so \( \varphi \) is pointwise integrable.

If \( \nu \) is a finite measure, and there exists \( M \in \mathbb{R} \) such that \( |\varphi_t(A)| \leq M \) for every \( A \in \mathcal{B}(X) \) and \( t \in \Omega \), then \( \varphi \) is pointwise integrable because we can use the above arguments applied to \( \varphi \) and \( \rho = M \cdot 1_{\Omega} \).

The next result consists of an application of Proposition 2.5 to answer a concern raised in Section 1; namely, under fairly general conditions, we can use the corollary to assign a meaning to integrals of the form \( \int_{0}^{\infty} T_t \mu \, dt \), where \( s \in [0, +\infty) \), \( \mu \in \mathcal{M}(X) \), and \( (T_t)_{t \in [0, +\infty)} \) is the semigroup of operators on \( \mathcal{M}(X) \) generated by a transition probability.

**Corollary 2.6.** Let \( (P_t)_{t \in [0, +\infty)} \) be a transition function defined on \((X, d)\), assume that \( (P_t)_{t \in [0, +\infty)} \) satisfies the s.m.a., let \((S_t, T_t))_{t \in [0, +\infty)} \) be the family of Markov pairs defined by \( (P_t)_{t \in [0, +\infty)} \), and let \( L \) be a Lebesgue measurable subset of \([0, +\infty)\) that has finite Lebesgue measure. Then, for every \( \mu \in \mathcal{M}(X) \), the map \( t \mapsto T_t \mu \), \( t \in L \), is \( \mathcal{M}(X) \)-pointwise integrable.

**Proof.** Let \( \mu \in \mathcal{M}(X) \), and let \( \varphi : L \to \mathcal{M}(X) \), \( \varphi_t = T_t \mu \) for every \( t \in L \). We have to prove that \( \varphi \) is \( \mathcal{M}(X) \)-pointwise integrable.

We first note that \( \varphi \) is \( \mathcal{M}(X) \)-pointwise measurable. Indeed, let \( A \in \mathcal{B}(X) \), and note that, since \( (P_t)_{t \in [0, +\infty)} \) satisfies the s.m.a., we can apply Proposition 5.2.1-(a), p. 159 of Cohn [1] to the measure spaces \((X, \mathcal{B}(X), \mu)\) and \((L, \mathcal{L}(L), \lambda)\), where \( \mathcal{L}(L) \) is the \( \sigma \)-algebra of all Lebesgue measurable subsets of \( L \) and \( \lambda \) is the Lebesgue measure, and to the map \( (t, x) \mapsto P_t(x, A) \), \( (t, x) \in L \times X \), in order to obtain that the map \( t \mapsto \varphi_t(A) = \int_X P_t(x, A) \, d\mu(x) \),
Let of Cohn [1], and we obtain that
\( \varphi_t(A) \) is a simple function.

We will prove that (2.3) holds true in three steps: first for
\( f \in B_b(X) \), then for the case when
\( f \) is integrable.

\[ \int_{[0, \infty)} \varphi_t(A) d\mu = \|T_t\mu\| = \|\mu\| \quad \text{for every } t \in L \text{ and } A \in \mathcal{B}(X). \]

Since \( L \) has finite Lebesgue measure, using Proposition 2.5 we obtain that \( \varphi \) is \( M(X) \)-pointwise integrable.

\( \square \)

In the next result we discuss a useful relationship between the \( B_b(X) \)-pointwise integral \( P_\cdot \int_L S_t f \, dt \) and the \( M(X) \)-pointwise integral \( P_\cdot \int_L T_t \mu \, dt \) in the case in which we deal with a family \( ((S_t, T_t))_{t \in [0, \infty)} \) of Markov pairs defined by a transition function \( (P_t)_{t \in [0, \infty)} \) that satisfies the s.m.a., \( f \in B_b(X) \), \( \mu \in M(X) \), and \( L \) is a Lebesgue measurable subset of \([0, \infty)\) that has finite Lebesgue measure.

**Proposition 2.7.** Let \( (P_t)_{t \in [0, \infty)} \) be a transition function defined on \((X, d)\), assume that \( (P_t)_{t \in [0, \infty)} \) satisfies the s.m.a., let \( ((S_t, T_t))_{t \in [0, \infty)} \) be the family of Markov pairs defined by \( (P_t)_{t \in [0, \infty)} \), and let \( L \) be a Lebesgue measurable subset of \([0, \infty)\) that has finite Lebesgue measure. Then
\[ (2.3) \quad \left\langle P_\cdot \int_L S_t f \, dt, \mu \right\rangle = \left\langle f, P_\cdot \int_L T_t \mu \, dt \right\rangle \]
for every \( f \in B_b(X) \) and \( \mu \in M(X) \).

Note that (2.3) makes sense because, by Proposition 2.2, the \( B_b(X) \)-pointwise integral \( P_\cdot \int_L S_t f \, dt \) exists while the \( M(X) \)-pointwise integral \( P_\cdot \int_L T_t \mu \, dt \) exists by Corollary 2.6.

**Proof.** We first note that it is easy to see that it is enough to prove the proposition under the assumption that \( \mu \geq 0 \). Thus, assume that \( \mu \geq 0 \).

We will prove that (2.3) holds true in three steps: first for \( f = 1_A \) for some \( A \in \mathcal{B}(X) \), then for the case when \( f \) is a simple \( \mathcal{B}(X) \)-measurable function and, finally for the general case when \( f \in B_b(X) \) is not necessarily a simple function.

**Step 1.** Assume that \( f = 1_A \) for some \( A \in \mathcal{B}(X) \). The fact that \( (P_t)_{t \in [0, \infty)} \) satisfies the s.m.a. allows us to apply Proposition 5.2.1-(b), p. 159 of Cohn [1], and we obtain that
\[ \left\langle P_\cdot \int_L S_t 1_A \, dt, \mu \right\rangle = \int_X \left( P_\cdot \int_L S_t 1_A(x) \, dt \right) \, d\mu(x) = \]
\[ = \int_X \left( \int_L P_t(x, A) \, dt \right) \, d\mu(x) = \int_L \left( \int_X P_t(x, A) \, d\mu(x) \right) \, dt = \]
\[ = P_\cdot \int_L T_t \mu(A) \, dt = \left\langle 1_A, P_\cdot \int_L T_t \mu \, dt \right\rangle. \]
Step 2. If $f$ is a $B(X)$-measurable simple function (that is, if there exist $m \in \mathbb{N}$, $m$ measurable subsets $A_1, A_2, \ldots, A_m$ of $X$ and $m$ real numbers $a_1, a_2, \ldots, a_m$ such that $f = \sum_{i=1}^{m} a_i1_{A_i}$), then, using Step 1, we obtain easily that (2.3) holds true for $f$.

Step 3. Assume now that $f \in B_b(X)$. Clearly, it is enough to prove that (2.3) is true under the assumption that $f \geq 0$.

Thus, assume that $f \geq 0$. Then there exists a monotone nondecreasing sequence $(f_n)_{n \in \mathbb{N}}$ of real-valued positive simple $B(X)$-measurable functions on $X$ such that $(f_n)_{n \in \mathbb{N}}$ converges uniformly to $f$. Therefore, the sequence $(\langle f_n, P-\int_{L} T(t) d\mu(t) \rangle)_{n \in \mathbb{N}}$ converges to $\langle f, P-\int_{L} T(t) d\mu(t) \rangle$.

Since $S_t$ is a positive contraction of $B_b(X)$, the sequence $(S_t f_n)_{n \in \mathbb{N}}$ converges uniformly on $X$ for every $t$. Thus, for every $x \in X$, the sequence of positive functions $(\xi_n^{(x)})_{n \in \mathbb{N}}$, $\xi_n^{(x)} : L \rightarrow \mathbb{R}$, $\xi_n^{(x)}(t) = S_t f_n(x)$ for every $t \in L$ and $n \in \mathbb{N}$, is monotone nondecreasing and converges pointwise on $L$ to $\xi^{(x)} : L \rightarrow \mathbb{R}$, $\xi^{(x)}(t) = S_t f(x)$ for every $t \in L$; accordingly, by the monotone convergence theorem, the sequence $(\int_{L} \xi_n^{(x)}(t) d\mu(t))_{n \in \mathbb{N}}$ converges to $\int_{L} \xi^{(x)}(t) d\mu(t)$. The fact that $(\int_{L} \xi_n^{(x)}(t) d\mu(t))_{n \in \mathbb{N}}$ converges to $\int_{L} \xi^{(x)}(t) d\mu(t)$ for every $x \in X$ means that the sequence of functions $(P-\int_{L} S_t f_n d\mu d\mu)_{n \in \mathbb{N}}$ converges pointwise on $X$ to $P-\int_{L} S_t f d\mu d\mu$. Since $(P-\int_{L} S_t f_n d\mu d\mu)_{n \in \mathbb{N}}$ is a monotone nondecreasing sequence of positive $B(X)$-measurable functions that converges pointwise on $X$ to $(P-\int_{L} S_t f d\mu d\mu)$, using the monotone convergence theorem in the space $(X, B(X), \mu)$ we deduce that the sequence $(\int_{X} (P-\int_{L} S_t f_n(x) d\mu(x)) d\mu(x))_{n \in \mathbb{N}}$ converges to $\int_{X} (P-\int_{L} S_t f(x) d\mu(x)) d\mu(x)$; that is, the sequence $(\langle P-\int_{L} S_t f_n d\mu, \mu \rangle)_{n \in \mathbb{N}}$ converges to $\langle P-\int_{L} S_t f d\mu, \mu \rangle$.

Since by Step 2 $\langle P-\int_{L} S_t f_n d\mu, \mu \rangle = \langle f_n, P-\int_{L} T(t) d\mu(t) \rangle$ for every $n \in \mathbb{N}$, we conclude that $\langle P-\int_{L} S_t f d\mu, \mu \rangle = \langle f, P-\int_{L} T(t) d\mu(t) \rangle$. □

If $L$ is an interval with endpoints $a$ and $b$, $a \leq b$, and if $\varphi$ is a $B_b(X)$-valued function on $L$, then we denote the $B_b(X)$-pointwise integral and the Bochner integral of $\varphi$ by $P-\int_{a}^{b} \varphi d\mu$ and $P-\int_{a}^{b} \varphi d\lambda$, respectively; similarly, if $\varphi$ is an $M(X)$-valued function on $L$, then $P-\int_{a}^{b} \varphi d\mu$ stands for the $M(X)$-pointwise integral of $\varphi$.

In Section 1 we pointed out that, in order to obtain a weak* mean ergodic theorem dealing with the averages $\frac{1}{t} \int_{a}^{t} T_s \mu ds$, $t > 0$, where $\mu \in M(X)$ and $(T_t)_{t \in [0, +\infty)}$ is the semigroup of operators defined on $M(X)$ by a transition function, we have to assign a meaning to the integrals $\int_{a}^{t} T_s \mu ds$, $t \geq 0$. In view of our discussion so far, we define these integrals to be $M(X)$-pointwise integrals; by Corollary 2.6, the integrals exist whenever the transition function
defining \((T_t)_{t \in [0, +\infty)}\) satisfies the s.m.a. Naturally, the integrals are denoted \(P_\ast \int_0^t T_s \mu \, ds\), \(t \geq 0\).

As usual, given a function \(\varphi : (0, +\infty) \to \mathcal{M}(X)\), we say that \(\varphi\) converges in the weak* topology of \(\mathcal{M}(X)\) as \(t \to +\infty\) if there exists \(\mu \in \mathcal{M}(X)\) such that the limit \(\lim_{t \to +\infty} \langle f, \varphi(t) \rangle\) exists and is equal to \(\langle f, \mu \rangle\) for every \(f \in C_0(X)\); in such a case we also say that the limit of \(\varphi\) in the weak* topology of \(\mathcal{M}(X)\) exists as \(t \to +\infty\) and is equal to \(\mu\). We call \(\mu\) the weak* limit of \(\varphi\) as \(t \to +\infty\) and we denote \(\mu\) by \(w^\ast \lim_{t \to +\infty} \varphi(t)\).

Our discussion so far allows us to state the weak* mean ergodic theorem mentioned in Section 1 that can be used to prove the pointwise mean ergodic theorem for transition functions (Theorem 1.1).

**Theorem 2.8 (Weak* Mean Ergodic Theorem for Transition Functions).**

Let \((P_t)_{t \in [0, +\infty)}\) be a Feller transition function that is \(C_0(X)\)-strongly continuous and \(C_0(X)\)-equicontinuous. Then, for every \(\mu \in \mathcal{M}(X)\), the limit \(w^\ast \lim_{t \to +\infty} \frac{1}{t} P_\ast \int_0^t T_s \mu \, ds\) exists and is an invariant element for \((T_t)_{t \in [0, +\infty)}\) in the sense that if \(\mu^* = w^\ast \lim_{t \to +\infty} \frac{1}{t} P_\ast \int_0^t T_s \mu \, ds\), then \(T_t \mu^* = \mu^*\) for every \(t \in [0, +\infty)\); if \(\mu \geq 0\), then \(\mu^* \geq 0\), as well.

We will not prove the theorem here; a proof will be given in the book that we mentioned earlier that we are currently writing.

Note that if \(\mu = \delta_x\), \(x \in X\) and \(L = [0, s], s \in [0, +\infty)\), then (2.3) becomes

\[
(2.4) \quad \int_0^s S_t f(x) \, dt = \left\langle P_\ast \int_0^s S_t f \, dt, \delta_x \right\rangle = \left\langle f, P_\ast \int_0^s T_t \delta_x \, dt \right\rangle
\]

for every \(f \in B_b(X)\). Using (2.4) we can easily see that, indeed, as mentioned before stating Theorem 2.8, the truth of the theorem implies that Theorem 1.1 holds true. Equation (2.3) of Proposition 2.7 is also used in the proof of Theorem 2.8 in order to “transfer” the study of \(\mathcal{M}(X)\)-pointwise integrals to \(B_b(X)\)-pointwise integrals; since under the conditions of Theorem 2.8, Corollary 2.4 can be used, we obtain that the \(B_b(X)\)-pointwise integrals under consideration are also Bochner integrals, so we can use the powerful machinery of Dunford and Schwartz [3], and then, using (2.3) again, we can “return” to \(\mathcal{M}(X)\)-pointwise integrals.

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REFERENCES


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