SOME RESULTS AND ALGORITHMS FOR A CLASS OF $\gamma$-INVEX EQUILIBRIUM PROBLEMS

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We consider two new classes of $(\gamma, \mu)$-preinvex functions and $\xi$-skew symmetric functions, respectively. Also, two new concepts of generalized pseudomonotonicity and generalized partially relaxed monotonicity of mappings are introduced. By applying the auxiliary principle technique, two predictor-corrector algorithms for solving mixed quasi $\gamma$-invex equilibrium problems are proposed and analyzed. The convergence of these methods is obtained under $(\sigma, \eta)$-pseudomonotonicity with respect to $\xi$-skew symmetry and $(\tau, \eta)$ partially relaxed monotonicity assumptions.

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1. INTRODUCTION

Equilibrium theory plays an important role in many fields such as mechanics, physics, nonlinear programming, economics and engineering sciences [1, 6, 9, 10, 16, 17].

Because of their wide applicability, interesting and intensively studied subjects are the analysis of solution existence and the development of an efficient iterative algorithm to compute the approximate solution [5, 18, 21].

The most known techniques, like projection method with its variant forms and Wiener-Hopf equations method require strongly monotonicity, Lipschitz continuity or differentiability conditions [2, 3, 8, 14]. This fact motivated the introduction of the auxiliary principle technique, first used for solving a variational inequality [7], then for many classes of variational inequalities and equilibrium problems [4, 12, 15]. The existence of this method encouraged the appearance of some extended properties of functions, such as convexity, monotonicity, coercivity [11, 13, 19, 20].

In this paper we introduce new concepts of generalized preinvexity and generalized skew symmetry. We also, consider a class of quasi $\gamma$-invex equilibrium problems, which involves $(\sigma, \eta)$-pseudomonotonicity with respect to a
ξ-skew symmetric function or \((\tau, \eta)\)-partially relaxed monotonicity. Here, we present strong and weak versions of these properties and show, by examples, the relations between them.

In order to solve the problems, we consider an auxiliary \(\gamma\)-invex equilibrium problem and prove the convergence of the iterative sequence for some new predictor-corrector methods.

## 2. DEFINITIONS AND SOME PRELIMINARY RESULTS

Let \(H\) be a real Hilbert space with norm \(\|\cdot\|\) and inner product \(\langle \cdot, \cdot \rangle\). Let \(K \subset H\) be a nonempty convex subset, \(f : K \to H\) and \(\eta(\cdot, \cdot) : K \times K \to H\) two continuous functions, and \(\gamma : K \times K \to \mathbb{R}\) a real function.

**Definition 2.1** (see [12]). Let \(u \in K\). Then \(K\) is said to be \(\gamma\)-invex at \(u\) with respect to \(\eta\) if \(u + t\gamma(v, u) \eta(v, u) \in K\) for all \(u, v \in K\) and \(t \in [0, 1]\).

If \(K\) is \(\gamma\)-invex at \(u\) with respect to \(\eta\) for any \(u \in K\), then \(K\) is called \(\gamma\)-invex with respect to \(\eta\) or \((\gamma, \eta)\)-connected.

**Remark 2.1.** For \(\gamma \equiv 1\) we say that \(K\) is an invex set with respect to \(\eta\) (see [20]).

In what follows we suppose that \(K\) is a nonempty closed \(\gamma\)-invex subset in \(H\) with respect to \(\eta\).

**Definition 2.2** (see [12]). A function \(f : K \to H\) is said to be \(\gamma\)-preinvex with respect to \(\eta\) if

\[
f(u + t\gamma(v, u) \cdot \eta(v, u)) \leq (1 - t)f(u) + tf(v)
\]

for all \(u, v \in K\) and \(t \in [0, 1]\).

**Remark 2.2.** For \(\gamma \equiv 1\) we say that \(K\) is preinvex with respect to \(\eta\) (see [20]).

**Definition 2.3.** Let \(\mu\) be a real number. We say that \(f\) is a \((\gamma, \mu)\)-preinvex function with respect to \(\eta\) if

\[
f(u + t\gamma(v, u) \cdot \eta(v, u)) \leq (1 - t)f(u) + tf(v) - t(1 - t)\mu \cdot \|\eta(v, u)\|^2
\]

for all \(u, v \in K\) and \(t \in [0, 1]\).

According as \(\mu > 0, \mu = 0\) or \(\mu < 0\), we say that \(f\) is \(\gamma\)-strongly preinvex, \(\gamma\)-preinvex or \(\gamma\)-weakly preinvex with respect to \(\eta\).

**Remark 2.3.** For \(\mu = 0\) we obtain Definition 2.2.

**Remark 2.4.** It is obvious that \(\gamma\)-strong preinvexity implies \(\gamma\)-preinvexity, which in turn implies \(\gamma\)-weak preinvexity, but the opposite implications do not hold.
Remark 2.5. In the differentiable case, the condition of \((\gamma, \mu)\)-preinvexity is equivalent to the inequality
\[
f(v) - f(u) \geq \gamma(v, u) \langle f'(u), \eta(v, u) \rangle + \mu \| \eta(v, u) \|^2, \quad \forall u, v \in K.
\]

Definition 2.4. Let \(\varphi : H \times H \rightarrow \mathbb{R} \cup \{+\infty\}\) be a bifunction and \(\xi\) a real number. We say that \(\varphi\) is \(\xi\) skew-symmetric with respect to \(\eta\) if
\[
\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq \xi \| \eta(v, u) \|^2
\]
for all \(u, v \in H\).

A bifunction \(\varphi\) is called strongly skew-symmetric, skew-symmetric or weakly skew-symmetric with respect to \(\eta\) if \(\xi > 0\), \(\xi = 0\) or \(\xi < 0\).

Thus, we extend the concept of skew-symmetry obtained for \(\xi = 0\).

Remark 2.6. It is clear that strong skew-symmetry implies skew-symmetry, which implies weak skew-symmetry with respect to \(\eta\), but the converse is not true. In order to show that, we give the example below.

Example 2.1. Let \(H = \mathbb{R}\), \(\varphi_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\), \(i = 1, 2, 3\), \(\varphi_1(u, v) = 2u^2v^2\), \(\varphi_2(u, v) = 2u^2 - v^2\), \(\varphi_3(u, v) = -u^2v^2\) be three continuous bifunctions and \(\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\), \(\eta(u, v) = u^2 - v^2\). Obviously, \(\varphi_1\) is strongly skew-symmetric for any \(\xi\) with \(0 < \xi \leq 2\), \(\varphi_2\) is skew-symmetric, but there is no \(\xi > 0\) such that \(\varphi_2\) is strongly skew-symmetric, \(\varphi_3\) is weakly skew-symmetric if \(\xi \leq -1\), but not skew-symmetric, with respect to the same \(\eta\).

Remark 2.7. In the case where \(\varphi\) is bilinear, for all \(u, v \in H\) we have
\[
\varphi(u, u) \geq \xi \| \eta(v, u + v) \|^2.
\]

In what follows we consider a continuous function \(F : K \times K \rightarrow \mathbb{R}\) and a continuous bifunction \(\varphi : K \times K \rightarrow \mathbb{R} \cup \{+\infty\}\). We are looking for a \(u \in K\) such that
\[
(2.1) \quad F(u, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K.
\]

This problem is called the mixed quasi \(\gamma\)-invex equilibrium problem. For \(\gamma \equiv 1\) we get the mixed quasi invex equilibrium problem (see [15]).

If \(F(u, v) = \langle Tu, \eta(v, u) \rangle\), where \(T : H \rightarrow H\) and \(\eta\) as above, this problem is equivalent to the mixed quasi variational-like problem, namely, find \(u \in K\) such that \(\langle Tu, \eta(v, u) \rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \forall v \in K\).

Now, we define some classes of operators.

Definition 2.5. Let \(\sigma\) be a real number, \(F : K \times K \rightarrow \mathbb{R}\) a real function and \(\varphi : K \times K \rightarrow \mathbb{R} \cup \{+\infty\}\). We say that \(F\) is \((\sigma, \eta)\)-pseudomonotone with respect to \(\varphi\) if
\[
F(u, v) + \varphi(v, u) - \varphi(u, u) \geq 0 \Rightarrow -F(v, u) + \varphi(v, u) - \varphi(u, u) \geq \sigma \| \eta(v, u) \|^2
\]
for all \( u, v \in K \). If \( \sigma > 0, \sigma = 0 \) or \( \sigma < 0 \), \( F \) is said to be strongly \( \eta \)-pseudomonotone, \( \eta \)-pseudomonotone or weakly \( \eta \)-pseudomonotone with respect to \( \varphi \). The case \( \sigma = 0 \) is known as pseudomonotonicity with respect to \( \varphi \).

**Remark 2.8.** It is clear that strong \( \eta \)-pseudomonotonicity implies \( \eta \)-pseudomonotonicity, which implies weak \( \eta \)-pseudomonotonicity with respect to a bifunction, but the converse is not true.

**Example 2.2.** Let \( K = \mathbb{R}, F_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2, 3, F_1(u, v) = -(u^2 - v^2)^2, F_2(u, v) = u - v, F_3(u, v) = \frac{1}{4} (u^2 - v^2)^2 \), \( \varphi_i, i = 1, 2, 3 \) and \( \eta \) as in Example 2.1. It is easy to show that \( F_1 \) is \( \eta \)-strongly pseudomonotone with respect to \( \varphi_3 \) for any \( \sigma_1 \) such that \( 0 < \sigma_1 \leq 2 \), while \( F_2 \) is \( \eta \)-pseudomonotone with respect to \( \varphi_2 \), but there is no \( \varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) such that \( F_2 \) is strongly \( \eta \)-pseudomonotone with respect to \( \varphi \). We can also prove that \( F_3 \) is \( \eta \)-weakly pseudomonotone with respect to \( \varphi_1 \) for any \( \sigma_3 \leq -1 \), but it is not pseudomonotone with respect to any \( \varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \).

**Definition 2.6.** Let \( \tau \) be a real number. A function \( F \) is called \((\tau, \eta)\)-partially relaxed monotone if

\[
F(u, v) + F(v, z) \leq \tau \|\eta(z, u)\|^2
\]

for all \( u, v, z \in K \). If \( \tau < 0, \tau = 0 \) or \( \tau > 0 \) we say that \( F \) is strongly \( \eta \)-partially relaxed monotone, partially relaxed monotone or weakly \( \eta \)-partially relaxed monotone. For \( \tau = 0 \) and \( z = u \) with \( \eta(u, u) = 0 \), we get the definition of monotonicity (see [15]):

\[
F(u, v) + F(v, u) \leq 0, \quad \forall u, v \in K.
\]

**Remark 2.8.** It is easy to show that strong \( \eta \)-partially relaxed monotonicity implies partially relaxed monotonicity, which implies weak \( \eta \)-partially relaxed monotonicity, but the opposite implications do not hold.

**Example 2.3.** Let \( K = \mathbb{R}, F_i, \eta_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2, 3, F_1(u, v) = u^4 - v^4, F_2(u, v) = -u^2 - v^2, F_3(u, v) = -u^4 - v^4, \eta_1(u, v) = u^2 + v^2, \eta_2(u, v) = v - u, \eta_3(u, v) = u^2 - v^2 \). It is clear that \( F_1 \) is weakly \( \eta_1 \)-partially relaxed monotone for \( \tau_1 = 1 \), but it is not partially relaxed monotone, \( F_2 \) is partially relaxed monotone but it is not strongly partially relaxed monotone for any \( \tau < 0 \), while \( F_3 \) is strongly \( \eta_3 \)-partially relaxed monotone for \( \tau = -1 \).

**Definition 2.7.** A function \( F : K \times K \rightarrow \mathbb{R} \) is called hemicontinuous if the mapping \( \Psi(t) : [0, 1] \rightarrow \mathbb{R}, \Psi(t) = F(u + t\eta(v, u), v) \) is continuous at \( t = 0 \) for all \( u, v \in K \).

Relative to problem (2.1) we state
LEMMA 2.1. Let $K$, $\gamma$, $\eta$, $F$ and $\varphi$ as above and $\mu_1$, $\mu_2$, $\sigma$ be three real numbers. Assume that

- $(i_1)$ $F$ is $(\sigma, \eta)$-pseudomonotone with respect to $\varphi$;
- $(i_2)$ the function $u \mapsto F(v, u)$ is $(\gamma, \mu_1)$-preinvex for any $v \in K$;
- $(i_3)$ the function $v \mapsto \varphi(v, u)$ is $(\gamma, \mu_2)$-preinvex for any $u \in K$;
- $(i_4)$ $K$ is a $\gamma$-inveex set;
- $(i_5)$ $\|\eta(v_1, u)\| = t\|\eta(v, u)\|$ for any $v_1 = u + t\gamma(v, u)\eta(v, u)$, where $u, v \in K$ and $t \in [0, 1]$;
- $(i_6)$ $\lim_{t \searrow 0} \frac{F(v_t, u)}{t} = 0$ for any $u \in K$ solution of (2.1) and all $v \in K$;
- $(i_7)$ $F$ is hemicontinuous;
- $(i_8)$ $\mu_1 + \mu_2 \geq 0$.

Then problem (2.1) is equivalent to finding $u \in K$ such that

$$
F(v, u) + \varphi(u, u) - \varphi(v, u) \leq \sigma\|\eta(v, u)\|^2
$$

for all $v \in K$.

Proof. Let $u \in K$ be a solution of the $\gamma$-inveex equilibrium problem (2.1). Then for all $v \in K$ we have

$$
F(u, v) + \varphi(v, u) - \varphi(u, u) \geq 0.
$$

By (i_1), $F$ is $(\gamma, \eta)$-pseudomonotone with respect to $\varphi$ and it follows from (2.1) that

$$
-F(v, u) + \varphi(v, u) - \varphi(u, u) \geq \sigma\|\eta(v, u)\|^2,
$$

for any $v \in K$, so $u$ is a solution of (2.2).

Conversely, let $u \in K$ be a solution of problem (2.2). Hence

$$
-F(v, u) + \varphi(v, u) - \varphi(u, u) \geq \sigma\|\eta(u, v)\|^2.
$$

Since $K$ is a $\gamma$-inveex set with respect to $\eta$, we have $v_t = u + t\gamma(v, u)\eta(v, u) \in K$ for all $u, v \in K$ and $t \in [0, 1]$. Taking $v = v_t$ in (2.4), for any $v_t \in K$ we have

$$
F(v_t, u) + \varphi(u, u) - \varphi(v_t, u) \leq -\sigma\|\eta(v_t, u)\|^2.
$$

Now, by (i_3) and (i_5) we have

$$
F(v_t, u) + \varphi(u, u) \leq t\varphi(v, u) + (1-t)\varphi(u, u) - t(1-t)\mu_2\|\eta(v, u)\|^2 - \sigma t^2\|\eta(v, u)\|^2.
$$

From this relation we get

$$
F(v_t, u) \leq t\{\varphi(v, u) - \varphi(u, u) - (1-t)\mu_2\|\eta(v, u)\|^2 - t\sigma\|\eta(v, u)\|^2\}.
$$

Using (i_2) we deduce that

$$
F(v_t, v) \leq tF(v_t, v) + (1-t)F(v_t, u) - t(1-t)\mu_1\|\eta(v, u)\|^2
$$
and now, by (2.5), we have
\[
F(v_t, v_t) \leq tF(v_t, v) + (1 - t)t\{\varphi(v, u) - \varphi(u, u) - (1 - t)\mu_2\|\eta(v, u)\|^2 - \\
- \sigma t\|\eta(v, u)\|^2\} - t(1 - t)\mu_1\|\eta(v, u)\|^2,
\]
which is equivalent to
\[
F(v_t, v_t) \leq tF(v_t, v) + (1 - t)t\{\varphi(v, u) - \varphi(u, u) - (1 - t)\mu_2\|\eta(v, u)\|^2 - \\
- \sigma t\|\eta(v, u)\|^2 - \mu_1\|\eta(v, u)\|^2\}.
\]
Dividing by \(t\), we obtain
\[
\frac{F(v_t, v_t)}{t} \leq F(v_t, v) + (1 - t)t\{\varphi(v, u) - \varphi(u, u) - (1 - t)\mu_2\|\eta(v, u)\|^2 - \\
- \sigma t\|\eta(v, u)\|^2 - \mu_1\|\eta(v, u)\|^2\}.
\]
From (i6) and (i7), for any \(v \in K\) we get,
\[
F(u, v) + \{\varphi(v, u) - \varphi(u, u) - \|\eta(v, u)\|^2(\mu_1 + \mu_2)\} \geq 0.
\]
Since \(\mu_1 + \mu_2 \geq 0\) by (i8), \(u \in K\) is a solution of (2.1). Thus the lemma is proved.

**Corollary 2.1** (see [15]). Let \(F\) be pseudomonotone, hemicontinuous and preinvex in the second argument and let \(\varphi\) be preinvex in the first argument. Then problems (2.1) and (2.2) are equivalent.

**Proof.** Take \(\mu_1 = \mu_2 = \sigma = 0\) in Lemma 2.1. \(\square\)

**Remark 2.9.** Problem (2.2) is also a mixed \(\gamma\)-invex quasi equilibrium problem, called the dual problem of (2.1).

**Remark 2.10.** It follows from (i1) that \(F(u, u) \leq 0\) for any \(u \in K\). Thus, assumption (i6) is justified.

**Remark 2.11.** If in Lemma 2.1 we replace (i2) and (i3) by \(F\) strongly preinvex and \(\varphi\) weakly preinvex or by \(F\) weakly preinvex and \(\varphi\) strongly preinvex respectively, then the result still holds.

### 3. ALGORITHMS AND CONVERGENCE

In this section we consider two classes of iterative methods for solving (2.1) by using the auxiliary principle technique, and study the convergence under generalized pseudomonotonicity and preinvexity assumptions.

Let \(u \in K, E: K \rightarrow H\) be a differentiable and \((\gamma, \mu)\)-preinvex function with respect to \(\eta\), let \(\mu\) and \(\beta\) be real numbers, and \(\rho\) a positive constant.
Consider the auxiliary $\gamma$-invex equilibrium problem: find a unique $w \in K$ such that
\[
\rho F(w, v) + \gamma(v, w) \langle E'(w) - E'(u), \eta(v, w) \rangle \geq \rho \{ \varphi(w, w) - \varphi(v, w) \} + \beta \| \eta(w, u) \|^2
\]
for all $v \in K$.

**Algorithm 3.1.** For a given $u_0 \in H$, calculate the approximate solution $u_{n+1}$ by the iterative scheme
\[
\rho F(u_{n+1}, v) + \gamma(v, u_{n+1}) \langle E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1}) \rangle + 
\rho \{ \varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1}) \} \geq \beta \| \eta(u_{n+1}, u_n) \|^2,
\]
for all $v \in K$.

**Remark 3.1.** In the case where $\beta = 0$ or $\eta(u, u) = 0$ for any $u \in K$ we retrieve Algorithm 3.1 from [15]. We also get [15] for $\eta(v, u) = v - u$ and $\beta = 0$, or for $F(v, u) = \langle Tu, \eta(v, u) \rangle$ and $\beta = 0$. In these cases, the method can be applied for solving mixed quasi-equilibrium problems and quasi variational-like inequalities, respectively.

**Theorem 3.1.** Let $K$, $\varphi$, $\gamma$, $\eta$, $E$, $F$, $\mu$, $\beta$ and $\rho$ be as above, and let $\sigma$, $\xi$ be two real numbers. Suppose that
\begin{enumerate}
\item[(i)_1] $F$ is $(\sigma, \eta)$-pseudomonotone with respect to $\varphi$;
\item[(i)_2] $\varphi$ is $\xi$-skew symmetric with respect to $\eta$;
\item[(i)_3] $\gamma(u, v)\eta(u, v) = \gamma(u, z)\eta(u, z) + \gamma(z, v)\eta(z, v)$ for all $u, v, z \in H$;
\item[(i)_4] $\mu + \beta > 0$;
\item[(i)_5] $\xi - \sigma \geq 0$.
\end{enumerate}
Then the approximate solution $u_{n+1}$ obtained from Algorithm 3.1 converges to a solution $u \in K$ of the mixed quasi $\gamma$-invex equilibrium problem (2.1).

**Proof.** Let $u \in K$ be a solution of (2.1). Then for any $v \in K$ from (i)_1 we have
\[
F(v, u) + \varphi(v, u) - \varphi(u, u) \geq \sigma \| \eta(u, v) \|^2.
\]
If we take $v = u_{n+1}$, in (3.1) we get
\[
F(u_{n+1}, u) + \varphi(u_{n+1}, u) - \varphi(u, u) \geq \sigma \| \eta(u, u_{n+1}) \|^2.
\]
As in [22], we consider the function
\[
B(u, z) = E(u) - E(z) - \gamma(u, z) \langle E'(z), \eta(u, z) \rangle.
\]
Using the $(\gamma, \mu)$-preinvexity and differentiability of $E$ we deduce that
\[
B(u, z) \geq \mu \| \eta(u, z) \|^2.
\]
Now, by (i)_2, (i)_1, (3.2) and (3.3) we get
\[
B(u, u_n) - B(u, u_{n+1}) = E(u_{n+1}) - E(u_n) - \gamma(u, u_n) \langle E'(u_n), \eta(u, u_n) \rangle +
\]
Thus, \( u \) is a weak solution of (2.1). If this is not true, then it follows from (i_4) and (i_5) that \( B(u, u_n) \geq 0 \) and \( \lim_{n \to \infty} ||
abla u_n(u, u_n)|| = 0 \).

Now, we can prove the convergence of the sequence \( (u_n)_{n \geq 1} \) from Algorithm 3.1 by using the technique of Zhu and Marcotte [22]. The proof is complete. \( \square \)

For \( \gamma = 1 \), we get the following result.

**COROLLARY 3.1 (see [17]).** Suppose that one of the conditions from Remark 3.1 is satisfied and

- (i_4') \( K \) is an invex set;
- (i_2') \( F \) is pseudomonotone;
- (i_3') \( \mu > 0 \);
- (i_4') \( \varphi \) is skew-symmetric;
- (i_5') \( \eta(u, v) = \eta(u, z) + \eta(z, v) \) for all \( u, v, z \in K \).

Then the approximate solution \( u_{n+1} \) from Algorithm 3.1 converges to a solution \( u \in K \) of the corresponding invex equilibrium problem (2.1).

Another \( \gamma \)-invex auxiliary equilibrium problem which can be used for solving (2.1) is: find a unique \( w \in K \) such that

\[
\rho F(u, v) + \gamma(v, w)\langle E'(u) - E'(v), \eta(v, w) \rangle \geq \rho \{ \varphi(w, w) - \varphi(v, w) \} + \beta \eta(w, u) \]

for all \( v \in K \).

Based on this remark, we can give
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Algorithm 3.2. For a given $u_0 \in H$, calculate the approximate solution $u_{n+1}$ by the iterative scheme
\[
\rho F(u_n, v) + \gamma(v, u_{n+1}) \langle E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1}) \rangle + \rho [\varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1})] \geq \beta \|\eta(u_{n+1}, u)\|^2
\]
for all $v \in K$.

Remark 3.3. In the case where $\beta = 0$ or $\eta(u, u) = 0$ for any $u \in K$, we retrieve Algorithm 3.6 from [15].

Relative to Algorithm 3.2 we have the following result.

Theorem 3.2. Assume assumptions (i_2)–(i_3) from Theorem 3.1 hold and (j_1) $F$ is $(\tau, \eta)$-partially relaxed monotone, where $\tau$ is a real number;
(j_2) $\mu - \rho \tau > 0$;
(j_3) $\beta + \xi \geq 0$. Then the approximate solution $u_{n+1}$ from Algorithm 3.2 converges to a solution $u \in K$ of the mixed quasi $\gamma$-invex equilibrium problem (2.1).

Proof. As in Theorem 3.1, we get
\[
B(u, u_n) - B(u, u_{n+1}) \geq \mu \|\eta(u_{n+1}, u_n)\|^2 + \rho F(u_n, u) + \rho [\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})] + \beta \|\eta(u_{n+1}, u)\|^2.
\]
It follows from (i_2) and (j_1) that
\[
B(u, u_n) - B(u, u_{n+1}) \geq \mu \|\eta(u_{n+1}, u_n)\|^2 + \beta \|\eta(u_{n+1}, u)\|^2 + \rho \xi \|\eta(u_{n+1}, u)\|^2 - \varphi(u, u) + \varphi(u_{n+1}, u) - \tau \|\eta(u_{n+1}, u_n)\|^2 + F(u, u_{n+1})].
\]
But $u \in K$ is a solution of (2.1) and by taking $v = u_{n+1}$, in (2.1) we get
\[
F(u, u_{n+1}) + \varphi(u_{n+1}, u) - \varphi(u, u) \geq 0.
\]
Hence
\[
B(u, u_n) - B(u, u_{n+1}) \geq (\mu - \rho \tau) \|\eta(u_{n+1}, u_n)\|^2 + \rho (\beta + \xi) \|\eta(u_{n+1}, u_n)\|^2.
\]

Obviously, if $u_{n+1} = u_n$, $u_n$ is a solution for (2.1). If $u_n$ and $u_{n+1}$ are different, then it follows from (j_2) and (j_3) that $B(u, u_n) - B(u, u_{n+1})$ is strictly positive and, as in Theorem 3.1, we can prove the convergence of the iterative sequence $(u_n)_{n \geq 1}$ using the technique of Zhu and Marcotte [22]. Thus, the theorem is proved. □

Corollary 3.2 (see [17]). Suppose that one of the conditions from Remark 3.1 is satisfied, conditions (i'_1), (i'_3)–(i'_5) from Corollary 3.1 hold, and (j'_1) $F$ is partially relaxed strong monotone with $\tau > 0$;
(j'_2) $\mu > 0$;
Then the approximate solution \( u_{n+1} \) from Algorithm 3.6 in [15] converges to a solution \( u \in K \) of the invex equilibrium problem (2.1).

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