ON A PROCESS DERIVED FROM A FILTERED POISSON PROCESS

MARIO LEFEBVRE and JEAN-LUC GUILBAULT

A continuous-time and continuous-state stochastic process, denoted by $\{X(t), t \geq 0\}$, is defined from a process known as a filtered Poisson process. Various characteristics of the process, such as its mean and its variance at time t are calculated. We also use its conditional expectation, given the history of the process in the interval $[0, t]$, as an estimator of $X(t + \delta)$. An application of this estimator to forecast the flow of a river is presented.

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1. INTRODUCTION

Let $\{N(t), t \geq 0\}$ be a time-homogeneous Poisson process with rate λ. The stochastic process $\{Z(t), t \geq 0\}$ defined by

$$Z(t) = \sum_{n=1}^{N(t)} Y_n (t - \tau_n)^k \mathrm{e}^{-(t-\tau_n)/c}, \quad Z(t) = 0 \text{ if } N(t) = 0, \tag{1}$$

where the random variables $Y_1, Y_2, \ldots$ are independent and identically distributed, and independent of $N(t)$, the τ_n's are the arrival times of the events of the Poisson process, and $k \geq 0$ and $c > 0$ are constants, is a particular case of processes called filtered Poisson processes. The function

$$w(t, \tau_n, Y_n) := Y_n (t - \tau_n)^k \mathrm{e}^{-(t-\tau_n)/c}$$

is the *response function* of the process. This type of process is used in many applications [see Parzen ([5], p. 144)]. In particular, the process $\{Z(t), t \geq 0\}$ seems appropriate to describe the flow of a river (see Lefebvre *et al.* [4] and Lefebvre and Guilbault [3]. In this application, the τ_n's denote the times (the days, in practice) when intense precipitation events, either rain or snow, occurred and the Y_n's are the increases of the flow at the corresponding time instants τ_n. It is often assumed that the random variables Y_n have an exponential distribution with parameter μ.

In the case when the times between the successive events are long enough and the constant c is relatively small, we can neglect the events that occurred before the most recent one. It follows that

$$Z(t) \simeq Y_{N(t)}(t-\tau_{N(t)})^k \mathrm{e}^{-(t-\tau_{N(t)})/c} \quad \text{if } N(t) > 0.$$

In this paper, we define the stochastic process $\{X(t),\, t \geq 0\}$ by

$$X(t) = X(0) + w(t, \tau_{N(t)}, Y_{N(t)}), \quad t \geq 0, \tag{2}$$

where $w(t, \tau_{N(t)}, Y_{N(t)})$ is a response function. Generally, a response function is non-negative. Here, we only assume that

$$w(t, \tau_{N(t)}, Y_{N(t)}) = 0 \quad \text{if } N(t) = 0.$$

Note that, because $N(0) = 0$, we have

$$w(0, \tau_{N(0)}, Y_{N(0)}) = w(0, \tau_0, Y_0) = 0.$$

To be more general, we can suppose that $X(0)$ is a random variable [independent of $N(t)$].

The response function that will be used in an application of the model in hydrology is

$$w(t, \tau_{N(t)}, Y_{N(t)}) = Y_{N(t)}(t-\tau_{N(t)})^k \mathrm{e}^{-(t-\tau_{N(t)})/c}, \quad t \geq 0, \tag{3}$$

with $Y_0 = \tau_0 = 0$, so that $w(0, \cdot, \cdot) = 0$ if $N(t) = 0$.

In Section 2, we calculate, in particular, the mean and the variance of the random variable $X(t)$ for a general response function. Next, we consider the conditional expectation $E[X(t+\delta) \mid H(t)]$, which gives us the mean value of the process at time $t+\delta$, given the history $H(t)$ of the process in the interval $[0, t]$. We make use of this conditional expectation in Section 3 in an application of the model in hydrology. The problem of estimating the various parameters of the model is considered in Section 3 as well. Finally, we conclude this work with a few remarks in Section 4.

2. THEORETICAL RESULTS

First, we calculate the mean of the stochastic process $\{X(t),\, t \geq 0\}$.

PROPOSITION 2.1. *For the stochastic process defined in* (2),

$$E[X(t)] = E[X(0)] + \lambda \mathrm{e}^{-\lambda t} \int_0^t E[w(t, s, Y)] \mathrm{e}^{\lambda s} \mathrm{d}s.$$

Proof. We condition on $N(t)$:

$$E[X(t)] = \sum_{n=0}^{\infty} E[X(t) \mid N(t) = n] P[N(t) = n],$$

where

$$P[N(t) = n] = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \dots .$$

In the case when $N(t) > 0$, by linearity of the mathematical expectation,

$$\begin{aligned} E[X(t) \mid N(t) = n] &= E[X(0)] + E[w(t, \tau_{N(t)}, Y_{N(t)}) \mid N(t) = n] \\ &= E[X(0)] + E[w(t, \tau_n, Y_n) \mid N(t) = n] \end{aligned}$$

while

$$E[X(t) \mid N(t) = 0] = E[X(0)].$$

Now, given that $N(t) = n$, the arrival time τ_n of the last event in the interval $[0, t]$ is distributed like the maximum of n independent uniform $\mathrm{U}[0, t]$ random variables (see Lefebvre [2] or Ross [6]). That is,

$$f_{\tau_n}(s \mid N(t) = n\,(> 0)) = \frac{n}{t^n}s^{n-1}, \quad 0 \le s \le t.$$

It follows that

$$\begin{aligned} E[w(t, \tau_n, Y_n) \mid N(t) = n] &= \int_0^t E[w(t, s, Y)]\frac{n}{t^n}s^{n-1}\mathrm{d}s \\ &= \frac{n}{t^n}\int_0^t E[w(t, s, Y)]s^{n-1}\mathrm{d}s, \end{aligned}$$

where Y is a random variable distributed like the Y_n's. Thus,

$$\begin{aligned} E[X(t)] = E[X(0)]\mathrm{e}^{-\lambda t} &+ \sum_{n=1}^{\infty}\left\{E[X(0)] + \frac{n}{t^n}\int_0^t E[w(t, s, Y)]s^{n-1}\mathrm{d}s\right\}\mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!} \\ &= E[X(0)] + \sum_{n=1}^{\infty}\left\{\frac{n}{t^n}\int_0^t E[w(t, s, Y)]s^{n-1}\mathrm{d}s\right\}\mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!} \\ &= E[X(0)] + \mathrm{e}^{-\lambda t}\sum_{n=1}^{\infty}\frac{\lambda^n}{(n-1)!}\int_0^t E[w(t, s, Y)]s^{n-1}\mathrm{d}s. \end{aligned}$$

By the dominated convergence theorem,

$$\begin{aligned} E[X(t)] &= E[X(0)] + \mathrm{e}^{-\lambda t}\int_0^t \sum_{n=1}^{\infty}\frac{\lambda^n}{(n-1)!}E[w(t, s, Y)]s^{n-1}\mathrm{d}s \\ &= E[X(0)] + \lambda\mathrm{e}^{-\lambda t}\int_0^t E[w(t, s, Y)]\sum_{m=0}^{\infty}\frac{(\lambda s)^m}{m!}\mathrm{d}s \\ &= E[X(0)] + \lambda\mathrm{e}^{-\lambda t}\int_0^t E[w(t, s, Y)]\mathrm{e}^{\lambda s}\mathrm{d}s. \quad \square \end{aligned}$$

Corollary 2.1. *For the response function given in* (3)*, the mean of the stochastic process at time* t*, when* $k > -1$*, is*

$$E[X(t)] = E[X(0)] + \lambda E[Y] \left(\frac{c}{c\lambda + 1}\right)^{k+1} \gamma\left(k+1, \left(\lambda + \frac{1}{c}\right) t\right),$$

where $\gamma(\cdot\,,\cdot)$ *is the incomplete gamma function. In particular, when* $k=0,1,\ldots,$

$$E[X(t)] = E[X(0)] + \lambda k! E[Y] \left(\frac{c}{c\lambda + 1}\right)^{k+1} \left[1 - \mathrm{e}^{-(\lambda+\frac{1}{c})t} \sum_{m=0}^{k} \left(\lambda + \frac{1}{c}\right)^m \frac{t^m}{m!}\right].$$

Proof. By Proposition 2.1 and by independence,

$$\begin{aligned} E[X(t)] &= E[X(0)] + \lambda \mathrm{e}^{-\lambda t} \int_0^t E\left[Y(t-s)^k \mathrm{e}^{-\frac{(t-s)}{c}}\right] \mathrm{e}^{\lambda s} \mathrm{d}s \\ &= E[X(0)] + \lambda \mathrm{e}^{-\lambda t} E[Y] \int_0^t (t-s)^k \mathrm{e}^{-\frac{(t-s)}{c}} \mathrm{e}^{\lambda s} \mathrm{d}s. \end{aligned}$$

That is,

$$E[X(t)] = E[X(0)] + \lambda \mathrm{e}^{-(\lambda+\frac{1}{c})t} E[Y] \int_0^t (t-s)^k \mathrm{e}^{(\lambda+\frac{1}{c})s} \mathrm{d}s.$$

Now, if $k > -1$ then [Gradshteyn and Ryzhik ([1], p. 343)]

$$I_1 := \int_0^t (t-s)^k \mathrm{e}^{(\lambda+\frac{1}{c})s} \mathrm{d}s = \left(\frac{c}{c\lambda + 1}\right)^{k+1} \mathrm{e}^{(\lambda+\frac{1}{c})t} \gamma\left(k+1, \left(\lambda + \frac{1}{c}\right) t\right).$$

In particular, when $k = 0, 1, \ldots,$ [Gradshteyn and Ryzhik ([1], p. 890)],

$$I_1 = k! \left(\frac{c}{c\lambda + 1}\right)^{k+1} \mathrm{e}^{(\lambda+\frac{1}{c})t} \left[1 - \mathrm{e}^{-(\lambda+\frac{1}{c})t} \sum_{m=0}^{k} \left(\lambda + \frac{1}{c}\right)^m \frac{t^m}{m!}\right]. \quad \square$$

Next, we calculate the variance of the process $\{X(t),\, t \geq 0\}$.

Proposition 2.2. *For the stochastic process defined in* (2)*,*

$$\begin{aligned} V[X(t)] = V[X(0)] &+ \lambda \mathrm{e}^{-\lambda t} \int_0^t E[w^2(t,s,Y)] \mathrm{e}^{\lambda s} \mathrm{d}s \\ &- \lambda^2 \mathrm{e}^{-2\lambda t} \left\{\int_0^t E[w(t,s,Y)] \mathrm{e}^{\lambda s} \mathrm{d}s\right\}^2. \end{aligned}$$

Proof. By independence,

$$V[X(t)] = V[X(0)] + V[w(t, \tau_{N(t)}, Y_{N(t)})].$$

Conditioning on $N(t)$, we obtain

$$
\begin{aligned}
E[w^2(t,\tau_{N(t)},Y_{N(t)})] &= 0 + \sum_{n=1}^{\infty} E[w^2(t,\tau_{N(t)},Y_{N(t)}) \mid N(t)=n] \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_{n=1}^{\infty} E[w^2(t,\tau_n,Y_n) \mid N(t)=n] \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
E[w^2(t,\tau_n,Y_n) \mid N(t)=n(>0)] &= \int_0^t E[w^2(t,s,Y)]\frac{n}{t^n}s^{n-1}\mathrm{d}s \\
&= \frac{n}{t^n}\int_0^t E[w^2(t,s,Y)]s^{n-1}\mathrm{d}s.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E[w^2(t,\tau_{N(t)},Y_{N(t)})] &= \sum_{n=1}^{\infty}\left\{\frac{n}{t^n}\int_0^t E[w^2(t,s,Y)]s^{n-1}\mathrm{d}s\right\}\mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \lambda \mathrm{e}^{-\lambda t}\sum_{n=1}^{\infty}\left\{\int_0^t E[w^2(t,s,Y)]s^{n-1}\mathrm{d}s\right\}\frac{\lambda^{n-1}}{(n-1)!}.
\end{aligned}
$$

The result follows from the monotone convergence theorem, by which

$$
\begin{aligned}
E[w^2(t,\tau_{N(t)},Y_{N(t)})] &= \lambda \mathrm{e}^{-\lambda t}\int_0^t E[w^2(t,s,Y)]\sum_{n=1}^{\infty}\frac{(\lambda s)^{n-1}}{(n-1)!}\mathrm{d}s \\
&= \lambda \mathrm{e}^{-\lambda t}\int_0^t E[w^2(t,s,Y)]\mathrm{e}^{\lambda s}\mathrm{d}s,
\end{aligned}
$$

as well as from Proposition 2.1. □

COROLLARY 2.2. *When the response function is that defined in* (3)*, and* $k > -\frac{1}{2}$*, the variance of the stochastic process at time* t *is given by*

$$
\begin{aligned}
V[X(t)] = V[X(0)] &+ \lambda E[Y^2]\left(\frac{c}{c\lambda+2}\right)^{2k+1}\gamma\left(2k+1,\left(\lambda+\frac{2}{c}\right)t\right) \\
&- \left\{\lambda E[Y]\left(\frac{c}{c\lambda+1}\right)^{k+1}\gamma\left(k+1,\left(\lambda+\frac{1}{c}\right)t\right)\right\}^2.
\end{aligned}
$$

In particular, if $k = 0, \frac{1}{2}, 1, \ldots,$

$$
\gamma\left(2k+1,\left(\lambda+\frac{2}{c}\right)t\right) = (2k)!\left[1-\mathrm{e}^{-(\lambda+\frac{2}{c})t}\sum_{m=0}^{2k}\left(\lambda+\frac{2}{c}\right)^m\frac{t^m}{m!}\right]
$$

while, if $k = 0, 1, \dots,$

$$\gamma\left(k+1, \left(\lambda + \frac{1}{c}\right)t\right) = k!\left[1 - \mathrm{e}^{-(\lambda+\frac{1}{c})t}\sum_{m=0}^{k}\left(\lambda+\frac{1}{c}\right)^m \frac{t^m}{m!}\right]. \tag{4}$$

Proof. All is left to do is to calculate the integral

$$I_2 := \lambda \mathrm{e}^{-\lambda t}\int_0^t E\left[Y(t-s)^k \mathrm{e}^{-\frac{(t-s)}{c}}\right]^2 \mathrm{e}^{\lambda s}\mathrm{d}s.$$

By independence,

$$I_2 = \lambda \mathrm{e}^{-\lambda t} E[Y^2]\int_0^t (t-s)^{2k}\mathrm{e}^{-\frac{2(t-s)}{c}}\mathrm{e}^{\lambda s}\mathrm{d}s,$$

and when $k > -\frac{1}{2}$ [Gradshteyn and Ryzhik ([1], p. 343)],

$$\begin{aligned} I_2 &= \lambda \mathrm{e}^{-\lambda t}E[Y^2]\mathrm{e}^{-\frac{2t}{c}}\int_0^t (t-s)^{2k}\mathrm{e}^{(\lambda+\frac{2}{c})s}\mathrm{d}s \\ &= \lambda E[Y^2]\left(\frac{c}{c\lambda+2}\right)^{2k+1}\gamma\left(2k+1, \left(\lambda+\frac{2}{c}\right)t\right). \end{aligned}$$

In particular, when $k = 0, \frac{1}{2}, 1, \dots$ [Gradshteyn and Ryzhik ([1], p. 890)],

$$I_2 = \lambda(2k)!E[Y^2]\left(\frac{c}{c\lambda+2}\right)^{2k+1}\left[1 - \mathrm{e}^{-(\lambda+\frac{2}{c})t}\sum_{m=0}^{2k}\left(\lambda+\frac{2}{c}\right)^m\frac{t^m}{m!}\right]. \quad \square$$

We continue with the calculation of the conditional expectation $E[X(t+\delta) \mid H(t)]$, where $H(t)$ denotes the history of the process in the interval $[0, t]$. That is, if we know $H(t)$, then we know the value of $X(u)$ for any u in the interval $[0, t]$ as well as the arrival times τ_n of the events and the value of the random variables Y_n, $n = 0, 1, \dots$.

Proposition 2.3. *The conditional expectation of* $X(t+\delta)$ *given* $H(t)$ *is*

$$\begin{aligned} E[X(t+\delta) \mid H(t)] = X(0) &+ w(t+\delta, \tau_{N(t)}, Y_{N(t)})\mathrm{e}^{-\lambda\delta} \\ &+ \lambda \mathrm{e}^{-\lambda\delta}\int_0^\delta E[w(t+\delta, u+t, Y)]\mathrm{e}^{\lambda u}\mathrm{d}u. \end{aligned}$$

Proof. Proceeding as in the proofs of the previous propositions, we write $E[X(t+\delta) \mid H(t)] =$

$$\begin{aligned} &= \sum_{n=0}^{\infty} E[X(t+\delta) \mid H(t), N(t+\delta)-N(t)=n]P[N(t+\delta)-N(t)=n \mid H(t)] \\ &= \sum_{n=0}^{\infty} E[X(t+\delta) \mid H(t), N(t+\delta)-N(t)=n]\mathrm{e}^{-\lambda\delta}\frac{(\lambda\delta)^n}{n!} \end{aligned}$$

(because the increments of the Poisson process are independent). Now,

$$E[X(t+\delta) \mid H(t), N(t+\delta)-N(t)=0] = X(0) + w(t+\delta, \tau_{N(t)}, Y_{N(t)}).$$

Moreover, the knowledge of $H(t)$, apart from the value of $X(0)$, is not needed when $N(t+\delta)-N(t)=n>0$. Therefore,

$$\begin{aligned}
&E[X(t+\delta) \mid H(t), N(t+\delta)-N(t)=n(>0)] \\
&= E[X(t+\delta) \mid X(0), N(t+\delta)-N(t)=n(>0)] \\
&= E[X(0) + w(t+\delta, \tau_{N(t+\delta)}, Y_{N(t+\delta)}) \mid X(0), N(t+\delta)-N(t)=n(>0)] \\
&= X(0) + E[w(t+\delta, \tau_{N(t+\delta)}, Y_{N(t+\delta)}) \mid X(0), N(t+\delta)-N(t)=n(>0)] \\
&= X(0) + \int_t^{t+\delta} E[w(t+\delta, s, Y)] \frac{n}{\delta^n}(s-t)^{n-1} \mathrm{d}s \\
&= X(0) + \frac{n}{\delta^n} \int_0^\delta E[w(t+\delta, u+t, Y)] u^{n-1} \mathrm{d}u.
\end{aligned}$$

Finally,

$$\begin{aligned}
&E[X(t+\delta) \mid H(t)] \\
&= \{X(0) + w(t+\delta, \tau_{N(t)}, Y_{N(t)})\} \mathrm{e}^{-\lambda\delta} \\
&\quad + \sum_{n=1}^{\infty} \left\{ X(0) + \frac{n}{\delta^n} \int_0^\delta E[w(t+\delta, u+t, Y)] u^{n-1} \mathrm{d}u \right\} \mathrm{e}^{-\lambda\delta} \frac{(\lambda\delta)^n}{n!} \\
&= X(0) + w(t+\delta, \tau_{N(t)}, Y_{N(t)}) \mathrm{e}^{-\lambda\delta} \\
&\quad + \sum_{n=1}^{\infty} \left\{ \frac{n}{\delta^n} \int_0^\delta E[w(t+\delta, u+t, Y)] u^{n-1} \mathrm{d}u \right\} \mathrm{e}^{-\lambda\delta} \frac{(\lambda\delta)^n}{n!} \\
&= X(0) + w(t+\delta, \tau_{N(t)}, Y_{N(t)}) \mathrm{e}^{-\lambda\delta} \\
&\quad + \lambda \mathrm{e}^{-\lambda\delta} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \int_0^\delta E[w(t+\delta, u+t, Y)] u^{n-1} \mathrm{d}u \\
&= X(0) + w(t+\delta, \tau_{N(t)}, Y_{N(t)}) \mathrm{e}^{-\lambda\delta} \\
&\quad + \lambda \mathrm{e}^{-\lambda\delta} \int_0^\delta E[w(t+\delta, u+t, Y)] \mathrm{e}^{\lambda u} \mathrm{d}u
\end{aligned}$$

(by dominated convergence). □

From what precedes, we deduce the result below.

COROLLARY 2.3. *If the response function is that in* (3), *and* $k > -1$, *then the conditional expectation of* $X(t+\delta)$ *given* $H(t)$ *is*

$$E[X(t+\delta) \mid H(t)] = X(0) + [X(t) - X(0)]\mathrm{e}^{-(\lambda+\frac{1}{c})\delta}\left(1 + \frac{\delta}{t-\tau_{N(t)}}\right)^k + \lambda E[Y]\left(\frac{c}{c\lambda+1}\right)^{k+1}\gamma\left(k+1, \left(\lambda+\frac{1}{c}\right)\delta\right).$$

When $k = 0, 1, \ldots$, *the incomplete gamma function above is given by* (4).

Proof. First,

$$w(t+\delta, \tau_{N(t)}, Y_{N(t)}) = [X(t) - X(0)]\mathrm{e}^{-\frac{\delta}{c}}\left(1 + \frac{\delta}{t-\tau_{N(t)}}\right)^k.$$

Furthermore, by independence [see Gradshteyn and Ryzhik ([1], p. 343)],

$$\begin{aligned}\int_0^\delta E[w(t+\delta, u+t, Y)]\mathrm{e}^{\lambda u}\mathrm{d}u &= \int_0^\delta E[Y(\delta-u)^k\mathrm{e}^{-\frac{(\delta-u)}{c}}]\mathrm{e}^{\lambda u}\mathrm{d}u \\ &= \mathrm{e}^{-\frac{\delta}{c}}E[Y]\int_0^\delta (\delta-u)^k\mathrm{e}^{(\lambda+\frac{1}{c})u}\mathrm{d}u \\ &= \mathrm{e}^{-\frac{\delta}{c}}E[Y]\left(\frac{c}{c\lambda+1}\right)^{k+1}\mathrm{e}^{(\lambda+\frac{1}{c})\delta}\gamma\left(k+1, \left(\lambda+\frac{1}{c}\right)\delta\right).\end{aligned}$$

Finally, for formula (4) that gives the incomplete gamma function when $k = 0, 1, \ldots$, see Gradshteyn and Ryzhik ([1], p. 890). □

Remark. For the particular response function considered, it is sufficient to know $X(0)$, $X(t)$ and $\tau_{N(t)}$ to evaluate explicitly the conditional expectation $E[X(t+\delta) \mid H(t)]$, and when $k = 0$, the value of $\tau_{N(t)}$ is not required. Moreover, when $k = 0$,

$$E[X(t+\delta) \mid H(t)] = \left[X(0) + \lambda E[Y]\left(\frac{c}{c\lambda+1}\right)\right]\left(1 - \mathrm{e}^{-(\lambda+\frac{1}{c})\delta}\right) + \mathrm{e}^{-(\lambda+\frac{1}{c})\delta}X(t).$$

If we assume that $X(0) = 0$, then

$$E[X(t+\delta) \mid H(t)] = \mathrm{e}^{-(\lambda+\frac{1}{c})\delta}X(t) + \lambda E[Y]\left(\frac{c}{c\lambda+1}\right)\left(1 - \mathrm{e}^{-(\lambda+\frac{1}{c})\delta}\right). \tag{5}$$

It is this formula, with $\delta = 1$, that we will use in Section 3 to forecast the flow of the Delaware River the day after the most recent observation of this flow.

To conclude this section, we calculate a mathematical expectation that could be used, together with other results, to estimate the parameters of the model with response function (3). Suppose that $k = 0$ and that $X(0) = 0$. We then have

$$X(t) = Y_{N(t)}\mathrm{e}^{-(t-\tau_{N(t)})/c} \Rightarrow \ln X(t) = \ln Y_{N(t)} - \frac{t-\tau_{N(t)}}{c}.$$

Hence,

$$E[\ln X(t)] = E[\ln Y_{N(t)}] - \frac{1}{c}E[t-\tau_{N(t)}].$$

The quantity

$$A(t) := t - \tau_{N(t)}$$

is called the *age* of the Poisson process, which is a special renewal process. The average value of $A(t)$ over a long period can be calculated relatively easily for any renewal process, see Lefebvre ([2], p. 318). In the case of the Poisson process, using the fact that the time between the successive events follows an exponential distribution with parameter λ and that this distribution possesses the memoryless property, we deduce that in the limit the average age of the process is given by

$$\lim_{t\to\infty} E[A(t)] = E[\mathrm{Exp}(\lambda)] = \frac{1}{\lambda}.$$

If we now assume that the random variables Y_n are exponentially distributed with parameter μ, we must finally calculate

$$E[\ln Y_{N(t)}] = \int_0^\infty (\ln y)\cdot \mu\mathrm{e}^{-\mu y}\mathrm{d}y.$$

We find that

$$\int_0^\infty (\ln) y\cdot \mu\mathrm{e}^{-\mu y}\mathrm{d}y = -\ln\mu - \gamma,$$

where $\gamma \simeq 0{,}5772$ is the Euler-Mascheroni constant. Thus,

$$\lim_{t\to\infty} E[\ln X(t)] = -\ln\mu - \gamma - \frac{1}{c\lambda}. \tag{6}$$

In practice, if we assume that the process considered is in equilibrium, or in stationary regime, we can estimate $\lim_{t\to\infty} E[\ln X(t)]$ by the arithmetic mean of the natural logarithms of the observations over a long period of time. Since we can estimate the parameter λ by the average number of events by time unit during the period considered and, theoretically, the parameter μ by the inverse of the arithmetic mean of the observations of the random variables Y_n, we can use (6) to obtain a point estimate of c. However, in fact, it is not always easy to determine the value of the Y_n's, so that the problem of estimating the parameters λ, μ and c in the model is not trivial. Furthermore, in the general case, the constant k must also be determined.

In the next section, we will see that formula (5) enables us to obtain (more) precise forecasts of the flow of the Delaware River for the next day.

3. FORECAST OF THE FLOW OF THE DELAWARE RIVER

The authors considered in [3] the problem of modeling and forecasting the flow of the Delaware River, located in the North-Eastern part of the United States. To do so, they proposed as model a filtered Poisson process, such as that defined by (1). They found that the choice $k = \frac{1}{2}$ gave a very good fit of the model to the real data. However, for the forecasts of the flow one day ahead, it is almost necessary to limit ourselves to the basic case, namely that when $k = 0$. Otherwise, in practice, the formulas to estimate $Z(t+1)$ become inconvenient.

Based on model (1), with $k = 0$, the estimator of the flow $Z(t+1)$ of the river on day $t+1$, given the observed flow $Z(t)$, is the conditional expectation

$$E[Z(t+1) \mid Z(t)] = \mathrm{e}^{-1/c} Z(t) + \frac{\lambda c}{\mu}(1 - \mathrm{e}^{-1/c}), \tag{7}$$

where $\mu^{-1} = E[Y]$.

The observations of the flow for the period from October 1st, 2002 to September 30th, 2003 were used to estimate the parameters λ, μ and c; then, use was made of the model to forecast the daily flow of the river during a 101-day period in 2004. We found that the mean of the absolute values of the forecasting errors was 838 ft^3/s.

Now, we deduce from (5) that

$$E[X(t+1) \mid H(t)] = \mathrm{e}^{-(\lambda + \frac{1}{c})} X(t) + \lambda E[Y] \left(\frac{c}{c\lambda + 1} \right) \left(1 - \mathrm{e}^{-(\lambda + \frac{1}{c})} \right). \tag{8}$$

If we use the point estimates of the parameters λ, μ and c computed in Lefebvre and Guilbault [3], we find that the mean of the absolute values of the forecasting errors obtained with the above formula is about 752 ft^3/s, which is very good if we compare with results obtained with other estimators. Moreover, the forecast of the flow on the next day is better, 67 times out of 101, than that given by equation (7).

Next, if we consider the average of the forecasts provided by (7) and (8) as estimator of the flow on the next day, we find that the absolute values of the forecasting errors is reduced to 686 ft^3/s, which is even better than the previous results.

Remark. As mentioned in the introduction, the model proposed in this paper originates from a filtered Poisson process and can serve as an approximate model when the time between the successive events is sufficiently long. Consequently, if δ is small, for example if we are interested in the conditional expectation of $X(t+1)$ (that is, the flow on the next day, as above), we may assert that the probability of more than one event in the interval $(t, t+\delta]$ is negligible. Since

$$E[X(t+\delta) \mid H(t), N(t+\delta)-N(t)=1] \overset{k\equiv 0}{=} X(0) + E[Y]\frac{\mathrm{e}^{-\delta/c}}{\delta}\int_0^\delta \mathrm{e}^{u/c}\mathrm{d}u$$
$$= X(0) + E[Y]\frac{c}{\delta}(1-\mathrm{e}^{-\delta/c}),$$

we obtain

$$E[X(t+\delta) \mid H(t)] \overset{k\equiv 0}{=} E[X(t+\delta) \mid X(0), X(t)]$$
$$\simeq \{X(0) + [X(t)-X(0)]\mathrm{e}^{-\delta/c}\}\mathrm{e}^{-\lambda\delta} + \{X(0) + E[Y]\frac{c}{\delta}(1-\mathrm{e}^{-\delta/c})\}\lambda\delta\mathrm{e}^{-\lambda\delta}$$
$$= \mathrm{e}^{-\lambda\delta}\big\{X(0)(1+\lambda\delta) + [X(t)-X(0)]\mathrm{e}^{-\delta/c} + \lambda c E[Y](1-\mathrm{e}^{-\delta/c})\big\}.$$

If we assume that $X(0)=0$, then

$$E[X(t+\delta) \mid H(t)] \overset{k\equiv 0}{=} E[X(t+\delta) \mid X(t)]$$
$$\simeq \mathrm{e}^{-\lambda\delta}\{X(t)\mathrm{e}^{-\delta/c} + \lambda c E[Y](1-\mathrm{e}^{-\delta/c})\},$$

so that

$$E[X(t+1) \mid X(t)] \simeq \left\{\mathrm{e}^{-1/c}X(t) + \frac{\lambda c}{\mu}(1-\mathrm{e}^{-1/c})\right\}\mathrm{e}^{-\lambda}. \tag{9}$$

If we make again use of the point estimates of the parameters λ, μ and c computed in Lefebvre and Guilbault [3], then the forecast (7) of the flow on the next day is simply multiplied by the factor $\mathrm{e}^{-\lambda}$. Since $\hat{\lambda} \simeq 0{,}1458$, we multiply the forecasts of the flow by about 0,8643. The mean of the absolute values of the forecasting errors obtained with this estimator is approximately equal to 771 ft^3/s, and the forecast of the flow on the next day is better, 66 times out of 101, than that given by equation (7).

Finally, if we consider as estimator of the flow on the next day the mean of the forecasts provided by (7), (8) and (9), we now find that the mean of the absolute values of the forecasting errors is reduced to 669 ft^3/s.

4. CONCLUSION

We studied a stochastic process derived from a filtered Poisson process that can serve as an approximate model or as an (almost) exact model in certain practical situations. We saw how the parameters that appear in the process can be estimated in order to be able to use the model in real applications.

In Section 3, we saw that the estimator of the value of the random variable $X(t+1)$, given the most recent observation of the process [that is,

$X(t)$], enabled us to improve the accuracy of the forecasts in an application in hydrology. Because we assumed that $X(0) = 0$ and $k = 0$, the estimator in question only requires the value of $X(t)$. If $k > 0$, the formula used becomes more complicated and requires the knowledge of the arrival times τ_n of the events of the Poisson process.

In the application in hydrology, we assume that the system has been in operation long enough for the process to be in equilibrium. If the time between two consecutive events is very long, the value of $X(t)$ decreases to zero. In practice, the flow of the Delaware River never goes below a certain threshold. This threshold, which is the minimal flow over a long period, could be the (deterministic) value of $X(0)$ in the more general model. It would be sufficient then to define $X^*(t) = X(t) - X(0)$ to retrieve the process for which $X(0) = 0$.

Finally, another model that would be worth studying is the filtered Poisson process defined by

$$Z(t) = \sum_{n=1}^{N(t)} Y_n (t - \tau_n)^k \mathrm{e}^{-(t-\tau_n)/c} g(t, \tau_n), \quad Z(t) = 0 \text{ if } N(t) = 0,$$

where

$$g(t, \tau_n) = \begin{cases} 1 & \text{if } t - \tau_n \in [0, s], \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we would keep all the events that occurred at most s time units before time t, rather than keeping all the events in the interval $[0, t]$ or only the most recent event. The quantity s would be another parameter in the model that should be chosen in an appropriate way depending on the application considered.

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REFERENCES

[1] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 6th Ed. Academic Press, San Diego, 2000.

[2] M. Lefebvre, *Processus stochastiques appliqués*. Hermann, Paris, 2005.

[3] M. Lefebvre and J.L. Guilbault, *Using filtered Poisson processes to model a river flow.* Appl. Math. Model. **32** (2008), 2792–2805.

[4] M. Lefebvre, J. Ribeiro, J. Rousselle, O. Seidou and N. Lauzon, Probabilistic prediction of peak flood discharges. In: A. Der Kiureghian, S. Madanat and J.M. Pestana (Eds.), *Proc. ICASP* 9 *Conf.*, pp. 867–871. Millpress, Rotterdam, 2003.

[5] E. Parzen, *Stochastic Processes*. Holden-Day, San Francisco, 1962.
[6] S.M. Ross, *Introduction to Probability Models*, 8th Ed. Academic Press, San Diego, 2003.

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École Polytechnique de Montréal
Département de mathématiques et de génie industriel
C.P. 6079, Succursale Centre-ville
Montréal, Québec H3C 3A7, Canada
mario.lefebvre@polymtl.ca, jean-luc.guilbault@polymtl.ca