AN EXTENSION OF CAMBERN’S THEOREM

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The theorem of Cambern for the into isometries of $C(X,E)$ is generalized to complex strictly convex range spaces. This allows for an extension and also a new proof of the vector valued Banach-Stone theorem. As a consequence, complex strictly convex Banach spaces have the strong Banach-Stone property.

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1. INTRODUCTION

The theorems of Holsztyński and Cambern have been the source for our ideas and have provided a path to the generalizations which we obtain. We begin by recalling Holsztyński’s theorem on the characterization of the into isometries between spaces of continuous functions.

Theorem 1.1 (cf. [6]). If $X$ and $Y$ are compact and Hausdorff spaces and $T : C(X) \to C(Y)$ is a linear isometry, then there exist a closed subset $Y_0$ of $Y$, a surjective continuous map $\varphi : Y_0 \to X$, and $\alpha \in C(Y)$ with $\|\alpha\|_\infty = 1$ and $|\alpha(y)| = 1$ for every $y \in Y_0$, such that

$$T(f)(y) = f(\varphi(y))\alpha(y)$$

for every $f \in C(X)$ and $y \in Y_0$.

The classical Banach Stone theorem gives a representation for surjective isometries between $C(X)$ and $C(Y)$. The usual proof of the Banach Stone theorem relies on a characterization of the extreme points of the dual unit ball, cf. [1]. We show that the Banach-Stone characterization of surjective isometries can be derived from Holsztyński’s in a straightforward manner. This provides a proof of the Banach Stone theorem that only depends on basic topological properties of the spaces involved and does not require information about the extreme points of the dual ball. Our motivation for following this approach is related to an extension of Holsztyński’s theorem to spaces of vector valued continuous functions due to Cambern [2]. He proved Holsztyński’s theorem for into isometries between $C(X,E)$ and $C(Y,E_1)$ assuming that the
Banach spaces $E$ and $E_1$ are strictly convex. We generalize this theorem to the case whenever $E$ and $E_1$ are complex strictly convex. Moreover, following the path set forth in the scalar case, we obtain a generalization of the Banach-Stone theorem to the complex strictly convex setting. Our techniques are different from the standard approach available in the literature and, to our knowledge, the results are new.

2. REMARKS ON HOLSZTYŃSKI’S THEOREM

Holsztyński’s proof is given for complex valued functions, however it is mentioned in [6], as a footnote, that the same characterization is also valid for real valued functions. For completeness of exposition we provide the minor modification to Holsztyński’s proof for the real-valued case. Before giving the modification we recall some essential notation from [6]:

\[ S_x = \{ f \in C(X) : \|f\| = 1 \text{ and } |f(x)| = 1 \}, \quad x \in X, \]
\[ R_y = \{ g \in C(Y) : \|g\| = 1 \text{ and } |g(y)| = 1 \}, \quad y \in Y, \]
\[ Q_x = \{ y \in Y : T(S_x) \subset R_y \}, \quad x \in X. \]

If $C(X)$ refers to real-valued continuous functions on $X$, then steps (i)–(vi) in Holsztyński’s proof are valid with a minor modification necessary to show step (i). This first step asserts that if $f \in C(X)$ vanishes at $x \in X$ (i.e., $f(x) = 0$), then $T(f)(y) = 0$ for every $y \in Q_x$. Without loss of generality we may assume that $f$ has norm 1, and that $T(f)(y) \neq 0$ for some $y \in Q_x$. We set $g = \min \{1 + f, 1, 1 - f\}$. Therefore, $g(x) = 1$ and $\|g\| = 1$. This implies that $g$ and $g - f$ are in $S_x$. Hence $|T(g)(y)| = 1$ and $|T(g - f)(y)| = 1$. This implies that $T(f)(y) = 0$, contradicting our initial assumption.

We now show how Holsztyński’s theorem leads to an easy proof of the Banach-Stone theorem. For purposes of exposition, we recall the statement of that theorem, see [5].

**Theorem 2.1.** If $X$ and $Y$ are compact and Hausdorff spaces and $T : C(X) \to C(Y)$ is a surjective linear isometry, then there exist a homeomorphism $\varphi : Y \to X$ and $\alpha \in C(Y)$ with $\|\alpha\| = 1$ and $|\alpha(y)| = 1$ for every $y \in Y$, such that

\[ T(f)(y) = \alpha(y)f(\varphi(y)), \] (2.1)

for every $f \in C(X)$ and $y \in Y$.

We now provide the arguments necessary to complete the proof of the Banach-Stone theorem using Holsztyński’s theorem. The first step is to show that $Y = Y_0$. If there exists $z \in Y \setminus Y_0$, then let $f \in C(X)$ be a Uryshon function with values in $[0, 1]$ such that $f(z) = 1$ and $f(Y_0) = 0$. Since $T$ is
onto, there exists \( h \in C(X) \) such that \( T(h) = f \). Equation (2.1) implies that for every \( y \in Y \) we have

\[
f(y) = T(h)(y) = \alpha(y)h(\varphi(y)) = 0.
\]

Therefore, \( h(\varphi(y)) = 0 \). The surjectivity of \( \varphi \) implies that \( h \) is the zero function. This contradiction asserts that \( Y_0 = Y \). To show that \( \varphi \) is injective, we start by assuming the existence of two points in \( Y \), \( y_0 \neq y_1 \) such that \( \varphi(y_0) = \varphi(y_1) = x \) (i.e., \( y_0, y_1 \in Q_x \)). Let \( f \in C(X) \) be a Urysohn function with values in the interval \([0, 1]\) such that \( f(y_0) = 0 \) and \( f(y_1) = 1 \). Let \( h \in C(X) \) be such that \( T(h) = f \). Thus, we have \( T(h)(y) = f(y) = \alpha(y)h(\varphi(y)) \) for all \( y \in Y \).

In particular,

\[
\alpha(y_0)h(\varphi(y_0)) = f(y_0) = 0 \quad \text{and} \quad \alpha(y_1)h(\varphi(y_1)) = f(y_1) = 1.
\]

These equalities are incompatible. In \([6]\) is shown that, for every \( x \), \( Q_x \) is nonempty, then \( Q_x \) must reduce to a single point. This implies that \( \varphi \) is injective, thus a homeomorphism, which completes the proof.

3. SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS

In this section we follow the path set forward in the scalar case by first extending a vector valued version of Holsztyński’s theorem to a new setting, complex strictly convex spaces. We then show how this result can be generalized to obtain the full Banach-Stone theorem. This demonstrates that complex strictly convex spaces have the strong Banach-Stone property \([1]\).

The first vector valued version of Holsztyński’s theorem is due to Cambern. His theorem (see \([2]\)) gives the form of an into isometry between two spaces of vector valued continuous functions, \( C(X, E) \) and \( C(Y, E_1) \), with \( X \) and \( Y \) compact topological spaces, \( E \) and \( E_1 \) Banach spaces, and \( E_1 \) strictly convex. These spaces are equipped with the usual norm \( \| \cdot \|_{\infty} \). For reference purposes, the characterizations of surjective isometries due to Jerison and Cambern’s generalization for into isometries are given in the following statement.

**Theorem 3.1.** 1. (Jerison \([8]\)). If \( A \) is an isometry from \( C(X, E) \) onto \( C(Y, E) \), with \( E \) strictly convex, there exists a homeomorphism \( \tau \) of \( Y \) onto \( X \) and a continuous map \( y \to A_y \) from \( Y \) into the space of bounded operators on \( E \) equipped with the strong operator topology, such that \( A_y \) is an isometry of \( E \) for all \( y \in Y \) and

\[
A(F)(y) = A_y(F)(\tau(y))
\]

for \( F \in C(X, E) \), \( y \in Y \).

2. (Cambern \([2]\)). If \( E \) and \( E_1 \) are normed linear spaces with \( E_1 \) strictly convex and \( A \) an isometry from \( C(X, E) \) into \( C(Y, E_1) \), then there exist a subset \( B(A) \subset Y \) and a continuous function \( \phi : Y \to B(E, E_1) \) such that \( \phi(y) = A_y \)
(here, $B(E, E_1)$ denotes all bounded operators from $E$ into $E_1$ equipped with the strong operator topology) with $\|A_y\| \leq 1$ for all $y \in Y$ and $\|A_y\| = 1$ for all $y \in B(A)$, and there exists a continuous $\tau$ from $B(A)$ onto $X$ such that

$$A(F)(y) = A_y(F)(\tau(y))$$

for $F \in C(X, E)$, $y \in B(A)$. If $E$ is finite dimensional, then $B(A)$ is a closed subset of $Y$.

First, we show that Cambern’s characterization holds for $E_1$ complex strictly convex. Then we derive a generalization of Jerison’s theorem from Cambern’s theorem under the hypotheses that $E$ and $E_1$ are strictly convex or complex strictly convex Banach spaces. It follows, as a consequence, that complex strictly convex spaces have the strong Banach-Stone property (cf. [1]). We refer the reader to [7] for an extensive number of examples of complex strictly convex spaces. For completeness we first review relevant definitions.

Definition 3.2. A complex Banach space $E$ is said to be complex strictly convex if and only if given $x$ and $y \in E$ such that $\|x\| = \|e^{i\theta}y + x\| = 1$ for every $\theta \in \mathbb{R}$, then $y = 0$. Equivalently, given $x$ and $y \in E$ such that $\|x\| = \|\pm iy + x\| = 1$, then $y = 0$ (cf. [9]). A Banach space $E$ is said to be strictly convex if and only if given $x$ and $y \in E$ of norm 1 and such that $\|x + y\| = 1$, then $x = y$.

We set our notation to be used in this section as in Cambern’s paper [2]. $A : C(X, E) \rightarrow C(X, E_1)$ denotes an isometry such that

$$A(F)(y) = A_y(F)(\tau(y)),$$

as stated in Cambern’s theorem. The operators $A_y$ are defined by $A_y(e) = A(E)(y)$ with $E(x) = e$, the constant function in $C(X, E)$.

Theorem 3.3. Let $E$ and $E_1$ be Banach spaces with $E_1$ complex strictly convex and $A$ an isometry from $C(X, E)$ into $C(Y, E_1)$. Then there exist a subset $B(A) \subset Y$, and a continuous function $\phi : Y \rightarrow B(E, E_1)$ such that $\phi(y) = A_y$ with $\|A_y\| \leq 1$ for all $y \in Y$ and $\|A_y\| = 1$ for all $y \in B(A)$, and there exists a continuous $\tau$ from $B(A)$ onto $X$ such that

$$A(F)(y) = A_y(F)(\tau(y))$$

for $F \in C(X, E)$, $y \in B(A)$. If $E$ is finite dimensional, then $B(A)$ is a closed subset of $Y$.

The proof of this theorem is similar to Cambern’s proof. The difference occurs in the proof of Lemma 2 in [2]. Rather than reproduce all of Cambern’s lemmas we refer the reader to [2] and only state the crucial lemma and provide a proof of it under this new hypothesis. We replace the strict convexity of the range space $E_1$ by the complex strict convexity. In order to make the proof
of the lemma a little more self contained we record some of the basic notation from [2]:

\[ F_{e,x} = \{ F \in C(X,E) : F(x) = \|F\|_\infty \cdot e \}, \]

\[ B(e, x) = \{ y \in Y : \|(A(F))(y)\| = \|F\|_\infty \text{ for all } F \in F_{e,x} \}, \]

\[ B(x) = \bigcup B(e, x) \quad \text{and} \quad B(A) = \bigcup_{x \in X} B(x). \]

We now state Lemma 2 from [2] and provide the essential changes in the proof.

**Lemma 3.4.** If \( y \in B(x) \) then for each \( F \in C(X,E) \) we have \( (A(F))(y) = A_y(F(x)) \).

**Proof.** If \( y \in B(x) \) then \( y \in B(e, x) \) for some \( e \in E \) with \( \|e\| = 1 \). Suppose that \( F \in C(X,E) \) vanishes on some neighborhood \( U \) of \( x \). We want to show first that \( A(F)(y) = 0 \). Choose a nonnegative function \( f \in C(X) \) such that \( f(x) = \|f\|_\infty = \|F\|_\infty \), the support of which is contained in \( U \). Define \( F_1(z) = f(z)e \) for \( z \in X \). Then \( \|F_1(z)\|_E = |f(z)| \). As a consequence, \( \|F_1\|_\infty = \max_{z \in X} \|F(z)\|_E = \max_{z \in X} |f(z)| = f(x) \). This shows that \( F \in F_{e,x} \). Let \( \theta \in \mathbb{R} \) and set \( F_\theta = e^{i\theta}F + F_1 \). Then

\[ \|F_\theta\| = \max_{z \in X} \{ \|e^{i\theta}F(z) + F_1(z)\|_E \} = \max_{z \in X} \{ \|F(z)\|_E, \|F_1(z)\|_E \}, \]

and \( \|F_\theta\| = f(x) \). Since \( y \in B(e, x) \), we have \( \|(AF_1)(y)\|_{E_1} = \|F_1\| = f(x) \)

and

\[ \|(AF_\theta)(y)\|_{E_1} = \|F_\theta\| = f(x). \]

Thus,

\[ \left\| e^{i\theta} \frac{(A(F))(y)}{f(x)} + \frac{(AF_1)(y)}{f(x)} \right\| = 1 \]

for all \( \theta \in \mathbb{R} \). The complex strict convexity of \( E_1 \) implies that \( A(F)(y) = 0 \). The remainder of the proof is exactly as in [2]. \( \square \)

We now have sufficient machinery to prove that complex strictly convex Banach spaces have the “strong-Banach Stone property.”

**Theorem 3.5.** If \( E \) and \( E_1 \) are strictly convex (or complex strictly convex) Banach spaces and \( A \) is an isometry from \( C(X,E) \) onto \( C(Y,E_1) \), then \( B(A) = Y \), \( \tau \) is a homeomorphism from \( Y \) onto \( X \), and \( A_y \) is an onto isometry for every \( y \in Y \) such that

\[ A(F)(y) = A_y(F)(\tau(y)) \]

for \( F \in C(X,E) \) and \( y \in Y \).

We start by proving the next result that encompasses the essential features needed for the proof of the theorem.
PROPOSITION 3.6. If $E$ and $E_1$ are strictly convex Banach spaces and $A$ is an isometry from $C(X, E)$ onto $C(Y, E_1)$, then $B(A) = Y$, $\tau$ is a homeomorphism from $Y$ onto $X$, $A_y$ is an onto isometry for every $y \in Y$.

Proof. The proof is presented in a sequence of steps (i)–(vii).

(i) For every $y \in B(A)$, $A_y$ is onto.

Let $\xi \in E_1$ be such that $\|\xi\|_{E_1} = 1$. We define $G : Y \to E_1$ such that $G(y) = \xi$. Clearly, $G \in C(Y, E_1)$ and $\|G\|_\infty = 1$. Since $A$ is onto, there exists $F \in C(X, E)$, $\|F\|_\infty = 1$, such that $A(F) = G$. For every $y \in B(A)$ we have $A(F)(y) = G(y) = A_y(F(\tau(y)))$. Hence $\xi = A_y(F(\tau(y)))$ or $\xi \in \text{Range}(A_y)$.

(ii) If $G \in C(Y, E_1)$ is such that $\|G\|_\infty = 1$ and $\|G(y)\|_{E_1} = 1$ for every $y \in B(A)$, then if $F \in C(X, E)$ is such that $A(F) = G$, we have $\|F(x)\|_E = 1$ for every $x \in X$.

Since $A(F) = G$ for every $y \in B(A)$, we have $A(F)(y) = G(y) = A_y(F(\tau(y)))$. Therefore, $1 = \|G(y)\|_{E_1} = \|A_y\| \|F(y)\|_E = \|F(y)\|_E$. Since $\|F\|_\infty = 1$ and $\tau$ is onto, we have $\|F(x)\|_E = 1$ for every $x \in X$.

(iii) If $E$ is strictly convex, then $B(A) = Y$.

Let $z \in Y \setminus B(A)$ and let $\beta : Y \to [0, 1]$ be a continuous function such that $\beta(B(A)) = 1$ and $\beta(z) = 0$. We choose a vector $e_1 \in E_1$ of norm 1 and set $G(y) = \beta(y) \cdot e_1$ for every $y \in Y$. Clearly, $\|G\|_\infty = 1$. Since $A$ is onto, there exists a unique $F \in C(X, E)$ of norm 1 such that $A(F) = G$. By statement (ii) for every $x \in X$ we have $\|F(x)\|_E = 1$. Define $G_\ast(y) = (2\beta(y) - 1) \cdot e_1$. There exists $F_\ast$ such that $A(F_\ast) = G_\ast$. Let $F_0$ in $C(X, E)$ be such that $A(F_0) = E_1$, with $E_1$ the constant function everywhere equal to $e_1$. Set $F_\ast = 2F - F_0$.

Statement (ii) implies that $\|F_\ast(x)\|_E = \|F_0(x)\|_E = 1$ for every $x$. Therefore, $\|\frac{1}{2}(2F(x) - F_0(x)) + F_0(x)\|_E = 1$ or, equivalently, $\|\frac{1}{2}(F_\ast(x) + F_0(x))\|_E = 1$. Since $E$ is strictly convex, we have $F_\ast(x) = F_0(x)$ for all $x$. This implies that $F = F_0$, hence $AF = AF_0$ or $G = E_1$. This contradicts the fact that $g(z) = 0 \neq E_1(z) = e_1$ and shows that $B(A) = Y$.

(iv) $A_y$ is injective for every $y \in B(A)$.

Suppose there exists $y_0 \in B(A)$ and $\xi \in E$ of norm 1 such that $A_{y_0}(\xi) = 0$. Since $A$ is an onto isometry, there exists $A^\ast : C(Y, E_1) \to C(X, E)$ such that $AA^\ast = \text{Id}_{C(Y, E_1)}$ and $A^\ast A = \text{Id}_{C(X, E)}$. Define $\mathbb{X} \in C(X, E)$ by $\mathbb{X}(x) = \xi$. Cambern’s theorem asserts the existence of $B(A^\ast) \subset X$ and a surjective and continuous map $\tau_1 : B(A^\ast) \to Y$ such that $A^\ast[G](x) = A^\ast_y[G(\tau_1(x))]$ for every $x \in B(A^\ast)$. Given $y_0 \in B(A)$, there exists $x_0 \in B(A^\ast)$ such that $\tau_1(x_0) = y_0$.

Therefore,

$$\xi = A^\ast \mathbb{X}(x_0) = A^\ast_{y_0} [\mathbb{X}(y_0)] = A^\ast_{y_0} [A_{y_0}(\mathbb{X}(y_0))] = A^\ast_{y_0} [A_{y_0} \xi] = 0.$$ 

Hence $A_{y_0}(\xi) = 0$ implies $\xi = 0$.

(v) For every $x \in X$ and $e \in E$ (of norm 1), $B(x, e) = B(x) = \bigcup \{B(x, e) : e \in E \text{ and } \|e\|_E = 1\}$ has exactly one element.
Assume that $B(x, e)$ has at least two elements $y_1$ and $y_2$. Define a continuous map $\beta : Y \to [0, 1]$ such that $\beta(y_1) = 1$ and $\beta(y_2) = 0$. Set $f = A_{y_1}(e)$ ($f \neq 0$ since $A_{y_1}$ is injective). Denote $G(y) = \beta(y) \cdot f$. There exists $F$, $\|F\|_\infty = \|f\|_{E_1} = \|G\|_\infty$, such that $A(F) = G$. Since $A(F)(y_1) = G(y_1) = f = A_{y_1}(F(x))$, we have $\|f\|_{E_1} \leq \|A_{y_1}\| \cdot \|F(x)\|_E \leq \|F(x)\|_E$. On the other hand, $A(F)(y_2) = G(y_2) = 0 = A_{y_2}(F(x))$ implies that $F(x) = 0$ ($A_{y_2}$ is injective). This proves that $B(x, e)$ has at most one element. It was shown in [2] that $B(x, e)$ is nonempty. Observe that a similar proof allows us to conclude that $B(x) = \bigcup \{B(x, e) : e \in E \text{ and } \|e\|_E = 1\}$ has exactly one element.

(vi) $\tau : B(A) \to X$ is a homeomorphism.

First, observe that $\tau$ and $\tau_1 : B(A^*) \to Y$ are continuous and bijective. We have shown in (v) that $B(x) = \{y\}$ and $B(y) = \{x_0\}$. We now prove that $x_0 = x$. Given $F \in C(X, E)$ such that $F(x) = \|F\|_\infty \cdot e$ for some norm 1 vector $e \in E$, we have $\|A(F)(y)\| = \|F\|_\infty = \|AF\|_\infty$. Therefore, $\|F(x)\| = \|F(x_0)\| = \|A^*(AF)(x_0)\| = \|A(F)\|_\infty$, which implies that $x = x_0, B(y) = \{x\}$, $\tau \circ \tau_1 = \text{Id}_{B(A^*)}$, and $\tau_1 \circ \tau = \text{Id}_{B(A)}$. Hence $\tau$ is a homeomorphism and $B(A)$ is a compact subset of $X$. Statement (iii) implies that $B(A) = X$.

(vii) For every $y \in Y$, $A_y$ is an isometry.

We need to show that we have $\|A_y(e)\| = \|e\|$ for every $e \in E$. Without loss of generality, we assume $e$ to be of norm 1. Suppose that there exist $y_0$ and $e$ such that $\|A_{y_0}(e)\| < 1$. Denote by $F$ the constant function everywhere equal to $e$ ($\|F\|_\infty = \|e\|_E = 1$). Since $A$ is an isometry, we have $\|A(F)\|_\infty = 1$ and $A(F)(y) = A_y(e)$. Therefore, $\sup_y \|A_y(e)\|_{E_1} = 1$. Assuming that $\|A_{y_0}(e)\| < 1$, there exists an open neighborhood $W_{y_0}$ such that

$$\|A_z(e)\| < \frac{1 + \|A_{y_0}(e)\|}{2} < 1$$

for every $z \in W_{y_0}$. There also exists an open neighborhood $O_{y_0}$ such that $y_0 \in O_{y_0} \subset \overline{O}_{y_0} \subset W_{y_0}$. Select $\beta$ continuous, with values in the interval $[0, 1]$, such that $\beta(x) = 1$ for every $x \in \tau(\overline{O}_{y_0})$ and $\beta(x) = 0$ for every $x \notin \tau(W_{y_0})$. Set $G(x) = \beta(x) \cdot e$. Therefore, we have $\|G\|_\infty = 1 = \|A(G)\|_\infty$ and

$$A(G)(y) = A_y(G(\tau(y))) = \begin{cases} \beta(\tau(y)) \cdot A_y(e) & \text{if } y \in W_{y_0}, \\ 0 & \text{if } y \notin W_{y_0}. \end{cases}$$

This implies that $\|A(G)\|_\infty < 1$ since

$$\|A(G)(y)\| = \begin{cases} \|\beta(\tau(y))\| \cdot \|A_y(e)\| & \leq \frac{1 + \|A_{y_0}(e)\|}{2} < 1 & \text{if } y \in W_{y_0}, \\ 0 & \text{if } y \notin W_{y_0}. \end{cases}$$

This contradiction proves the statement.

We have shown that given an isometry $A$ from $C(X, E)$ onto $C(Y, E_1)$, with $E$ a strictly convex Banach space, $X$ and $Y$ compact topological spaces,
we have $B(A) = Y$, $\tau$ is a homeomorphism from $Y$ onto $X$, and $A(F)(y) = A_Y(F(\tau(y)))$ for every $F \in C(X,E)$ and $y \in Y$. Moreover, $A_y$ is an onto isometry in $B(E,E_1)$ for every $y \in Y$. This completes the proof. □

We recall that strictly convex Banach spaces are also complex strictly convex. As a consequence, to complete the proof of Theorem 3.5, we only need to modify the argument in statement (iii) in the proof of Proposition 3.6.

Proof. If $\|F(x)\| = 1$ for every $x$, then $\|iF(x)\| = 1$. Therefore, we have $\|(-i)(F(x) - F_*(x)) + iF(x)\| = \|iF_*(x)\| = 1$ and $\|F(x) - F_*(x)\| = \|2F(x) - F_*(x)\| = 1$. Since $E$ is complex strictly convex, we have $F = F_*$. The remainder of the proof of Theorem 3.5 follows from the previous proposition. □

Remark 3.7. It would be interesting to know whether Cambern’s theorem holds for range spaces $E$ which have trivial multipliers, cf. [1].

REFERENCES