HOMOGENIZATION OF A PARABOLIC PROBLEM WITH AN IMPERFECT INTERFACE

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We describe heat diffusion in a two-component composite conductor with an ε-periodic interface. Due to an imperfect contact on the interface, the heat flow through the interface is proportional to the jump of the temperature field by a factor of order ε^γ. We study the limit behaviour of this parabolic problem when the parameter ε tends to zero. We describe the different homogenized (limit) problems, according to the value of γ. When $\gamma = 1$, a memory effect appears on the limit problem.

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1. INTRODUCTION

The aim of this paper is to study the homogenization of a parabolic problem whose setting is a two-component composite conductor with an ε-periodic interface. In particular, the physical problem involves heat transfer. Due to an imperfect contact on the interface, the heat flow through the interface is proportional to the jump of the temperature field by a factor of order ε^γ (see [6] for the physical model).

We look closely to the domain in $\mathbb{R}^n$ which is $\Omega = \Omega_{1\varepsilon} \cup \overline{\Omega_{2\varepsilon}}$ with $\Omega_{1\varepsilon}$ connected and $\Omega_{2\varepsilon}$ a disconnected union of εY-periodic sets of size εY_2. We consider $Y = Y_1 \cup \overline{Y_2}$ to be the reference cell. The same domain were considered by Monsurrò [24] and Donato and Monsurrò [15] for the elliptic case and by Donato, Faella and Monsurrò [14] for the hyperbolic case. This work which is devoted to the parabolic case completes the investigation.

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Mathematically speaking, we can model this physical problem as

$$
(1.1)\qquad \begin{cases}
u'_{1\varepsilon} - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = f_{1\varepsilon} + P_1^{\varepsilon *}(g) & \text{in } \Omega_{1\varepsilon}\times]0,T[,\\
u'_{2\varepsilon} - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = f_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}\times]0,T[,\\
A^\varepsilon \nabla u_{1\varepsilon}\cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon}\cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon\times]0,T[,\\
A^\varepsilon \nabla u_{1\varepsilon}\cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon\times]0,T[,\\
u_{1\varepsilon} = 0 & \text{on } \partial\Omega\times]0,T[,\\
u_{1\varepsilon}(x,0) = U^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon},\\
u_{2\varepsilon}(x,0) = U^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon},
\end{cases}
$$

where $T > 0$ and $\gamma \le 1$, $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$, $i = 1, 2$, P_1^ε is a suitable extension operator and $P_1^{\varepsilon *}$ its adjoint. Here, $A^\varepsilon(x) := A(x/\varepsilon)$, with A a periodic, bounded and positive definite matrix field while $h^\varepsilon(x) := h(x/\varepsilon)$, with a positive, bounded and periodic function h. The data $U^0_{i\varepsilon}$ and $f_{i\varepsilon}$, $i = 1, 2$, are given in $L^2(\Omega_{i\varepsilon})$ and $L^2(0, T;\, L^2(\Omega_{i\varepsilon}))$, respectively, while $g \in L^2(0, T;\, H^{-1}(\Omega))$.

We study the limit behaviour of this problem when the parameter ε tends to zero. We describe the different homogenized (limit) problems, according to the value of γ. We restrict our study to the case $\gamma \le 1$ because, otherwise, as shown by Hummel [20], one cannot have boundedness in the solution.

It is but natural to impose some convergence on the data in order to get homogenization results. We work under the assumptions

$$
\begin{cases}
\widetilde{U^0_\varepsilon} := (\widetilde{U^0_{1\varepsilon}}, \widetilde{U^0_{2\varepsilon}}) \rightharpoonup U^0 := (\theta_1 U^0_1, \theta_2 U^0_2) & \text{weakly in } L^2(\Omega)\times L^2(\Omega),\\
\widetilde{f_\varepsilon} := (\widetilde{f_{1\varepsilon}}, \widetilde{f_{2\varepsilon}}) \rightharpoonup (\theta_1 f_1, \theta_2 f_2) & \text{weakly in } [L^2(0,T;\, L^2(\Omega))]^2,
\end{cases}
$$

where θ_i, $i = 1, 2$, is the proportion of material occupying $\Omega_{i\varepsilon}$ and $\widetilde{\ }$ denotes the zero extension to the whole of Ω.

In Section 4, we prove step by step (see Theorem 4.1) the homogenization results

$$
(1.2)\qquad \begin{cases}
P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 & \text{weakly in } L^2(0,T;\, H^1_0(\Omega)),\\
\widetilde{u_{2\varepsilon}} \rightharpoonup u_2 & \text{weakly* in } L^\infty(0,T;\, L^2(\Omega)),
\end{cases}
$$

for $\gamma \le 1$, where P_1^ε is a suitable extension operator and u_1 is the solution of a homogenized problem which changes according to the value of γ.

In particular, we show for the case $\gamma < 1$ that u_1 is the unique solution of the problem

$$
\begin{cases}
u'_1 - \operatorname{div}(A^0_\gamma \nabla u_1) = \theta_1 f_1 + \theta_2 f_2 + g & \text{in } \Omega\times]0,T[,\\
u_1 = 0 & \text{on } \partial\Omega\times]0,T[,\\
u_1(0) = \theta_1 U^0_1 + \theta_2 U^0_2 & \text{in } \Omega.
\end{cases}
$$

In addition, we prove in this case that $u_2 = \theta_2 u_1$. This only means that the second component converges to the same limit as the first component, up to the proportionality constant occupied by the material.

For the case $\gamma = 1$, we prove that the couple (u_1, u_2) is the unique solution of the problem (a PDE coupled with an ODE)

$$\begin{cases} \theta_1 u_1' - \operatorname{div}(A_\gamma^0 \nabla u_1) + c_h(\theta_2 u_1 - u_2) = \theta_1 f_1 + g & \text{in } \Omega \times]0, T[, \\ u_2' - c_h(\theta_2 u_1 - u_2) = \theta_2 f_2 & \text{in } \Omega \times]0, T[, \\ u_1 = 0 & \text{on } \partial\Omega \times]0, T[, \\ u_1(0) = U_1^0,\ u_2(0) = \theta_2 U_2^0 & \text{in } \Omega, \end{cases}$$

where $c_h = \frac{1}{Y_2} \int_\gamma h(y)\, \mathrm{d}\sigma_y$. Moreover, solving the ODE and replacing in the PDE yields

$$\theta_1 u_1' - \operatorname{div}(A_\gamma^0 \nabla u_1) + c_h \theta_2 u_1 - c_h^2 \theta_2 \int_0^t \mathcal{K}(t,s) u_1(s)\, \mathrm{d}s = F(x,t),$$

where $\mathcal{K}$ is an exponential kernel which is explicitly computed, together with F. This means that a memory effect appears in the limit for the case $\gamma = 1$, which is an interesting one. A similar phenomenon was observed in the hyperbolic case but, in that case, the kernel is periodic.

Here, A_γ^0 is a constant positive definite matrix, called the homogenized or effective matrix. In our case, it varies according to whether $\gamma < -1$, $\gamma = -1$ and $-1 < \gamma \leq 1$. The same findings as in [15] and [24] regarding the homogenized matrices have been found, as follows.

When $\gamma < -1$, the matrix A_γ^0 is described in terms of the classical periodic solution of a problem for a composite occupying the whole Ω without jump on the interface (see for instance [2]).

For the case $\gamma = -1$, the homogenized matrix A_γ^0 is described in terms of the periodic solution of a problem posed in two sub-domains of the reference cell separated by an interface. At the interface, a conormal derivative proportional to the jump of the solution is prescribed.

Last, when $-1 < \gamma \leq 1$, the matrix A_γ^0 is the same as that obtained for the homogenization of an elliptic problem in the perforated domain $\Omega_{1\varepsilon}$ with a Neumann condition on the boundary (see [8]).

Aside from (1.2), we also prove the convergence

$$A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} + A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup A_\gamma^0 \nabla u_1 \quad \text{weakly in } L^2(0,T;\, [L^2(\Omega)]^n).$$

This result describes the contribution of both components in the limit.

More precisely, when $-1 < \gamma \leq 1$,

$$\begin{cases} A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \rightharpoonup A_\gamma^0 \nabla u_1 & \text{weakly in } L^2(0,T;\, [L^2(\Omega)]^n), \\ A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup 0 & \text{weakly in } L^2(0,T;\, [L^2(\Omega)]^n). \end{cases}$$

Hence, for this case, the contributions given by the component $\Omega_{1\varepsilon}$ and the component $\Omega_{2\varepsilon}$ can be identified separately.

One of the main difficulties in the parabolic case is that we only have convergences (1.2) and no compactness of $P_1^\varepsilon u_{1\varepsilon}$ in $L^2(0,T;\, L^2(\Omega))$, which makes it different from the hyperbolic case. This observation had been seen in the case of perforated domains in [16]. To overcome this difficulty, we adapt some lemmas already used in [16]. Also, one need to check that the initial conditions $u_1(0)$ and $u_2(0)$ make sense. Finally, a specific difficulty in the case $\gamma = 1$ is that one has no longer $u_2 = \theta_2 u_1$. The identification of u_2 and its initial conditions requires technical arguments, too.

In Section 2, we set up notation and terminologies. We state the variational formulation of problem (1.1) in a suitable Sobolev space H_γ^ε for which we also give some characterizations. Moreover, we recall some technical results needed in proving the main result.

Section 3 provides a detailed exposition of existence and uniqueness of the solution to (1.1) using a result from an abstract Galerkin's method. After that, we compute the a priori estimates which are needed to establish the necessary convergences. We also recall the classes of suitable test functions (see [15]) that will be used to identify the limit problem via Tartar's oscillating test functions method (see [28]). These test functions were used for the boundary terms to cancel.

Some results of this paper were announced without proofs in [19]. Useful results in the homogenization of heat equations in perforated domains can be attributed to Donato and Nabil [16, 17]. Other works in parabolic problems include those of Spagnolo [26] and Brahim-Otsman, Francfort and Murat [4].

The classical work on the elliptic problem corresponding to (1.1) has been done by Lipton [21] for $\gamma = 0$. For the different values of γ we refer to Monsurrò [24] and Donato and Monsurrò [15]. For the same elliptic problem in other geometries, see Auriault and Ene [1], Pernin [25], Canon and Pernin [5], Ene and Polisevski [18], Hummel [20] and, for optimal bounds, Lipton and Vernescu [22].

For the treatment of the wave equation, we refer to [14], [2] and [11]. See also [9] for the classical case in a fixed domain with oscillating coefficients and [8] for the case of perforated domain with Neumann conditions.

For the pioneer works on linear memory effects in the homogenization of parabolic problems, we refer to Mascarenhas [23] and Tartar [28].

2. STATEMENT OF THE PROBLEM

Let Ω denote an open and bounded set in $\mathbb{R}^n$ and $\{\varepsilon\}$ be a sequence of positive real numbers that converges to zero. Suppose Y_1 and Y_2 are two nonempty open sets such that Y_1 is connected and Y_2 has a Lipschitz continuous boundary Γ. We let

$$Y =]0, \ell_1[\times \cdots \times]0, \ell_n[\text{ be the representative cell with } Y = Y_1 \cup \overline{Y_2}.$$

For any $k \in \mathbb{Z}^n$, let

$$Y_i^k := k_l + Y_i,\ \Gamma_k := k_l + \Gamma \text{ where } k_l = (k_1 l_1, \dots, k_n l_n) \text{ and } i = 1, 2.$$

For any given ε, let K_ε be the set of n-tuples such that εY_i^k is included in Ω, that is,

$$K_\varepsilon := \{k \in \mathbb{Z}^n \mid \varepsilon Y_i^k \cap \Omega \neq \phi,\ i = 1, 2\}.$$

Also, we define the two components of Ω and the interface, as

$$\Omega_{i\varepsilon} := \Omega \cap \left\{ \bigcup_{k \in K_\varepsilon} \varepsilon Y_i^k \right\},\ i = 1, 2 \quad \text{and} \quad \Gamma^\varepsilon = \partial \Omega_{2\varepsilon},$$

respectively. Here we assume that

$$\partial\Omega \cap \left(\bigcup_{k \in \mathbb{Z}^n} (\varepsilon \Gamma_k) \right) = \phi. \tag{2.1}$$

Therefore, $\Omega_{1\varepsilon}$ is connected and $\Omega_{2\varepsilon}$ is a union of ε^{-n} disjoint translated sets of εY_2. Obviously, $\partial\Omega \cap \Gamma^\varepsilon = \phi$. Figure 1 shows the domain.

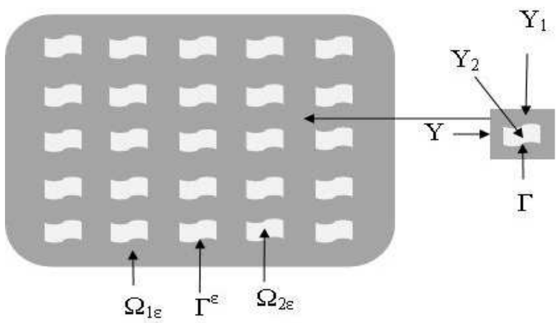

Fig. 1. The domain.

From now on, we follow the usual notation:

$$\begin{cases} \chi_\omega, \text{ the characteristic function of any open set } \omega \subset \mathbb{R}^n, \\ m_\omega(v) = \frac{1}{|\omega|} \int_\omega v \, \mathrm{d}x, \text{ the mean value of } v \text{ over a measurable set } \omega, \\ \widetilde{v}, \text{ the zero extension to } \mathbb{R}^n \text{ of any function } v \text{ defined on } \Omega_{i\varepsilon} \text{ or } Y_i \text{ for } i = 1, 2. \end{cases}$$

It is a fact that (for instance see [9])

$$\chi_{\Omega_{i\varepsilon}} \rightharpoonup \theta_i := \frac{|Y_i|}{|Y|},\ i = 1, 2, \quad \text{weakly in } L^2(\Omega). \tag{2.2}$$

We will consider the two spaces V^ε and H^ε_γ defined by

$$V^\varepsilon := \{v_1 \in H^1(\Omega_{1\varepsilon}) \mid v_1 = 0 \text{ on } \partial\Omega\}$$

with the corresponding norm

$$\|v_1\|_{V^\varepsilon} := \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})} \tag{2.3}$$

and

$$H^\varepsilon_\gamma := \{v = (v_1, v_2) \mid v_1 \in V^\varepsilon \text{ and } v_2 \in H^1(\Omega_{2\varepsilon})\} \tag{2.4}$$

for all $\gamma \in \mathbb{R}$, with the corresponding norm

$$\|v\|^2_{H^\varepsilon_\gamma} := \|\nabla v_1\|^2_{L^2(\Omega_{1\varepsilon})} + \|\nabla v_2\|^2_{L^2(\Omega_{2\varepsilon})} + \varepsilon^\gamma \|v_1 - v_2\|^2_{L^2(\Gamma^\varepsilon)}. \tag{2.5}$$

Note that if $\gamma_1 \leq \gamma_2$ then $\|v\|^2_{H^\varepsilon_{\gamma_2}} \leq \|v\|^2_{H^\varepsilon_{\gamma_1}}$. Hence, in particular, for all $\gamma \leq 1$ we have

$$\|v\|_{H^\varepsilon_1} \leq \|v\|_{H^\varepsilon_\gamma}. \tag{2.6}$$

Now, we have the functional setting to introduce our parabolic problem. For the coefficient matrix, let A be an $n \times n$ Y-periodic matrix-valued function in $L^\infty(Y)$ such that $\forall \lambda \in \mathbb{R}^n$ and, a.e. in Y,

$$\begin{cases} (A(x)\lambda, \lambda) \geq \alpha|\lambda|^2, \\ |A(x)\lambda| \leq \beta\lambda, \end{cases} \tag{2.7}$$

where $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$. Now, for any $\varepsilon > 0$ set

$$A^\varepsilon(x) := A\left(\frac{x}{\varepsilon}\right). \tag{2.8}$$

Furthermore, consider a Y-periodic function h such that

$$h \in L^\infty(\Gamma),\ \exists\, h_0 \in \mathbb{R} \text{ such that } 0 < h_0 < h(y),\ y \text{ a.e. in } \Gamma, \tag{2.9}$$

and assume

$$h^\varepsilon(x) := h\left(\frac{x}{\varepsilon}\right). \tag{2.10}$$

We suppose that

$$\begin{cases} g \in L^2(0, T;\, H^{-1}(\Omega)), \\ U^0_\varepsilon := (U^0_{1\varepsilon}, U^0_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}), \\ f_\varepsilon := (f_{1\varepsilon}, f_{2\varepsilon}) \in L^2(0, T;\, L^2(\Omega_{1\varepsilon})) \times L^2(0, T;\, L^2(\Omega_{2\varepsilon})). \end{cases} \tag{2.11}$$

For $T > 0$ and $\gamma \leq 1$ consider the problem

$$
(2.12) \qquad \begin{cases}
u'_{1\varepsilon} - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = f_{1\varepsilon} + P_1^{\varepsilon *}(g) & \text{in } \Omega_{1\varepsilon} \times]0, T[, \\
u'_{2\varepsilon} - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = f_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times]0, T[, \\
A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times]0, T[, \\
A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times]0, T[, \\
u_{1\varepsilon} = 0 & \text{on } \partial\Omega \times]0, T[, \\
u_{1\varepsilon}(x, 0) = U^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\
u_{2\varepsilon}(x, 0) = U^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon},
\end{cases}
$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$, $i = 1, 2$, P_1^ε is a suitable extension operator given in Lemma 2.7 and $P_1^{\varepsilon *}$ its adjoint.

By definition, $P_1^{\varepsilon *} \in \mathcal{L}(L^2(0, T;\, H^{-1}(\Omega)); L^2(0, T;\, (V^\varepsilon)'))$ and for $g \in L^2(0, T;\, H^{-1}(\Omega))$, $P_1^{\varepsilon *} g$ is given by

$$
(2.13) \qquad P_1^{\varepsilon *} g : \; v \in L^2(0, T;\, V^\varepsilon) \mapsto \int_0^T \langle g, P_1^\varepsilon v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \mathrm{d}s.
$$

Our goal is to describe the limit behaviour of the above problem when the parameter ε tends to zero. We describe the different homogenized (limit) problems, according to the value of γ. To do this, we consider its variational formulation which is

$$
(2.14) \qquad \begin{cases}
\text{Find } u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}) \text{ in } W^\varepsilon \text{ such that} \\
\langle u'_{1\varepsilon}, v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle u'_{2\varepsilon}, v_2 \rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} + \displaystyle\int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla v_1 \, \mathrm{d}x \\
+ \displaystyle\int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla v_2 \, \mathrm{d}x + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon})(v_1 - v_2) \, \mathrm{d}\sigma_x \\
= \displaystyle\int_{\Omega_{1\varepsilon}} f_{1\varepsilon} v_1 \, \mathrm{d}x + \langle g, P_1^\varepsilon v_1 \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} v_2 \, \mathrm{d}x \text{ in } \mathcal{D}'(0, T), \\
\text{for every } (v_1, v_2) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}), \\
u_{1\varepsilon}(x, 0) = U^0_{1\varepsilon} \text{ in } \Omega_{1\varepsilon} \text{ and } u_{2\varepsilon}(x, 0) = U^0_{2\varepsilon} \text{ in } \Omega_{2\varepsilon},
\end{cases}
$$

where

$$
\begin{aligned}
W^\varepsilon := \{ v = (v_1, v_2) \in L^2(0, T;\, V^\varepsilon) \times L^2(0, T;\, H^1(\Omega_{2\varepsilon})) \text{ s.t.} \\
v' \in L^2(0, T;\, (V^\varepsilon)') \times L^2(0, T;\, (H^1(\Omega_{2\varepsilon}))') \}
\end{aligned}
$$

equipped with the norm

$$
\begin{aligned}
\|v\|_{W^\varepsilon} = \|v_1\|_{L^2(0, T;\, V^\varepsilon)} + \|v_2\|_{L^2(0, T;\, H^1(\Omega_{2\varepsilon}))} + \\
+ \|v'_1\|_{L^2(0, T;\, (V^\varepsilon)')} + \|v'_2\|_{L^2(0, T;\, (H^1(\Omega_{2\varepsilon}))')}.
\end{aligned}
$$

Remark 2.1. The existence and uniqueness of the solution of our problem are proved using a theorem based on an abstract Galerkin method while the a priori estimates are obtained using Gronwall's Lemma. These are discussed in Section 3. The asymptotic behaviour is then established in Section 4.

We end this section with some technical lemmas that will be used in the sequel. Let us recall some characterizations of the norm in the above spaces introduced by Monsurrò.

LEMMA 2.2 ([24]). *For every fixed ε, the norms of H^ε_γ and $V^\varepsilon \times H^1(\Omega_{2\varepsilon})$ are equivalent. Furthermore, there exist constants $C_1, C_2 > 0$ independent of ε such that*

(i) $\|v\|_{L^2(\Omega_{1\varepsilon})\times L^2(\Omega_{2\varepsilon})} \leq C_1 \|v\|_{H^\varepsilon_1}$, $\forall\, v \in H^\varepsilon_1$;

(ii) $\|v_i\|^2_{L^2(\Gamma^\varepsilon)} \leq C_2(\varepsilon^{-1}\|v_i\|^2_{L^2(\Omega_{i\varepsilon})} + \varepsilon\|\nabla v_i\|^2_{L^2(\Omega_{i\varepsilon})})$, $\forall\, v_i \in H^1(\Omega_{i\varepsilon})$, $i = 1, 2$.

The next result is an immediate consequence of Lemma 2.2.

LEMMA 2.3. *There exist two positive constants C_1, C_2 independent of ε such that*

$$C_1\|v\|_{H^\varepsilon_1} \leq \|v\|_{V^\varepsilon\times H^1(\Omega_{2\varepsilon})} \leq C_2\|v\|_{H^\varepsilon_1}, \quad \forall\, v \in H^\varepsilon_1.$$

LEMMA 2.4 ([12, 24]). *There exists a positive constant C independent of ε such that*

$$\|v\|^2_{L^2(\Omega_{2\varepsilon})} \leq C(\varepsilon\|v\|^2_{L^2(\Gamma^\varepsilon)} + \varepsilon^2\|\nabla v\|^2_{L^2(\Omega_{2\varepsilon})}), \quad \forall\, v \in L^2(\Omega_{2\varepsilon}).$$

The following extension results due to D. Cioranescu and J. Saint Jean Paulin will be useful.

LEMMA 2.5 ([10]). (i) *There exists a linear continuous operator*

$$Q_1 \in \mathcal{L}(H^1(Y_1); H^1(Y)) \cap \mathcal{L}(L^2(Y_1); L^2(Y))$$

such that

$$\|Q_1 v_1\|_{L^2(Y)} \leq C\|v_1\|_{L^2(Y_1)} \quad \textit{and} \quad \|\nabla Q_1 v_1\|_{L^2(Y)} \leq C\|\nabla v_1\|_{L^2(Y_1)}$$

for some positive constant C and for all $v_1 \in H^1(Y_1)$.

(ii) *There exists a linear and continuous extension operator*

$$Q_2 \in \mathcal{L}(H^1(Y_2); H^1_{per}(Y))$$

such that

$$\|Q_2 v_2\|_{H^1(Y)} \leq C\|v_2\|_{H^1(Y_2)}$$

for some positive constant C and for every $v_2 \in H^1(Y_2)$.

(iii) *There exists an extension operator*

$$Q^\varepsilon_1 \in \mathcal{L}(L^2(\Omega_{1\varepsilon}); L^2(\Omega)) \cap \mathcal{L}(V^\varepsilon; H^1_0(\Omega))$$

such that

$$\|Q_1^\varepsilon v_1\|_{L^2(\Omega)} \le C\|v_1\|_{L^2(\Omega_{1\varepsilon})} \quad \textit{and} \quad \|\nabla Q_1^\varepsilon v_1\|_{L^2(\Omega)} \le C\|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})}$$

for some positive constant C independent of ε.

Remark 2.6. Observe that Lemma 2.5 implies that there is a Poincaré inequality in V^ε, i.e., there exists a constant $C > 0$ such that

$$\|v\|_{L^2(\Omega_{1\varepsilon})} \le C\|\nabla v\|_{L^2(\Omega_{1\varepsilon})}, \quad \forall\, v \in V^\varepsilon.$$

On the other hand, D. Cioranescu and P. Donato proved

LEMMA 2.7 ([8]). *There exists a linear continuous extension operator $P_1^\varepsilon \in \mathcal{L}(L^2(0,T;V^\varepsilon); L^2(0,T;H_0^1(\Omega))) \cap \mathcal{L}(L^2(0,T;\, L^2(\Omega_{1\varepsilon})); L^2(0,T;\, L^2(\Omega)))$ such that for some positive constant C independent of ε and for any $\varphi \in L^2(0,T;V^\varepsilon)$ with $\varphi' \in L^2(0,T;L^2(\Omega_{1\varepsilon}))$ we have*

$$\begin{cases} P_1^\varepsilon\varphi = \varphi \ \ \textit{in}\ \Omega_{1\varepsilon}\times]0,T[, \\ P_1^\varepsilon\varphi' = (P_1^\varepsilon\varphi)' \ \ \textit{in}\ \Omega\times]0,T[, \\ \|P_1^\varepsilon\varphi\|_{L^2(0,T;\,L^2(\Omega))} \le C\|\varphi\|_{L^2(0,T;\,L^2(\Omega_{1\varepsilon}))}, \\ \|P_1^\varepsilon\varphi'\|_{L^2(0,T;\,L^2(\Omega))} \le C\|\varphi'\|_{L^2(0,T;\,L^2(\Omega_{1\varepsilon}))}, \\ \|P_1^\varepsilon\varphi(t)\|_{H_0^1(\Omega)} \le C\|\nabla\varphi(t)\|_{L^2(\Omega_{1\varepsilon})},\ \forall\, t \in\,]0,T[\,, \\ \|\nabla(P_1^\varepsilon\varphi)\|_{L^2(0,T;\,[L^2(\Omega)]^n)} \le C\|\nabla\varphi\|_{L^2(0,T;\,[L^2(\Omega_{1\varepsilon})]^n)}. \end{cases}$$

We adapt Lemma 2.1 given in [3] to state

COROLLARY 2.8. *If (v_ε) and $(v_\varepsilon)'$ are bounded in $L^2(0,T;\, H_0^1(\Omega))$ and $L^2(0,T;\, L^2(\Omega))$, respectively, with $v_\varepsilon \to v$ strongly in $L^2(0,T;\, L^2(\Omega))$, then*

$$P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}}) \rightharpoonup v \quad \textit{weakly in}\ L^2(0,T;\, L^2(\Omega)).$$

Proof. By Lemma 2.7 we have

$$\begin{cases} P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}}) & \text{bounded in } L^2(0,T;\, H_0^1(\Omega)), \\ (P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}}))' & \text{bounded in } L^2(0,T;\, L^2(\Omega)). \end{cases}$$

So, by compactness, up to a subsequence, we get

$$\begin{cases} P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}}) \rightharpoonup v_1 & \text{weakly in } L^2(0,T;\, H_0^1(\Omega)), \\ P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}}) \to v_1 & \text{strongly in } L^2(0,T;\, L^2(\Omega)). \end{cases}$$

On the other hand, $P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}})\chi_{\Omega_{1\varepsilon}} = v_\varepsilon\chi_{\Omega_{1\varepsilon}}$. As $\varepsilon \to 0$, using (2.2), we have

$$\begin{cases} P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}})\chi_{\Omega_{1\varepsilon}} \rightharpoonup v_1\theta_1 & \text{weakly in } L^2(0,T;\, L^2(\Omega)), \\ v_\varepsilon\chi_{\Omega_{1\varepsilon}} \rightharpoonup v\theta_1 & \text{weakly in } L^2(0,T;\, L^2(\Omega)). \end{cases}$$

Hence $v_1 = v$ and the whole sequence $P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}})$ converges. □

The next result is a direct extension of Lemma 3.3 in [13] which is an adaptation to the case of a disconnected set of Lemma 3.1 in [7].

Lemma 2.9 ([13, 7]). *Suppose that Γ is of class C^2. Let g be a function in $L^\infty(\Gamma)$ and set $c_g = \frac{1}{|Y_2|}\int_\Gamma g(y)\,\mathrm{d}\sigma_y$. For every ε, let v_ε be a function in $L^2(H^1(\Omega_{2\varepsilon}))$ such that for some positive constant C one has*

$$\begin{cases} \|v_\varepsilon\|_{L^2(0,T;H^1(\Omega_{2\varepsilon}))} \le C, \\ \widetilde{v_\varepsilon} \rightharpoonup v \text{ weakly in } L^2(0,T;\, L^2(\Omega)). \end{cases}$$

Then

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^T \int_{\Gamma^\varepsilon} g(x/\varepsilon) v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = c_g \int_0^T \int_\Omega v\,\mathrm{d}x\,\mathrm{d}t.$$

The last lemma of this section is very significant in this paper as it overcomes the technical difficulty in passing to the limit with weak convergences of products when one factor is independent of t. This result is due to Donato and Nabil [16].

Lemma 2.10 ([16]). *Let $(h_\varepsilon) \subset L^p(0,T;W_0^{1,q}(\Omega))$ and $(g_\varepsilon) \subset L^{q'}(\Omega)$ with $p, q \ge 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ be two sequences such that*

$$\begin{cases} h_\varepsilon \rightharpoonup h \quad \text{weakly in } L^p(0,T;W_0^{1,q}(\Omega)), \\ g_\varepsilon \rightharpoonup g \quad \text{weakly in } L^{q'}(\Omega). \end{cases}$$

Then $h_\varepsilon g_\varepsilon \rightharpoonup hg$ weakly in $L^p(0,T;L^1(\Omega))$.

3. PRELIMINARY RESULTS

In this section, the existence and uniqueness of the solution of problem (2.12) are ensured and a priori estimates are given. Moreover, the suitable test functions necessary in describing the limit problems are presented.

3.1. Existence and uniqueness of solution

The result below is based on an abstract Galerkin method (see [29]).

Theorem 3.1 ([29]). *Let (V, H, V') be an evolution triple, that is,*

(i) $V \subseteq H \subseteq V'$;

(ii) *V is a real, separable and reflexive Banach space;*

(iii) *H is a real, separable Hilbert space, endowed with a scalar product $(\cdot\,,\cdot)_H$;*

(iv) *V is dense in H and the embedding $V \subseteq H$ is continuous.*

Let $a : V \times V \times]0, T[\to \mathbb{R}$ *be a mapping on* $]0, T[$ *for all* $u, v \in V$ *such that for every* $t \in]0, T[$*, the map* $a(\cdot\,, \cdot\,, t) : V \times V \to \mathbb{R}$ *is bilinear and* $\forall\, u, v \in V$*, the function* $t \mapsto a(u, v, t)$ *is measurable on* $]0, T[$*. Moreover,* $\forall\, u, v \in V$*,* $\forall\, t \in]0, T[$*, suppose that*

(v) $\exists$ *a positive constant* a_1 *such that* $|a(u, v, t)| \le a_1 \|u\|_V \|v\|_V$ *(boundedness);*

(vi) $\exists$ $a_2 > 0$ *and* $a_3 \ge 0$ *such that* $|a(u, u, t)| \ge a_2 \|u\|_V^2 - a_3 \|u\|_H^2$ *(Garding inequality).*

Let $U_0 \in H$ *and* $b \in L^2(0, T; V')$*. Then there exists a unique solution to the problem*

$$\begin{cases} \text{Find } u \in L^2(0, T; V) \text{ such that } u' \in L^2(0, T; V') \text{ and} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(u(t), v)_H + a(u(t), v)_H = \langle b(t), v \rangle_{V', V},\, u(0) = U_0 \in H, \\ \text{for all } v \in V \text{ and for almost all } t \in]0, T[. \end{cases}$$

The following classical result (see for instance [29]) gives meaning to the initial condition $u(0)$ in Theorem 3.1.

Proposition 3.2 ([29]). *Let* (V, H, V') *be an evolution triple. If*

$$W_2^1(0, T; V, H) = \{u \in L^2(0, T; V) \mid u' \in L^2(0, T; V')\}$$

is equipped with the norm

$$\|u\|_{W_2^1(0,T;V,H)} = \|u\|_{L^2(0,T;V)} + \|u'\|_{L^2(0,T;V')},$$

then one has the continuous embedding $W_2^1(0, T; V, H) \subset C([0, T], H)$.

We now derive the existence and uniqueness results of problem (2.14).

Theorem 3.3. *Let* $T > 0$*,* $\gamma \le 1$ *and* $H_\gamma^\varepsilon, A^\varepsilon, h^\varepsilon$ *be defined by* (2.4), (2.8) *and* (2.10)*, respectively. Then under assumption* 2.11*, problem* (2.14) *has a unique solution.*

Proof. We apply Theorem 3.1 with $V = H_\gamma^\varepsilon$ and $H = L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$ which clearly satisfy (i)–(iii). Now, for $\gamma \le 1, H_\gamma^\varepsilon \subseteq L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$ is a continuous embedding since, $\forall\, v \in H_\gamma^\varepsilon$, from (2.6) and Lemma 2.2(i) we have

$$\|v\|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})} \le C \|v\|_{H_\gamma^\varepsilon}.$$

Hence (V, H, V') is an evolution triple.

Consequently, it follows from Proposition 3.2 and (2.11) that

$$W^\varepsilon \subset C([0, T]; L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})),$$

so that the initial conditions in (2.14) make sense for $u_\varepsilon \in W^\varepsilon$.

Second, suppose $b = f_\varepsilon + \langle g, P_1^\varepsilon v_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$, $u_0 = U_\varepsilon^0$ and $a(u,v,t) = a^\varepsilon(u,v)$, where

$$a^\varepsilon(u,v) = \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_1 \nabla v_1 \, \mathrm{d}x + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_2 \nabla v_2 \, \mathrm{d}x + \\ + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)(v_1 - v_2) \, \mathrm{d}\sigma_x,$$

for every $u = (u_1, u_2), v = (v_1, v_2) \in H_\gamma^\varepsilon$.

We verify (v). Observe that for every $u, v \in H_\gamma^\varepsilon$ we have

$$|a(u,v,t)| \leq \left| \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_1 \nabla v_1 \, \mathrm{d}x \right| + \left| \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_2 \nabla v_2 \, \mathrm{d}x \right| + \\ + \left| \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)(v_1 - v_2) \, \mathrm{d}\sigma_x \right|. \tag{3.1}$$

By (2.8) and Hölder inequality,

$$\left| \int_{\Omega_{i\varepsilon}} A^\varepsilon \nabla u_i \nabla v_i \, \mathrm{d}x \right| \leq \beta \|\nabla u_i\|_{L^2(\Omega_{i\varepsilon})} \|\nabla v_i\|_{L^2(\Omega_{i\varepsilon})}, \quad i = 1, 2.$$

On the other hand,

$$\left| \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)(v_1 - v_2) \, \mathrm{d}\sigma_x \right| \leq \varepsilon^\gamma h^\varepsilon \|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)} \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}.$$

Let

$$C = \left(\beta^{\frac{1}{2}} \|\nabla u_1\|_{L^2(\Omega_{1\varepsilon})}, \beta^{\frac{1}{2}} \|\nabla u_2\|_{L^2(\Omega_{2\varepsilon})}, \varepsilon^{\frac{\gamma}{2}} h^{\frac{\varepsilon}{2}} \|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)} \right)$$

and

$$D = \left(\beta^{\frac{1}{2}} \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})}, \beta^{\frac{1}{2}} \|\nabla v_2\|_{L^2(\Omega_{2\varepsilon})}, \varepsilon^{\frac{\gamma}{2}} h^{\frac{\varepsilon}{2}} \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)} \right).$$

With the Euclidean norm in $\mathbb{R}^3$, we have $C \cdot D \leq \|C\| \, \|D\|$, so that

$$\beta \|\nabla u_1\|_{L^2(\Omega_{1\varepsilon})} \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})} + \beta \|\nabla u_2\|_{L^2(\Omega_{2\varepsilon})} \|\nabla v_2\|_{L^2(\Omega_{2\varepsilon})} + \\ + \varepsilon^\gamma h^\varepsilon \|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)} \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)} \leq \\ \leq \left(\beta \|\nabla u_1\|^2_{L^2(\Omega_{1\varepsilon})} + \beta \|\nabla u_2\|^2_{L^2(\Omega_{2\varepsilon})} + \varepsilon^\gamma h^\varepsilon \|u_1 - u_2\|^2_{L^2(\Gamma^\varepsilon)} \right)^{\frac{1}{2}} \times \\ \times \left(\beta \|\nabla v_1\|^2_{L^2(\Omega_{1\varepsilon})} + \beta \|\nabla v_2\|^2_{L^2(\Omega_{2\varepsilon})} + \varepsilon^\gamma h^\varepsilon \|v_1 - v_2\|^2_{L^2(\Gamma^\varepsilon)} \right)^{\frac{1}{2}}. \tag{3.2}$$

So, choosing $a_1 = \max\{\beta, h^\varepsilon\}$, it follows from (3.1) and (3.2) that

$$|a| \leq a_1 \|u\|_V \|v\|_V, \quad \forall\, u, v \in H_\gamma^\varepsilon, \ \forall\, t \in]0, T[.$$

We now verify (vi). By definition,

$$|a(u,u,t)| = \left| \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_1 \nabla u_1 \,\mathrm{d}x + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_2 \nabla u_2 \,\mathrm{d}x + \right.$$
$$\left. + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)^2 \,\mathrm{d}\sigma_x \right|.$$

By (2.8) we have

$$\begin{cases} \displaystyle\int_{\Omega_{i\varepsilon}} A^\varepsilon \nabla u_i \nabla u_i \,\mathrm{d}x \geq \alpha \|\nabla u_i\|^2_{L^2(\Omega_{i\varepsilon})}, \ i = 1,2, \\ \displaystyle\varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)^2 \,\mathrm{d}\sigma_x \geq \varepsilon^\gamma h_0 \|u_1 - u_2\|^2_{L^2(\Gamma^\varepsilon)}. \end{cases}$$

So, choosing $a_2 = \min\{\alpha, h_0\}$ and $a_3 = 0$ we have (vi). □

Remark 3.4. Under the same assumptions as in Theorem 3.3, it can be concluded further that there is a continuous dependence on the data (see [29]). We need to check the uniformity in ε of these estimates which will be shown in the next subsection.

3.2. **A priori estimates**

In order to get uniform estimates and homogenization results, it is necessary to make convergence assumptions on U^0_ε and f_ε, that is,

$$\begin{cases} \widetilde{U^0_\varepsilon} \rightharpoonup U^0 := (\theta_1 U^0_1, \theta_2 U^0_2) \quad \text{weakly in } L^2(\Omega) \times L^2(\Omega), \\ \widetilde{f_\varepsilon} \rightharpoonup (\theta_1 f_1, \theta_2 f_2) \quad \text{weakly in } L^2(0,T;\, L^2(\Omega)) \times L^2(0,T;\, L^2(\Omega)), \end{cases} \tag{3.3}$$

where θ_i, $i = 1,2$ are given by (2.2).

Proposition 3.5. *Let A^ε and h^ε be defined as in Theorem* 3.3 *and suppose* (2.11), (2.13) *and* (3.3) *hold. Let u_ε be the solution of problem* (2.14) *with $\gamma \leq 1$. Then $\|u_\varepsilon\|_{H^\varepsilon_\gamma}$ is bounded, that is, there exists a constant c independent of ε such that*

(i) $\|u_{1\varepsilon}\|_{L^2(0,T;\, V^\varepsilon)} + \|u_{1\varepsilon}\|_{L^\infty(0,T;\, L^2(\Omega_{1\varepsilon}))} < c,$
(ii) $\|u_{2\varepsilon}\|_{L^2(0,T;\, H^1(\Omega_{2\varepsilon}))} + \|u_{2\varepsilon}\|_{L^\infty(0,T;\, L^2(\Omega_{2\varepsilon}))} < c,$
(iii) $\|u_{1\varepsilon} - u_{2\varepsilon}\|_{L^2(0,T;\, L^2(\Gamma^\varepsilon))} < c\varepsilon^{-\frac{\gamma}{2}}.$

Proof. In the variational formulation (2.14), choose $v = (u_{1\varepsilon}, u_{2\varepsilon})$ as test function. Integrating by parts and using the Hölder inequality we have

$$\frac{1}{2}\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon} \,\mathrm{d}x \,\mathrm{d}s +$$

$$+\int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon} \,\mathrm{d}x \,\mathrm{d}s + \varepsilon^\gamma \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2 \,\mathrm{d}\sigma_x \,\mathrm{d}s =$$

$$= \frac{1}{2}\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} u_{1\varepsilon} \,\mathrm{d}x \,\mathrm{d}s +$$

$$+ \int_0^t \langle g, P_1^\varepsilon u_{1\varepsilon} \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} u_{2\varepsilon} \,\mathrm{d}x \,\mathrm{d}s \leq$$

$$\leq \frac{1}{2}\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \|f_{1\varepsilon}\|_{L^2(\Omega_{1\varepsilon})} \|u_{1\varepsilon}\|_{L^2(\Omega_{1\varepsilon})} \,\mathrm{d}s +$$

$$+ \int_0^t \|g\|_{H^{-1}(\Omega)} \|P_1^\varepsilon u_{1\varepsilon}\|_{H^1_0(\Omega)} \,\mathrm{d}s + \int_0^t \|f_{2\varepsilon}\|_{L^2(\Omega_{2\varepsilon})} \|u_{2\varepsilon}\|_{L^2(\Omega_{2\varepsilon})} \,\mathrm{d}s.$$

This, together with the properties of A^ε and h^ε given in (2.7)–(2.10), implies that

$$\frac{1}{2}\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \alpha \int_0^t \int_{\Omega_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2 \,\mathrm{d}x \,\mathrm{d}s + \tag{3.4}$$

$$+ \alpha \int_0^t \int_{\Omega_{2\varepsilon}} |\nabla u_{2\varepsilon}|^2 \,\mathrm{d}x \,\mathrm{d}s + \varepsilon^\gamma h_0 \int_0^t \int_{\Gamma^\varepsilon} |u_{1\varepsilon} - u_{2\varepsilon}|^2 \,\mathrm{d}\sigma_x \,\mathrm{d}s \leq$$

$$\leq \frac{1}{2}\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \|f_{1\varepsilon}\|_{L^2(\Omega_{1\varepsilon})} \|u_{1\varepsilon}\|_{L^2(\Omega_{1\varepsilon})} \,\mathrm{d}s +$$

$$+ \int_0^t \|g\|_{H^{-1}(\Omega)} \|P_1^\varepsilon u_{1\varepsilon}\|_{H^1_0(\Omega)} \,\mathrm{d}s + \int_0^t \|f_{2\varepsilon}\|_{L^2(\Omega_{2\varepsilon})} \|u_{2\varepsilon}\|_{L^2(\Omega_{2\varepsilon})} \,\mathrm{d}s.$$

Now, observe that

$$\int_0^t \|f_{i\varepsilon}\|_{L^2(\Omega_{i\varepsilon})} \|u_{i\varepsilon}\|_{L^2(\Omega_{i\varepsilon})} \,\mathrm{d}s \leq \frac{1}{2} \int_0^t \left(\|f_{i\varepsilon}\|^2_{L^2(\Omega_{i\varepsilon})} + \|u_{i\varepsilon}\|^2_{L^2(\Omega_{i\varepsilon})} \right) \mathrm{d}s \tag{3.5}$$

for $i = 1, 2$. By Lemma 2.7 and with $\eta = 2\alpha$ in the Young inequality $ab \leq \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2$, $\forall \eta > 0$, we get

$$\int_0^t \|g\|_{H^{-1}(\Omega)} \|P_1^\varepsilon u_{1\varepsilon}\|_{H^1_0(\Omega)} \,\mathrm{d}s \leq \int_0^t C \|g\|_{H^{-1}(\Omega)} \|\nabla u_{1\varepsilon}\|_{L^2(\Omega_{1\varepsilon})} \,\mathrm{d}s \leq \tag{3.6}$$

$$\leq \int_0^t \frac{C^2}{4\alpha} \|g\|^2_{H^{-1}(\Omega)} \,\mathrm{d}s + \alpha \int_0^t \|\nabla u_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} \,\mathrm{d}s =$$

$$= \frac{C^2}{4\alpha} \int_0^t \|g\|^2_{H^{-1}(\Omega)} \,\mathrm{d}s + \alpha \int_0^t \int_{\Omega_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2 \,\mathrm{d}x \,\mathrm{d}s.$$

From (3.4)–(3.6) we obtain

$$\frac{1}{2}\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \alpha \int_0^t \int_{\Omega_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2 \,\mathrm{d}x \,\mathrm{d}s \leq$$

$$\leq \frac{1}{2}\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} + \frac{1}{2}\int_0^t \Big(\|f_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \|f_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})}\Big)\,\mathrm{d}s +$$

$$+\frac{1}{2}\int_0^t \Big(\|u_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \|u_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})}\Big)\,\mathrm{d}s + \frac{C^2}{4\alpha}\int_0^t \|g\|^2_{H^{-1}(\Omega)}\,\mathrm{d}s +$$

$$+\,\alpha\int_0^t\int_{\Omega_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}s =$$

$$= \frac{1}{2}\|U^0_{\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})\times L^2(\Omega_{2\varepsilon})} + \frac{1}{2}\|f_{\varepsilon}\|^2_{L^2(0,T;\,L^2(\Omega_{1\varepsilon}))\times L^2(0,T;\,L^2(\Omega_{2\varepsilon}))} +$$

$$+\frac{C^2}{4\alpha}\|g\|^2_{L^2(0,T;\,H^{-1}(\Omega))} + \frac{1}{2}\int_0^t \Big(\|u_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \|u_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})}\Big)\,\mathrm{d}s +$$

$$+\,\alpha\int_0^t\int_{\Omega_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Simplifying the expression, $\forall\, t \in [0,T]$ we have

$$\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} \leq \tag{3.7}$$

$$\leq \gamma_\varepsilon + \int_0^t \Big(\|u_{1\varepsilon}(\tau)\|^2_{L^2(\Omega_{1\varepsilon})} + \|u_{2\varepsilon}(\tau)\|^2_{L^2(\Omega_{2\varepsilon})}\Big)\,\mathrm{d}\tau,$$

where

$$\gamma_\varepsilon = \|U^0_{\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})\times L^2(\Omega_{2\varepsilon})} + \|f_{\varepsilon}\|^2_{L^2(0,T;\,L^2(\Omega_{1\varepsilon}))\times L^2(0,T;\,L^2(\Omega_{2\varepsilon}))} +$$

$$+\frac{C^2}{2\alpha}\|g\|^2_{L^2(0,T;\,H^{-1}(\Omega))}.$$

Applying Gronwall's Lemma in (3.7), we deduce that $\forall\, t \in [0,T]$ and c independent of ε we have

$$\|u_{1\varepsilon}\|^2_{L^\infty(0,T;\,L^2(\Omega_{1\varepsilon}))} + \|u_{2\varepsilon}\|^2_{L^\infty(0,T;\,L^2(\Omega_{2\varepsilon}))} \leq \mathrm{e}^T\gamma_\varepsilon \leq c, \tag{3.8}$$

since γ_ε is bounded by (3.3).

Now, using arguments similar to those that allowed to arrive at (3.6) with $\eta = \alpha$ in the Young inequality, we get

$$\int_0^t \|g\|_{H^{-1}(\Omega)}\|P_1^\varepsilon u_{1\varepsilon}\|_{H_0^1(\Omega)}\,\mathrm{d}s \leq \tag{3.9}$$

$$\leq \frac{C}{2\alpha}\int_0^t \|g\|^2_{H^{-1}(\Omega)}\,\mathrm{d}s + \frac{\alpha}{2}\int_0^t\int_{\Omega_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Consequently, by (3.4), (3.3), (3.8) and (3.9) we claim that

$$\alpha \int_0^T \|\nabla u_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} \,\mathrm{d}s + \alpha \int_0^T \|\nabla u_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} \,\mathrm{d}s + \\ + \varepsilon^\gamma h_0 \int_0^T \|u_{1\varepsilon} - u_{2\varepsilon}\|^2_{L^2(\Gamma^\varepsilon)} \,\mathrm{d}\sigma_x \,\mathrm{d}s = \\ = \alpha \int_0^T \int_{\Omega_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2 \,\mathrm{d}x \,\mathrm{d}s + \alpha \int_0^T \int_{\Omega_{2\varepsilon}} |\nabla u_{2\varepsilon}|^2 \,\mathrm{d}x \,\mathrm{d}s + \\ + \varepsilon^\gamma h_0 \int_0^T \int_{\Gamma^\varepsilon} |u_{1\varepsilon} - u_{2\varepsilon}|^2 \,\mathrm{d}\sigma_x \,\mathrm{d}s \le c.$$

This completes the proof. □

Corollary 3.6. *Under the conditions stated in Proposition* 3.5, *there exists a subsequence* (*still denoted by* ε) *such that for some* $u_1 \in L^2(0,T;\, H_0^1(\Omega))$ *and* $u_2 \in L^2(0,T;\, L^2(\Omega))$ *we have*

(i) $P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1$ *weakly in* $L^2(0,T;\, H_0^1(\Omega))$,

(ii) $\widetilde{u_{1\varepsilon}} \rightharpoonup \theta_1 u_1$ *weakly in* $L^2(0,T;\, L^2(\Omega))$ *and weakly* in* $L^\infty(0,T;\, L^2(\Omega))$,

(iii) $\widetilde{u_{2\varepsilon}} \rightharpoonup u_2$ *weakly in* $L^2(0,T;\, L^2(\Omega))$ *and weakly* in* $L^\infty(0,T;\, L^2(\Omega))$.

Proof. Convergence (iii) is a direct consequence of Proposition 3.5(ii) while (i) follows from Proposition 3.5(i) and Lemma 2.7. To prove (ii), we use Lemma 2.10. Let $g_\varepsilon = \chi_{\Omega_{1\varepsilon}}$, $h_\varepsilon = P_1^\varepsilon u_{1\varepsilon}$ and $p = 2$. Since

$$\begin{cases} P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 & \text{weakly in } L^2(0,T;\, H_0^1(\Omega)), \\ \chi_{\Omega_{1\varepsilon}} \rightharpoonup \theta_1 & \text{weakly in } L^2(\Omega), \end{cases}$$

by Lemma 2.10, we obtain

$$\widetilde{u_{1\varepsilon}} = \chi_{\Omega_{1\varepsilon}} P_1^\varepsilon u_{1\varepsilon} \rightharpoonup \theta_1 u_1 \quad \text{weakly in } L^2(0,T;\, L^2(\Omega)).$$

This gives (ii) by Proposition 3.5(i). □

Another important consequence of Proposition 3.5 is the result below that makes use of Lemmas 2.4 and 2.5.

Proposition 3.7. *Under the assumptions of Proposition* 3.5, *we have*

$$\|P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}\|_{L^2(0,T;\, L^2(\Omega_{2\varepsilon}))} \to 0 \quad \textit{for all } \gamma < 1.$$

Proof. Let $v = P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}$. By Lemma 2.4 we have

$$\|P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} \le C(\varepsilon \|u_{1\varepsilon} - u_{2\varepsilon}\|^2_{L^2(\Gamma^\varepsilon)} + \varepsilon^2 \|\nabla(P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon})\|^2_{L^2(\Omega_{2\varepsilon})}).$$

Integrating both sides over $]0,T[$ and taking into account 2.5(iii) and Proposition 3.5, we obtain

$$\int_0^T \|P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} \mathrm{d}s \leq$$

$$\leq C\varepsilon \int_0^T \|u_{1\varepsilon} - u_{2\varepsilon}\|^2_{L^2(\Gamma^\varepsilon)} \,\mathrm{d}s + C\varepsilon^2 \int_0^T (\|\nabla P_1^\varepsilon u_{1\varepsilon}\|_{L^2(\Omega)} + \|\nabla u_{2\varepsilon}\|_{L^2(\Omega_{2\varepsilon})})^2 \,\mathrm{d}s$$

$$\leq C\varepsilon(C_1\varepsilon^{-\gamma}) + C_2\varepsilon^2 \int_0^T (\|\nabla u_{1\varepsilon}\|_{L^2(\Omega_{1\varepsilon})} + \|\nabla u_{2\varepsilon}\|_{L^2(\Omega_{2\varepsilon})})^2 \,\mathrm{d}s \leq C_3(\varepsilon^{1-\gamma} + \varepsilon^2).$$

Letting $\varepsilon \to 0$, we get $\|P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}\|_{L^2(0,T;\, L^2(\Omega_{2\varepsilon}))} \to 0$ since $\gamma < 1$. $\square$

3.3. Some classes of test functions

We apply Tartar's oscillating test functions method (see [27]) to identify the limit problems and so we recall the definitions of two classes of suitable test functions. These functions were already used to study the elliptic cases in [15] and [24] and the hyperbolic cases in [14].

We recall first the one introduced in [15]. Consider the couple $(w_{1\lambda}, w_{2\lambda}) \in H^1(Y_1) \times H^1(Y_2)$ satisfying the system

$$\begin{cases} -\mathrm{div}({}^tA\nabla w_{1\lambda}) = 0 & \text{in } Y_1, \\ -\mathrm{div}({}^tA\nabla w_{2\lambda}) = 0 & \text{in } Y_2, \\ {}^tA\nabla w_{1\lambda}\cdot n_1 = -{}^tA\nabla w_{2\lambda}\cdot n_2 & \text{on } \Gamma, \\ {}^tA\nabla w_{1\lambda}\cdot n_1 = -\varepsilon^{\gamma+1}h(w_{1\lambda} - w_{2\lambda}) & \text{on } \Gamma, \\ \lambda\cdot y - w_{1\lambda} & Y\text{-periodic}, \\ m_{Y_1}(\lambda\cdot y - w_{1\lambda}) = 0, \end{cases} \tag{3.10}$$

whose variational formulation is

$$\begin{cases} \text{Find}(w_{1\lambda}, w_{2\lambda}) \text{ in } H^1(Y_1)\times H^1(Y_2) \\ \text{such that } \lambda\cdot y - w_{1\lambda} \in W_{\mathrm{per}}(Y_1) \text{ and} \\ \displaystyle\int_{Y_1} {}^tA\nabla w_{1\lambda}\cdot\nabla v_1\,\mathrm{d}y + \int_{Y_2} {}^tA\nabla w_{2\lambda}\cdot\nabla v_2\,\mathrm{d}y + \\ \displaystyle +\,\varepsilon^{\gamma+1}\int_\Gamma h(w_{1\lambda} - w_{2\lambda})(v_1 - v_2)\,\mathrm{d}\sigma_y = 0, \\ \forall\,(v_1, v_2) \in W_{\mathrm{per}}(Y_1)\times H^1(Y_2), \end{cases} \tag{3.11}$$

where

$$W_{\mathrm{per}}(Y_1) = \left\{u \in H^1_{\mathrm{per}}(Y_1) \mid m_{Y_1}(u) = 0\right\}$$

with the norm $\|u\|_{W_{\mathrm{per}}} := \|\nabla u\|_{L^2(Y_1)}$.

By definition, $w^\varepsilon_{i\lambda}$ is given by

$$w^\varepsilon_{i\lambda}(x) := \lambda x - \varepsilon(Q_i(\chi_{i\lambda})(x/\varepsilon)), \quad \chi_{i\lambda} = \lambda \cdot y - w_{i\lambda}(y), \quad i = 1, 2, \tag{3.12}$$

where Q_i is given in Lemma 2.5 and with $w_{i\lambda}$ solutions of problem (3.10).

Note that due to the periodicity of the functions defined in (3.12), classical arguments apply and one obtains

$$\begin{cases} w^\varepsilon_{i\lambda} \rightharpoonup \lambda \cdot x & \text{weakly in } H^1(\Omega), \\ w^\varepsilon_{i\lambda} \to \lambda \cdot x & \text{strongly in } L^2(\Omega). \end{cases} \tag{3.13}$$

Furthermore, let us recall that setting

$$\eta^\varepsilon_\lambda := (\eta^\varepsilon_{1\lambda}, \eta^\varepsilon_{2\lambda}) = ({}^tA^\varepsilon \nabla w^\varepsilon_{1\lambda}, {}^tA^\varepsilon \nabla w^\varepsilon_{2\lambda}), \tag{3.14}$$

it can be easily checked that, $\forall\, v = (v_1, v_2) \in H^\varepsilon_\gamma$, the function η^ε_λ satisfies

$$\begin{aligned} &\int_{\Omega_{1\varepsilon}} \eta^\varepsilon_{1\lambda} \cdot \nabla v_1 \, \mathrm{d}x + \int_{\Omega_{2\varepsilon}} \eta^\varepsilon_{2\lambda} \cdot \nabla v_2 \, \mathrm{d}x + \\ &+ \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (w^\varepsilon_{1\lambda} - w^\varepsilon_{2\lambda})(v_1 - v_2) \, \mathrm{d}\sigma = 0. \end{aligned} \tag{3.15}$$

Moreover, the convergences below hold:

- If $\gamma \le -1$ (see [15, 24]) then

$$\widetilde{\eta^\varepsilon_{1\lambda}} + \widetilde{\eta^\varepsilon_{2\lambda}} \rightharpoonup {}^tA^0_\gamma \lambda \quad \text{weakly in } [L^2(\Omega)]^n. \tag{3.16}$$

If $\gamma < -1$, A^0_γ is defined by

$${}^tA^0_\gamma \lambda := m_Y({}^tA \nabla W_\lambda) \quad \text{for all } \gamma < -1, \tag{3.17}$$

where, for any $\lambda \in \mathbb{R}^n$, $W_\lambda \in H^1(Y)$ is the solution of the classical problem:

$$\begin{cases} -\mathrm{div}({}^tA \nabla W_\lambda) = 0 & \text{in } Y, \\ W_\lambda - \lambda \cdot y & Y\text{-periodic}, \\ \dfrac{1}{|Y|} \displaystyle\int_Y (W_\lambda - \lambda \cdot y) \, \mathrm{d}y = 0. \end{cases} \tag{3.18}$$

This auxilliary problem is used in the classical homogenization of the stationary heat equation in the whole domain studied in [2] (see also [9]).

On the other hand, if $\gamma = -1$, A^0_γ is defined by

$${}^tA^0_\gamma \lambda := m_Y({}^tA(\widetilde{\nabla w_{1\lambda}} + \widetilde{\nabla w_{2\lambda}})) \quad \text{for } \gamma = -1, \tag{3.19}$$

where $(w_{1\lambda}, w_{2\lambda}) \in H^1(Y_1) \times H^1(Y_2)$ is the solution of the problem

$$\begin{cases} -\mathrm{div}({}^tA_1\nabla w_{1\lambda} = 0 & \text{in } Y_1,\\ -\mathrm{div}({}^tA_2\nabla w_{2\lambda} = 0 & \text{in } Y_2,\\ {}^tA_1\nabla w_{1\lambda}\cdot n_1 = -{}^tA_2\nabla w_{2\lambda}\cdot n_2 & \text{on } \Gamma,\\ {}^tA_1\nabla w_{1\lambda}\cdot n_1 = -h(w_{1\lambda}-w_{2\lambda}) & \text{on } \Gamma,\\ \lambda\cdot y - w_{1\lambda} & Y\text{-periodic},\\ m_{Y_1}(\lambda\cdot y - w_{1\lambda}) = 0. \end{cases} \tag{3.20}$$

• If $-1 < \gamma \le 1$ (see [15]) then

$$\begin{cases} \text{(i)}\ \ \widetilde{\eta^{\varepsilon}_{1\lambda}} \rightharpoonup {}^tA^0_\gamma\lambda & \text{weakly in } [L^2(\Omega)]^n,\\ \text{(ii)}\ \ \varepsilon^{-\frac{\gamma+1}{2}}\widetilde{\eta^{\varepsilon}_{2\lambda}} \rightharpoonup 0 & \text{weakly in } [L^2(\Omega)]^n, \end{cases} \tag{3.21}$$

with A^0_γ defined by

$${}^tA^0_\gamma\lambda := m_Y({}^t\widetilde{A\nabla w_\lambda}) \quad \text{for all} -1 < \gamma \le 1, \tag{3.22}$$

where, for any $\lambda \in \mathbb{R}^n, w_\lambda \in H^1(Y_1)$ is the solution of the problem

$$\begin{cases} -\mathrm{div}({}^tA\nabla w_\lambda) = 0 & \text{in } Y_1,\\ ({}^tA\nabla w_\lambda)\cdot n_1 = 0 & \text{in } \Gamma,\\ w_\lambda - \lambda\cdot y & Y\text{-periodic},\\ \dfrac{1}{|Y_1|}\displaystyle\int_{Y_1}(w_\lambda - \lambda\cdot y)\,\mathrm{d}y = 0. \end{cases} \tag{3.23}$$

By definition, w^ε_λ is given by

$$w^\varepsilon_\lambda(x) := \lambda x - \varepsilon(Q_1(\chi_{1\lambda})(x/\varepsilon)), \quad \chi_{1\lambda} = \lambda\cdot y - w_\lambda(y), \tag{3.24}$$

where Q_1 is given in Lemma 2.5 and $w_\lambda(y)$ is the solution of problem (3.23). By a change in scale (see [10]), it can be shown that

$$\int_{\Omega_{1\varepsilon}} {}^tA^\varepsilon\nabla w^\varepsilon_\lambda\cdot\nabla v_1\,\mathrm{d}x = 0, \quad \forall\, v_1 \in V^\varepsilon, \tag{3.25}$$

and

$$\begin{cases} w^\varepsilon_\lambda \rightharpoonup \lambda\cdot x & \text{weakly in } H^1(\Omega),\\ w^\varepsilon_\lambda \to \lambda\cdot x & \text{strongly in } L^2(\Omega),\\ {}^tA^\varepsilon\widetilde{\nabla w^\varepsilon_\lambda} \rightharpoonup {}^tA^0_\gamma\lambda & \text{weakly in } [L^2(\Omega)]^n, \end{cases} \tag{3.26}$$

where A^0_γ is defined by (3.22).

We note that w^ε_λ is used in the classical homogenization of the stationary heat equation on a perforated domain with a Neumann condition on the boundary of the holes studied in [10].

4. THE MAIN RESULT

In this section we present the homogenization result and provide a step by step proof.

4.1. The homogenization theorem

THEOREM 4.1. *Let A^ε and h^ε be defined by* (2.8) *and* (2.10) *respectively. Let u_ε be the solution of problem* (2.12) *for $\gamma \leq 1$. Moreover, suppose that* (2.11) *and* (3.3) *hold. Then there exists a suitable extension operator $P_1^\varepsilon \in \mathcal{L}(L^2(0,T;V^\varepsilon);L^2(0,T;H_0^1(\Omega))) \cap \mathcal{L}(L^2(0,T;\,L^2(\Omega_{1\varepsilon}));L^2(0,T;\,L^2(\Omega)))$ such that*

$$\begin{cases} P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 & \text{weakly in } L^2(0,T;\,H_0^1(\Omega)), \\ \widetilde{u_{2\varepsilon}} \rightharpoonup u_2 & \text{weakly* in } L^\infty(0,T;\,L^2(\Omega)) \end{cases} \tag{4.1}$$

and

$$A^\varepsilon\widetilde{\nabla u_{1\varepsilon}} + A^\varepsilon\widetilde{\nabla u_{2\varepsilon}} \rightharpoonup A_\gamma^0 \nabla u_1 \quad \text{weakly in } L^2(0,T;\,[L^2(\Omega)]^n) \quad \forall\gamma \leq 1, \tag{4.2}$$

where A_γ^0 is defined by (3.17), (3.19) *and* (3.22) *for the cases $\gamma < -1$, $\gamma = -1$ and $-1 < \gamma \leq 1$ respectively.*

In particular, if $-1 < \gamma \leq 1$ then

$$\begin{cases} A^\varepsilon\widetilde{\nabla u_{1\varepsilon}} \rightharpoonup A_\gamma^0 \nabla u_1 & \text{weakly in } L^2(0,T;\,[L^2(\Omega)]^n), \\ A^\varepsilon\widetilde{\nabla u_{2\varepsilon}} \rightharpoonup 0 & \text{weakly in } L^2(0,T;\,[L^2(\Omega)]^n). \end{cases} \tag{4.3}$$

Moreover, the limit functions u_1 and u_2 can be described as follows:

- *Case $\gamma < 1$. We have $u_2 = \theta_2 u_1$.*

The factor θ_2 is given by (2.2) *and u_1 is the unique solution of the homogenized problem*

$$\begin{cases} u_1' - \operatorname{div}(A_\gamma^0 \nabla u_1) = \theta_1 f_1 + \theta_2 f_2 + g & \text{in } \Omega\times]0,T[, \\ u_1 = 0 & \text{on } \partial\Omega\times]0,T[, \\ u_1(0) = \theta_1 U_1^0 + \theta_2 U_2^0 & \text{in } \Omega. \end{cases} \tag{4.4}$$

- *Case $\gamma = 1$. The couple (u_1, u_2) is the unique solution of the problem* (*a PDE coupled with an ODE*)

$$\begin{cases} \theta_1 u_1' - \operatorname{div}(A_\gamma^0 \nabla u_1) + c_h(\theta_2 u_1 - u_2) = \theta_1 f_1 + g & \text{in } \Omega\times]0,T[, \\ u_2' - c_h(\theta_2 u_1 - u_2) = \theta_2 f_2 & \text{in } \Omega\times]0,T[, \\ u_1 = 0 & \text{on } \partial\Omega\times]0,T[, \\ u_1(0) = U_1^0,\ u_2(0) = \theta_2 U_2^0 & \text{in } \Omega, \end{cases} \tag{4.5}$$

where $c_h = \frac{1}{|Y_2|}\int_\gamma h(y)\,\mathrm{d}\sigma_y$. *Moreover, solving the ODE and replacing in the PDE yield*

$$\theta_1 u_1' - \operatorname{div}(A^0_\gamma \nabla u_1) + c_h\theta_2 u_1 - c_h^2\theta_2 \int_0^t \mathcal{K}(t,s)u_1(s)\,\mathrm{d}s = F(x,t), \tag{4.6}$$

where

$$F(x,t) = \theta_1 f_1(x,t) + g + c_h\theta_2 U_2^0 \mathrm{e}^{-c_h t} + c_h \int_0^t \mathcal{K}(t,s)\theta_2 f_2(x,s)\,\mathrm{d}s \tag{4.7}$$

and $\mathcal{K}$ *is an exponential kernel given by*

$$\mathcal{K}(t,s) = e^{c_h(s-t)}. \tag{4.8}$$

4.2. **Proof of the main result**

We first prove the next result which play an essential role in proving the main result.

LEMMA 4.2. *Let* $\varphi \in \mathcal{D}(0,T)$ *and* $v \in \mathcal{D}(\Omega)$. *Suppose* P_1^ε *is the extension operator described in Lemma* 2.7. *If* $\{u_\varepsilon\} = (u_{1\varepsilon}, u_{2\varepsilon})$ *is the subsequence given in Corollary* 3.6, *then*

$$\lim_{\varepsilon\to 0}\left(\int_0^T\int_\Omega \widetilde{\eta_{1\lambda}^\varepsilon}\nabla v P_1^\varepsilon u_{1\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \widetilde{\eta_{2\lambda}^\varepsilon}\nabla v\widetilde{u_{2\varepsilon}}\varphi\,\mathrm{d}x\,\mathrm{d}t\right) =$$

$$= \int_0^T\int_\Omega {}^t A^0_\gamma \lambda \nabla v u_1 \varphi\,\mathrm{d}x\,\mathrm{d}t$$

for every $\gamma \le 1$ *and for all* $\lambda \in \mathbb{R}^n$.

Proof. Case $-1 < \gamma \le 1$. Observe that by (3.21)(ii) we have

$$\|\widetilde{\eta_{2\lambda}^\varepsilon}\|_{L^2(\Omega)} \le \varepsilon^{\frac{\gamma+1}{2}}.$$

Since $\gamma + 1 > 0$, we get

$$\|\widetilde{\eta_{2\lambda}^\varepsilon}\|_{L^2(\Omega)} \to 0.$$

By Hölder inequality and Proposition 3.5 we have

$$\left|\int_0^T\int_\Omega \widetilde{\eta_{2\lambda}^\varepsilon}\nabla v\widetilde{u_{2\varepsilon}}\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \to 0.$$

It remains to show that

$$\int_0^T\int_\Omega \widetilde{\eta_{1\lambda}^\varepsilon}\nabla v P_1^\varepsilon u_{1\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega {}^t A^0_\gamma \lambda \nabla v u_1 \varphi\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\lambda \in \mathbb{R}^n.$$

Using Lemma 2.10 with $h_\varepsilon = P_1^\varepsilon u_{1\varepsilon}$ and $g_\varepsilon = \widetilde{\eta_{1\lambda}^\varepsilon}$ (independent of t), by (3.21)(i) and Corollary 3.6 we have

$$\begin{cases} \widetilde{\eta_{1\lambda}^\varepsilon} \rightharpoonup {}^t A_\gamma^0 \lambda & \text{weakly in } [L^2(\Omega)]^n, \\ P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 & \text{weakly in } L^2(0,T;\, H_0^1(\Omega)). \end{cases}$$

Therefore, letting $\varepsilon \to 0$ we get the desired result.

Case $\gamma \leq -1$. Note that

$$\int_0^T \int_\Omega \widetilde{\eta_{1\lambda}^\varepsilon} \nabla v P_1^\varepsilon u_{1\varepsilon} \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \widetilde{\eta_{2\lambda}^\varepsilon} \nabla v \widetilde{u_{2\varepsilon}} \varphi \,\mathrm{d}x\,\mathrm{d}t =$$

$$= \int_0^T \int_\Omega (\widetilde{\eta_{1\lambda}^\varepsilon} + \widetilde{\eta_{2\lambda}^\varepsilon}) \nabla v P_1^\varepsilon u_{1\varepsilon} \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \widetilde{\eta_{2\lambda}^\varepsilon} \nabla v (\widetilde{u_{2\varepsilon}} - P_1^\varepsilon u_{1\varepsilon}) \varphi \,\mathrm{d}x\,\mathrm{d}t.$$

But, by Proposition 3.7,

$$\|P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}\|_{L^2(0,T;\, L^2(\Omega_{2\varepsilon}))} \to 0.$$

Hence, by Hölder inequality and (3.13),

$$\int_0^T \int_\Omega \widetilde{\eta_{2\lambda}^\varepsilon} \nabla v (\widetilde{u_{2\varepsilon}} - P_1^\varepsilon u_{1\varepsilon}) \varphi \,\mathrm{d}x\,\mathrm{d}t \to 0.$$

To conclude, we use again Lemma 2.10. Let $h_\varepsilon = P_1^\varepsilon u_{1\varepsilon}$ and $g_\varepsilon = \widetilde{\eta_{1\lambda}^\varepsilon} + \widetilde{\eta_{2\lambda}^\varepsilon}$ (independent of t). Then, by (3.16) and Corollary 3.6,

$$\begin{cases} \widetilde{\eta_{1\lambda}^\varepsilon} + \widetilde{\eta_{2\lambda}^\varepsilon} \rightharpoonup {}^t A_\gamma^0 \lambda & \text{weakly in } [L^2(\Omega)]^n, \\ P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 & \text{weakly in } L^2(0,T;\, H_0^1(\Omega)). \end{cases}$$

Letting $\varepsilon \to 0$, for all $\lambda \in \mathbb{R}^n$ we have

$$\int_0^T \int_\Omega (\widetilde{\eta_{1\lambda}^\varepsilon} + \widetilde{\eta_{2\lambda}^\varepsilon}) \nabla v P_1^\varepsilon u_{1\varepsilon} \varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega {}^t A_\gamma^0 \lambda \nabla v u_1 \varphi \,\mathrm{d}x\,\mathrm{d}t. \qquad \square$$

In proving Theorem 4.1, both cases will be considered simultaneously. We'll treat them separately only when necessary. Let u_ε be the solution of problem (2.12) for any value of $\gamma \leq 1$. We adapt some techniques used in [14] and [15]. Set

$$\xi^\varepsilon := (\xi_{1\varepsilon}, \xi_{2\varepsilon}) = (A^\varepsilon \nabla u_{1\varepsilon}, A^\varepsilon \nabla u_{2\varepsilon}). \tag{4.9}$$

If follows from Proposition 3.5 that, up to a subsequence,

$$\widetilde{\xi_{i\varepsilon}} \rightharpoonup \xi_i \quad \text{weakly* in } L^2(0,T;\, [L^2(\Omega)]^n), \quad i = 1,2. \tag{4.10}$$

The main idea of the proof is to identify ξ_1 and ξ_2 and find the limit functions u_1 and u_2 which change according to the values of γ.

Proof of Theorem 4.1. The proof will be divided into six steps.

Step 1. Determine the equation satisfied by $\xi_1+\xi_2$.

Let $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$. In the variational formulation (2.14), use $(v|_{\Omega_{1\varepsilon}}\varphi, v|_{\Omega_{2\varepsilon}}\varphi)$ as test function, so that

$$\langle u'_{1\varepsilon}, v\varphi\rangle_{(V^\varepsilon)',V^\varepsilon} + \langle u'_{2\varepsilon}, v\varphi\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} +$$
$$+\int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon}\nabla v\,\varphi\,\mathrm{d}x + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon}\nabla v\,\varphi\,\mathrm{d}x =$$
$$= \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} v\varphi\,\mathrm{d}x + \langle g, P_1^\varepsilon((v\varphi)|_{\Omega_{1\varepsilon}})\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} v\varphi\,\mathrm{d}x.$$

Integrating both sides with respect to t and, extending to the whole of Ω, by (4.9) we have

$$-\int_0^T\int_\Omega (\widetilde{u_{1\varepsilon}}+\widetilde{u_{2\varepsilon}})v\varphi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \widetilde{\xi_{1\varepsilon}}\nabla v\,\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \widetilde{\xi_{2\varepsilon}}\nabla v\,\varphi\,\mathrm{d}x\,\mathrm{d}t =$$
$$= \int_0^T\int_\Omega (\widetilde{f_{1\varepsilon}}+\widetilde{f_{2\varepsilon}})v\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, P_1^\varepsilon((v\varphi)|_{\Omega_{1\varepsilon}})\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}t$$

for every $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$.

Letting $\varepsilon \to 0$, by (4.10), (3.3), Corollary 3.6 and Corollary 2.8 (with $v_\varepsilon = v\varphi$), we obtain

$$-\int_0^T\int_\Omega (\theta_1 u_1+u_2)v\varphi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \xi_1\nabla v\,\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \xi_2\nabla v\,\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{4.11}$$
$$= \int_0^T\int_\Omega (\theta_1 f_1+\theta_2 f_2)v\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, v\varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}t.$$

Since φ and v are arbitrary, the equation satisfied by $\xi_1+\xi_2$ is

$$\theta_1 u'_1 + u'_2 - \operatorname{div}(\xi_1+\xi_2) = \theta_1 f_1 + \theta_2 f_2 + g. \tag{4.12}$$

Step 2. Identify $\xi_1+\xi_2$.

Let $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$. In the variational formulation (2.14), choose $(vw^\varepsilon_{1\lambda}|_{\Omega_{1\varepsilon}}\varphi, vw^\varepsilon_{2\lambda}|_{\Omega_{2\varepsilon}}\varphi)$ as test function, where $w^\varepsilon_{i\lambda}$, $i=1,2$ are defined in (3.12). Then, for every $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$,

$$-\int_0^T\int_{\Omega_{1\varepsilon}} u_{1\varepsilon}w^\varepsilon_{1\lambda}v\varphi'\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Omega_{2\varepsilon}} u_{2\varepsilon}w^\varepsilon_{2\lambda}v\varphi'\,\mathrm{d}x\,\mathrm{d}t +$$
$$+\int_0^T\int_{\Omega_{1\varepsilon}} \xi_{1\varepsilon}\nabla v w^\varepsilon_{1\lambda}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_{1\varepsilon}} \xi_{1\varepsilon}\nabla w^\varepsilon_{1\lambda}v\varphi\,\mathrm{d}x\,\mathrm{d}t +$$
$$+\int_0^T\int_{\Omega_{2\varepsilon}} \xi_{2\varepsilon}\nabla v w^\varepsilon_{2\lambda}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_{2\varepsilon}} \xi_{2\varepsilon}\nabla w^\varepsilon_{2\lambda}v\varphi\,\mathrm{d}x\,\mathrm{d}t +$$
$$+\varepsilon^\gamma\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(w^\varepsilon_{1\lambda}-w^\varepsilon_{2\lambda})v\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t =$$

$$= \int_0^T \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} v w_{1\lambda}^\varepsilon \varphi \, dx \, dt + \int_0^T \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} v w_{2\lambda}^\varepsilon \varphi \, dx \, dt + \\ + \int_0^T \langle g, P_1^\varepsilon((v w_{1\lambda}^\varepsilon \varphi)|_{\Omega_{1\varepsilon}}) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt.$$

Now, put $(v_1, v_2) = (v|_{\Omega_{1\varepsilon}} u_{1\varepsilon}\varphi, v|_{\Omega_{2\varepsilon}} u_{2\varepsilon}\varphi)$ in (3.15) and integrate over $]0,T[$. Then

$$\int_0^T \int_{\Omega_{1\varepsilon}} \eta_{1\lambda}^\varepsilon \nabla v u_{1\varepsilon} \varphi \, dx \, dt + \int_0^T \int_{\Omega_{1\varepsilon}} \eta_{1\lambda}^\varepsilon \nabla u_{1\varepsilon} v \varphi \, dx \, dt + \\ + \int_0^T \int_{\Omega_{2\varepsilon}} \eta_{2\lambda}^\varepsilon \nabla v u_{2\varepsilon} \varphi \, dx \, dt \int_0^T \int_{\Omega_{2\varepsilon}} \eta_{2\lambda}^\varepsilon \nabla u_{2\varepsilon} v \varphi \, dx \, dt + \\ + \varepsilon^\gamma \int_0^T \int_{\Gamma^\varepsilon} h^\varepsilon (w_{1\lambda}^\varepsilon - w_{2\lambda}^\varepsilon)(u_{1\varepsilon} - u_{2\varepsilon}) v \varphi \, d\sigma_x \, dt = 0$$

for every $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$. Subtract the second equation from the first one and pass to the whole of Ω, recalling definitions (3.14) and (4.9). Since the boundary terms cancel, this will yield

$$-\int_0^T \int_\Omega (\widetilde{u_{1\varepsilon}} w_{1\lambda}^\varepsilon + \widetilde{u_{2\varepsilon}} w_{2\lambda}^\varepsilon) v \varphi' \, dx \, dt + \int_0^T \int_\Omega \widetilde{\xi_{1\varepsilon}} \nabla v w_{1\lambda}^\varepsilon \varphi \, dx \, dt + \\ + \int_0^T \int_\Omega \widetilde{\xi_{2\varepsilon}} \nabla v w_{2\lambda}^\varepsilon \varphi \, dx \, dt - \int_0^T \int_\Omega \widetilde{\eta_{1\lambda}^\varepsilon} \nabla v P_1^\varepsilon u_{1\varepsilon} \varphi \, dx \, dt - \int_0^T \int_\Omega \widetilde{\eta_{2\lambda}^\varepsilon} \nabla v \widetilde{u_{2\varepsilon}} \varphi \, dx \, dt \\ = \int_0^T \int_\Omega \widetilde{f_{1\varepsilon}} v w_{1\lambda}^\varepsilon \varphi \, dx \, dt + \int_0^T \int_\Omega \widetilde{f_{2\varepsilon}} v w_{2\lambda}^\varepsilon \varphi \, dx \, dt + \\ + \int_0^T \langle g, P_1^\varepsilon((v w_{1\lambda}^\varepsilon \varphi)|_{\Omega_{1\varepsilon}}) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt$$

for every $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$.

The next step is to let $\varepsilon \to 0$. All the limits are straightforward using (4.10), (3.13), (3.3), Corollary 3.6 and Corollary 2.8 (with $v_\varepsilon = v w_{1\lambda}^\varepsilon \varphi$). For the fourth and fifth integrals on the left-hand side, we apply Lemma 4.2 to obtain

$$-\int_0^T \int_\Omega (\theta_1 u_1 + u_2)(\lambda \cdot x) v \varphi' \, dx \, dt + \int_0^T \int_\Omega (\xi_1 + \xi_2) \nabla v (\lambda \cdot x) \varphi \, dx \, dt - \\ - \int_0^T \int_\Omega {}^t A_\gamma^0 \lambda \nabla v u_1 \varphi \, dx \, dt = \\ = \int_0^T \int_\Omega (\theta_1 f_1 + \theta_2 f_2)(\lambda \cdot x) v \varphi \, dx \, dt + \int_0^T \langle g, (\lambda \cdot x) v \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt$$

for every $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$, where A_γ^0 is defined according to the values of γ by (3.17), (3.19) and (3.22).

It then follows that

$$-\int_0^T\int_\Omega(\theta_1 u_1+u_2)(\lambda\cdot x)v\varphi'\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega(\xi_1+\xi_2)\nabla[v(\lambda\cdot x)]\varphi\,\mathrm{d}x\,\mathrm{d}t-$$
$$-\int_0^T\int_\Omega(\xi_1+\xi_2)v\lambda\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega {}^tA^0_\gamma\lambda\nabla u_1 v\varphi\,\mathrm{d}x\,\mathrm{d}t=$$
$$=\int_0^T\int_\Omega(\theta_1 f_1+\theta_2 f_2)(\lambda\cdot x)v\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\langle g,(\lambda\cdot x)v\varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}t.$$

On account of (4.11) written with $v(\lambda\cdot x)$ instead of v, we have

$$\int_0^T\int_\Omega(\xi_1+\xi_2)\lambda v\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega {}^tA^0_\gamma\lambda\nabla u_1 v\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Since λ, φ and v are arbitrary, we identify

$$\xi_1+\xi_2=A^0_\gamma\nabla u_1. \tag{4.13}$$

Step 3. Identify ξ_1 and ξ_2 separately for the case $-1<\gamma\le 1$.

As shown previously, $\xi_1+\xi_2=A^0_\gamma\nabla u_1$ holds for every $\gamma\le 1$. The task is now to describe ξ_1 and ξ_2 separately for the case $-1<\gamma\le 1$.

For this purpose, let $\varphi\in\mathcal{D}(0,T)$, $v\in\mathcal{D}(\Omega)$ and w^ε_λ be defined by (3.24). Take $((vw^\varepsilon_\lambda)|_{\Omega_{1\varepsilon}}\varphi,(v(\lambda\cdot x))|_{\Omega_{2\varepsilon}}\varphi)$ as test function in the variational formulation (2.14). For every $\varphi\in\mathcal{D}(0,T)$ and $v\in\mathcal{D}(\Omega)$ we have

$$-\int_0^T\int_{\Omega_{1\varepsilon}}u_{1\varepsilon}vw^\varepsilon_\lambda\varphi'\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\Omega_{2\varepsilon}}u_{2\varepsilon}v(\lambda\cdot x)\varphi'\,\mathrm{d}x\,\mathrm{d}t+$$
$$+\int_0^T\int_{\Omega_{1\varepsilon}}\xi_{1\varepsilon}\nabla vw^\varepsilon_\lambda\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\Omega_{1\varepsilon}}\xi_{1\varepsilon}\nabla w^\varepsilon_\lambda v\varphi\,\mathrm{d}x\,\mathrm{d}t+$$
$$+\int_0^T\int_{\Omega_{2\varepsilon}}\xi_{2\varepsilon}\nabla[v(\lambda\cdot x)]\varphi\,\mathrm{d}x\,\mathrm{d}t+\varepsilon^\gamma\int_0^T\int_{\Gamma^\varepsilon}h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(w^\varepsilon_\lambda-\lambda\cdot x)v\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t=$$
$$=\int_0^T\int_{\Omega_{1\varepsilon}}f_{1\varepsilon}vw^\varepsilon_\lambda\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\Omega_{2\varepsilon}}f_{2\varepsilon}v(\lambda\cdot x)\varphi\,\mathrm{d}x\,\mathrm{d}t+$$
$$+\int_0^T\langle g,P^\varepsilon_1((vw^\varepsilon_\lambda\varphi)|_{\Omega_{1\varepsilon}})\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}t.$$

Choose $v_1=v|_{\Omega_{1\varepsilon}}u_{1\varepsilon}\varphi$ in (3.25) and integrate over $]0,T[$ to get

$$\int_0^T\int_{\Omega_{1\varepsilon}}{}^tA^\varepsilon\nabla w^\varepsilon_\lambda\nabla vu_{1\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\Omega_{1\varepsilon}}{}^tA^\varepsilon\nabla w^\varepsilon_\lambda\nabla u_{1\varepsilon}v\varphi\,\mathrm{d}x\,\mathrm{d}t=0$$

for every $\varphi\in\mathcal{D}(0,T)$ and $v\in\mathcal{D}(\Omega)$.

Subtract the second from the first equation and extend to the whole of Ω taking into account (4.9). This yields

$$
\begin{aligned}
(4.14)\qquad & -\int_0^T\int_\Omega \widetilde{u_{1\varepsilon}} w^\varepsilon_\lambda v\varphi' \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \widetilde{u_{2\varepsilon}}(\lambda\cdot x)v\varphi' \,\mathrm{d}x\,\mathrm{d}t + \\
& + \int_0^T\int_\Omega \widetilde{\xi_{1\varepsilon}}\nabla v w^\varepsilon_\lambda \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \widetilde{\xi_{2\varepsilon}}\nabla[v(\lambda\cdot x)]\varphi \,\mathrm{d}x\,\mathrm{d}t - \\
& - \int_0^T\int_\Omega {}^tA^\varepsilon \widetilde{\nabla w^\varepsilon_\lambda}\nabla v P^\varepsilon_1 u_{1\varepsilon}\varphi \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^\gamma \int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(w^\varepsilon_\lambda-\lambda\cdot x)v\varphi \,\mathrm{d}\sigma_x\,\mathrm{d}t = \\
& = \int_0^T\int_\Omega \widetilde{f_{1\varepsilon}} v w^\varepsilon_\lambda \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \widetilde{f_{2\varepsilon}} v(\lambda\cdot x)\varphi \,\mathrm{d}x\,\mathrm{d}t + \\
& + \int_0^T \langle g, P^\varepsilon_1((v w^\varepsilon_\lambda\varphi)|_{\Omega_{1\varepsilon}})\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \,\mathrm{d}t
\end{aligned}
$$

for every $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$.

Our first concern is the behavior of the integral over Γ^ε. As $\varepsilon \to 0$ we show that

$$
(4.15)\qquad \varepsilon^\gamma \int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(w^\varepsilon_\lambda-\lambda\cdot x)v\varphi \,\mathrm{d}\sigma_x\,\mathrm{d}t \to 0.
$$

By definition (3.24), (2.10) and Proposition 3.5(iii), a change of scale yields

$$
\begin{aligned}
&\left|\varepsilon^\gamma \int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(w^\varepsilon_\lambda-\lambda\cdot x)v\varphi \,\mathrm{d}\sigma_x\,\mathrm{d}t\right| = \\
&= \left|\varepsilon^\gamma \int_0^T\int_{\Gamma^\varepsilon} \varepsilon h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})\chi_{1\lambda}(x/\varepsilon)v\varphi \,\mathrm{d}\sigma_x\,\mathrm{d}t\right| \le \\
&\le C\varepsilon^{\gamma+1}\varepsilon^{-\gamma/2}\|\chi_{1\lambda}(x/\varepsilon)\|_{L^2(\Gamma^\varepsilon)} \le C\varepsilon^{\gamma+1}\varepsilon^{-\gamma/2}\varepsilon^{-1/2} \to 0
\end{aligned}
$$

as $\varepsilon \to 0$, since $\gamma > -1$.

Now, let $\varepsilon \to 0$ in (4.14) and use (4.10), (3.26), (4.15), (3.3), Corollary 3.6, Lemma 2.10 (with $h_\varepsilon = P^\varepsilon_1 u_{1\varepsilon}$ and $g_\varepsilon = {}^tA^\varepsilon\widetilde{\nabla w^\varepsilon_\lambda}$) and Corollary 2.8 (with $v_\varepsilon = v w^\varepsilon_\lambda \varphi$) to obtain

$$
\begin{aligned}
& -\int_0^T\int_\Omega (\theta_1 u_1 + u_2)(\lambda\cdot x)v\varphi' \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \xi_1\nabla v(\lambda\cdot x)\varphi \,\mathrm{d}x\,\mathrm{d}t + \\
& + \int_0^T\int_\Omega \xi_2\nabla[v(\lambda\cdot x)]\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega {}^tA^0_\gamma\lambda\nabla v u_1\varphi \,\mathrm{d}x\,\mathrm{d}t = \\
& = \int_0^T\int_\Omega (\theta_1 f_1 + \theta_2 f_2)(\lambda\cdot x)v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, (\lambda\cdot x)v\varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \,\mathrm{d}t
\end{aligned}
$$

for every $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$, where A^0_γ is defined by (3.22). This gives

$$-\int_0^T\int_\Omega(\theta_1 u_1+u_2)(\lambda\cdot x)v\varphi'\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega\xi_1\nabla[v(\lambda\cdot x)]\varphi\,\mathrm{d}x\,\mathrm{d}t-$$

$$-\int_0^T\int_\Omega\xi_1\lambda v\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega\xi_2\nabla[v(\lambda\cdot x)]\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega {}^tA^0_\gamma\lambda\nabla u_1 v\varphi\,\mathrm{d}x\,\mathrm{d}t=$$

$$=\int_0^T\int_\Omega(\theta_1 f_1+\theta_2 f_2)(\lambda\cdot x)v\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\langle g,(\lambda\cdot x)v\varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,\mathrm{d}t$$

for every $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$ since A^0_γ is constant. It then follows from (4.11), written with $v(\lambda\cdot x)$ instead of v, that

$$\int_0^T\int_\Omega\xi_1\lambda v\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega {}^tA^0_\gamma\lambda\nabla u_1 v\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Since λ, φ and v are arbitrary, we have

$$\xi_1 = A^0_\gamma\nabla u_1. \tag{4.16}$$

By (4.13) and (4.16) it is obvious that

$$\xi_2 = 0. \tag{4.17}$$

Step 4. Describe the limit function u_2 in terms of u_1 for the two cases $\gamma<1$ and $\gamma=1$.

In the following we treat the two cases $\gamma<1$ and $\gamma=1$ separately. We employ the same method as in the proof of the elliptic and hyperbolic problem given in [14] and [15].

First, we consider the case $\gamma<1$. Observe that

$$\int_0^T\int_\Omega\widetilde{u_{2\varepsilon}}\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\Omega_{2\varepsilon}}(u_{2\varepsilon}-P^\varepsilon_1u_{1\varepsilon})\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\Omega_{2\varepsilon}}P^\varepsilon_1u_{1\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi\in L^2(0,T;\,L^2(\Omega))$. By Proposition 3.7, $\|P^\varepsilon_1u_{1\varepsilon}-u_{2\varepsilon}\|_{L^2(0,T;\,L^2(\Omega_{2\varepsilon}))}\to 0$. So, by Hölder inequality we have

$$\int_0^T\int_{\Omega_{2\varepsilon}}(u_{2\varepsilon}-P^\varepsilon_1u_{1\varepsilon})\varphi\,\mathrm{d}x\,\mathrm{d}t\to 0. \tag{4.18}$$

Using (4.18) and evaluating the limit by (2.2), Corollary 3.6 and Lemma 2.10 (with $h_\varepsilon = P^\varepsilon_1u_{1\varepsilon}$ and $g_\varepsilon=\chi_{\Omega_{2\varepsilon}}$), we have

$$\lim_{\varepsilon\to0}\int_0^T\int_\Omega\widetilde{u_{2\varepsilon}}\varphi\,\mathrm{d}x\,\mathrm{d}t=\lim_{\varepsilon\to0}\int_0^T\int_\Omega\chi_{\Omega_{2\varepsilon}}P^\varepsilon_1u_{1\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega\theta_2u_1\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Since φ is arbitrary,

$$\widetilde{u_{2\varepsilon}}\rightharpoonup\theta_2u_1. \tag{4.19}$$

By the uniqueness of the weak limit, $u_2 = \theta_2 u_1$. Replacing this in (4.12) and using (4.13) yields

$$u_1' - \operatorname{div}(A^0_\gamma \nabla u_1) = \theta_1 f_1 + \theta_2 f_2 + g. \tag{4.20}$$

Now, suppose that $\gamma = 1$. Let $\varphi \in \mathcal{D}(0,T)$ and $v \in \mathcal{D}(\Omega)$. Choose $(0, v|_{\Omega_{2\varepsilon}}\varphi)$ in the variational formulation (2.14) and extend to the whole of Ω so that

$$\begin{aligned} -\int_0^T\int_\Omega \widetilde{u_{2\varepsilon}} v\varphi' \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \nabla v \varphi \,\mathrm{d}x\,\mathrm{d}t - \\ -\varepsilon\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon} - u_{2\varepsilon}) v\varphi \,\mathrm{d}\sigma_x\,\mathrm{d}t = \int_0^T\int_\Omega \widetilde{f_{2\varepsilon}} v\varphi \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.21}$$

To evaluate the limit of the last integral on the left, we use Lemma 2.9 with $g = h^\varepsilon$, $v_\varepsilon = ((P_1^\varepsilon u_{1\varepsilon})|_{\Omega_{2\varepsilon}} - u_{2\varepsilon}) v|_{\Omega_{2\varepsilon}}\varphi$ and $c_h = \frac{1}{|Y_2|}\int_\Gamma h(y)\,\mathrm{d}\sigma_y$. Since

$$\widetilde{v_\varepsilon} = (\chi_{\Omega_{2\varepsilon}} P_1^\varepsilon u_{1\varepsilon} - \widetilde{u_{2\varepsilon}}) v\varphi \to (\theta_2 u_1 - u_2)v\varphi \quad \text{in } L^2(0,T;\, L^2(\Omega)),$$

we have

$$\lim_{\varepsilon\to 0} \varepsilon\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}) v\varphi \,\mathrm{d}\sigma_x\,\mathrm{d}t = c_h \int_0^T\int_\Omega (\theta_2 u_1 - u_2) v\varphi \,\mathrm{d}x\,\mathrm{d}t. \tag{4.22}$$

Letting $\varepsilon \to 0$ in (4.21) and using Corollary 3.6, (4.17), (4.22) and (3.3) will yield

$$-\int_0^T\int_\Omega u_2 v\varphi' \,\mathrm{d}x\,\mathrm{d}t - c_h\int_0^T\int_\Omega (\theta_2 u_1 - u_2) v\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \theta_2 f_2 v\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

By the arbitrariness of φ and v, we get

$$u_2' - c_h(\theta_2 u_1 - u_2) = \theta_2 f_2. \tag{4.23}$$

This, replaced in (4.12), and (4.13) yield

$$\theta_1 u_1' - \operatorname{div}(A^0_\gamma \nabla u_1) + c_h(\theta_2 u_1 - u_2) = \theta_1 f_1 + g. \tag{4.24}$$

Step 5. Verify if u_1 and u_2 satisfy the initial conditions given in (4.4) and (4.5), respectively.

First of all, let us check that $u_1(0)$ and $u_2(0)$ make sense in $L^2(\Omega)$. If $\gamma < 1$ then by (4.20) one has

$$u_1' = \operatorname{div}(A^0_\gamma \nabla u_1) + \theta_1 f_1 + \theta_2 f_2 + g \in L^2(0,T;\, H^{-1}(\Omega)).$$

Hence, by Proposition 3.2 written for $V = H^1_0(\Omega)$ and $H = L^2(\Omega)$, one has $u_1 \in C^0([0,T]; L^2(\Omega))$. So, $u_1(0)$ and $u_2(0) = \theta_2 u_1(0)$ are defined in $L^2(\Omega)$.

If $\gamma = 1$ then it follows from (4.23) that $u_2' = c_h(\theta_2 u_1 - u_2) + \theta_2 f_2$ is in $L^2(0,T;\, L^2(\Omega))$, since u_1 and u_2 are both in $L^2(0,T;\, L^2(\Omega))$. Classical embedding results imply that $u_2 \in C^0([0,T];L^2(\Omega))$ and $u_2(0)$ makes sense in $L^2(\Omega)$. On the other hand, from (4.24) we get

$$u_1' = \theta_1^{-1}(\operatorname{div}(A_\gamma^0 \nabla u_1) - c_h(\theta_2 u_1 - u_2) + \theta_1 f_1 + g) \in L^2(0,T;\, H^{-1}(\Omega)).$$

Again, it follows by Proposition 3.2 that $u_1 \in C^0([0,T];L^2(\Omega))$ and $u_1(0)$ is defined in $L^2(\Omega)$.

Suppose $v \in \mathcal{D}(\Omega)$ and $\varphi \in C^\infty([0,T])$ with $\varphi(T) = 0$ and $\varphi(0) = 1$. In the variational formulation (2.14), set $(v|_{\Omega_{1\varepsilon}}\varphi, v|_{\Omega_{2\varepsilon}}\varphi)$ as test function, so that

$$\int_0^T \langle u_{1\varepsilon}', v\rangle_{(V^\varepsilon)',V^\varepsilon}\varphi \,\mathrm{d}t + \int_0^T \langle u_{2\varepsilon}', v\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})}\varphi \,\mathrm{d}t + \tag{4.25}$$
$$+ \int_0^T\int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon}\nabla v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon}\nabla v\varphi \,\mathrm{d}x\,\mathrm{d}t =$$
$$= \int_0^T\int_{\Omega_{1\varepsilon}} f_{1\varepsilon} v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, P_1^\varepsilon(v|_{\Omega_{1\varepsilon}})\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi \,\mathrm{d}t + \int_0^T\int_{\Omega_{2\varepsilon}} f_{2\varepsilon} v\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

However, integrating by parts, since $\varphi(0) = 1$, we get

$$\begin{cases} \displaystyle\int_0^T \langle u_{1\varepsilon}', v\rangle_{(V^\varepsilon)',V^\varepsilon}\varphi \,\mathrm{d}t = -\int_{\Omega_{1\varepsilon}} U_{1\varepsilon}^0 v \,\mathrm{d}x - \int_0^T\int_{\Omega_{1\varepsilon}} u_{1\varepsilon} v\varphi' \,\mathrm{d}x\,\mathrm{d}t, \\ \displaystyle\int_0^T \langle u_{2\varepsilon}', v\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})}\varphi \,\mathrm{d}t = -\int_{\Omega_{2\varepsilon}} U_{2\varepsilon}^0 v \,\mathrm{d}x - \int_0^T\int_{\Omega_{2\varepsilon}} u_{2\varepsilon} v\varphi' \,\mathrm{d}x\,\mathrm{d}t. \end{cases} \tag{4.26}$$

Substituting (4.26) in (4.25) and extending to the whole of Ω, we can rewrite (4.25) as

$$-\int_\Omega \widetilde{U_{1\varepsilon}^0} v \,\mathrm{d}x - \int_\Omega \widetilde{U_{2\varepsilon}^0} v \,\mathrm{d}x - \int_0^T\int_\Omega \widetilde{u_{1\varepsilon}} v\varphi' \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \widetilde{u_{2\varepsilon}} v\varphi' \,\mathrm{d}x\,\mathrm{d}t +$$
$$+ \int_0^T\int_\Omega A^\varepsilon \widetilde{\nabla u_{1\varepsilon}}\nabla v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega A^\varepsilon \widetilde{\nabla u_{2\varepsilon}}\nabla v\varphi \,\mathrm{d}x\,\mathrm{d}t =$$
$$= \int_0^T\int_\Omega \widetilde{f_{1\varepsilon}} v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, P_1^\varepsilon(v|_{\Omega_{1\varepsilon}})\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi \,\mathrm{d}t + \int_0^T\int_\Omega \widetilde{f_{2\varepsilon}} v\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Letting $\varepsilon \to 0$ and using (2.2), (3.3), (4.13), Corollary 3.6 and Corollary 2.8 (with $v_\varepsilon = v$) we obtain

$$-\int_\Omega (\theta_1 U_1^0 + \theta_2 U_2^0) v \,\mathrm{d}x - \int_0^T\int_\Omega (\theta_1 u_1 + u_2) v\varphi' \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega A_\gamma^0 \nabla u_1 \nabla v\varphi \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_\Omega (\theta_1 f_1 + \theta_2 f_2) v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi \,\mathrm{d}t. \tag{4.27}$$

In the same manner, using (4.11) and (4.13) with $\varphi(0) = 1$, we see that

$$-\int_\Omega (\theta_1 u_1(0)+u_2(0))v\,\mathrm{d}x - \int_0^T\int_\Omega (\theta_1 u_1+u_2)v\varphi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega A_\gamma^0 \nabla u_1 \nabla v \varphi\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T\int_\Omega (\theta_1 f_1 + \theta_2 f_2)v\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi\,\mathrm{d}t. \tag{4.28}$$

Since v and φ are arbitrary, from (4.27) and (4.28) we conclude that

$$\theta_1 u_1(0) + u_2(0) = \theta_1 U_1^0 + \theta_2 U_2^0. \tag{4.29}$$

We now check the conditions separately for the cases $\gamma < 1$ and $\gamma = 1$.

Case $\gamma < 1$. Since $u_2 = \theta_2 u_1$ by (4.19), from (4.29) we have

$$u_1(0) = \theta_1 u_1(0) + \theta_2 u_1(0) = \theta_1 U_1^0 + \theta_2 U_2^0.$$

Case $\gamma = 1$. Let $v \in \mathcal{D}(\Omega)$ and $\varphi \in C^\infty([0,T])$ with $\varphi(T) = 0$ and $\varphi(0) = 1$. In the variational formulation (2.14), consider $(v|_{\Omega_{1\varepsilon}}\varphi, 0)$ as test function. Using arguments similar to those already used yields

$$-\int_\Omega \widetilde{U_{1\varepsilon}^0} v\,\mathrm{d}x - \int_0^T\int_\Omega \widetilde{u_{1\varepsilon}} v\varphi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \nabla v\varphi\,\mathrm{d}x\,\mathrm{d}t +$$

$$+\,\varepsilon\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon} - u_{2\varepsilon})v\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t =$$

$$= \int_0^T\int_\Omega \widetilde{f_{1\varepsilon}} v\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, P_1^\varepsilon(v|_{\Omega_{1\varepsilon}})\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi\,\mathrm{d}t.$$

Letting $\varepsilon \to 0$ by using (3.3), (4.10), (4.16), (4.22), Corollary 3.6 and Corollary 2.8 (with $v_\varepsilon = v$), we get

$$-\int_\Omega \theta_1 U_1^0 v\,\mathrm{d}x - \int_0^T\int_\Omega \theta_1 u_1 v\varphi'\,\mathrm{d}x\,\mathrm{d}t + \tag{4.30}$$

$$+\int_0^T\int_\Omega A_\gamma^0 \nabla u_1 \nabla v\varphi\,\mathrm{d}x\,\mathrm{d}t + c_h\int_0^T\int_\Omega (\theta_2 u_1 - u_2)v\varphi\,\mathrm{d}x\,\mathrm{d}t =$$

$$= \int_0^T\int_\Omega \theta_1 f_1 v\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi\,\mathrm{d}t.$$

Subtracting (4.28) from (4.30), we obtain

$$-\int_\Omega \theta_1 U_1^0 v\,\mathrm{d}x + c_h\int_0^T\int_\Omega (\theta_2 u_1 - u_2)v\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \theta_1 u_1(0)v\,\mathrm{d}x + \tag{4.31}$$

$$+\int_\Omega u_2(0)v\,\mathrm{d}x + \int_0^T\int_\Omega u_2 v\varphi'\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega \theta_2 f_2 v\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

By (4.29) and (4.23) we have, respectively,

$$
(4.32)\qquad \begin{cases} \int_\Omega \theta_1 u_1(0) v \,\mathrm{d}x + \int_\Omega u_2(0) v \,\mathrm{d}x = \int_\Omega (\theta_1 U_1^0 + \theta_2 U_2^0) v \,\mathrm{d}x, \\ c_h \int_0^T \int_\Omega (\theta_2 u_1 - u_2) v\varphi \,\mathrm{d}x \,\mathrm{d}t = \\ \quad = \int_0^T \int_\Omega u_2' v\varphi \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Omega \theta_2 f_2 v\varphi \,\mathrm{d}x \,\mathrm{d}t \\ \quad = -\int_0^T \int_\Omega u_2 v\varphi' \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Omega \theta_2 f_2 v\varphi \,\mathrm{d}x \,\mathrm{d}t. \end{cases}
$$

Since $\varphi(T) = 0$ and $\varphi(0) = 1$, replacing (4.32) in (4.31), we obtain

$$
\int_\Omega \theta_2 U_2^0 v \,\mathrm{d}x - \int_\Omega u_2(0) v \,\mathrm{d}x = 0.
$$

Therefore, $u_2(0) = \theta_2 U_2^0$. This together with (4.29) yield $u_1(0) = U_1^0$.

Step 6. Conclusion.

For both cases, Corollary 3.6 ensures (4.1) up to a subsequence. On the other hand, (4.13) shows (4.2) and, moreover, for the case $-1 < \gamma \leq 1$, (4.16) and (4.17) assert (4.3) for a subsequence.

Now, for the case $\gamma < 1$, Steps 4 and 5 imply that the limit function u_1 in (4.1) satisfies (4.4). The latter has a unique solution since the homogenized matrix A^0_γ is positive definite. Hence, all the convergences involved hold for the whole sequences.

Similarly, for the case $\gamma = 1$, Steps 4 and 5 show that the limit functions u_1 and u_2 satisfy (4.5). To complete the proof, we have to show that the solution (u_1, u_2) to (4.5) is unique.

To do that, observe that in the equation $u_2' - c_h(\theta_2 u_1 - u_2) = \theta_2 f_2$, u_2 can be computed in terms of u_1. Indeed,

$$
u_2(x,t) = \theta_2 U_2^0 \mathrm{e}^{-c_h t} + \int_0^t \mathcal{K}(t,s)(c_h \theta_2 u_1(x,s) + \theta_2 f_2(x,s)) \,\mathrm{d}s
$$

where $\mathcal{K}(t,s) = \mathrm{e}^{c_h(s-t)}$ and $c_h = \frac{1}{|Y_2|}\int_\gamma h(y)\,\mathrm{d}\sigma_y$. Substituting this in the first equation of (4.5), we get

$$
\begin{aligned} \theta_1 u_1' - \operatorname{div}(A^0_\gamma \nabla u_1) + c_h \theta_2 u_1 - c_h \theta_2 U_2^0 \mathrm{e}^{-c_h t} - \\ - c_h \int_0^t \mathcal{K}(t,s)(c_h \theta_2 u_1(x,s) + \theta_2 f_2(x,s)) \,\mathrm{d}s = \theta_1 f_1(x,t) + g. \end{aligned}
$$

Rewriting the expression yields

$$
\theta_1 u_1' - \operatorname{div}(A^0_\gamma \nabla u_1) + c_h \theta_2 u_1 - c_h^2 \theta_2 \int_0^t \mathcal{K}(t,s) u_1(x,s) \mathrm{d}s = F(x,t),
$$

where

$$F(x,t) = \theta_1 f_1(x,t) + g + c_h\theta_2 U_2^0 \mathrm{e}^{-c_h t} + c_h \int_0^t \mathcal{K}(t,s)\theta_2 f_2(x,s)\mathrm{d}s.$$

So (4.5) can be expressed as

$$(4.33) \quad \begin{cases} \theta_1 u_1' - \operatorname{div}(A_\gamma^0 \nabla u_1) + c_h\theta_2 u_1 - \\ \qquad -c_h^2\theta_2 \displaystyle\int_0^t \mathcal{K}(t,s)u_1(x,s)\mathrm{d}s = F(x,t) & \text{in } \Omega\times]0,T[, \\ u_2' - c_h(\theta_2 u_1 - u_2) = \theta_2 f_2 & \text{in } \Omega\times]0,T[, \\ u_1 = 0 & \text{on } \partial\Omega\times]0,T[, \\ u_1(0) = U_1^0,\ u_2(0) = \theta_2 U_2^0 & \text{in } \Omega. \end{cases}$$

We now use Theorem 3.1 to show that (4.33) has a unique solution. Accordingly, let

$$V = H_0^1(\Omega), \quad H = L^2(\Omega), \quad b = F, \quad u_0 = U_1^0 \in H_0^1(\Omega)$$

and define

$$a(u,v,t) = \int_\Omega A_\gamma^0 \nabla u \nabla v \,\mathrm{d}x + \int_\Omega c_h\theta_2 uv\,\mathrm{d}x - c_h^2\theta_2 \int_\Omega\int_0^t \mathcal{K}(t,s)uv\,\mathrm{d}s\,\mathrm{d}x$$

for every $u, v \in H_0^1(\Omega)$.

Since $g(u) = c_h^2\theta_2 \int_0^t \mathcal{K}(t,s)u(x,s)\mathrm{d}s$ is linear and continuous from $L^2(0,T;\,L^2(\Omega))$ into itself, by Theorem 3.1, there exists a unique solution to problem (4.33). This completes the proof. □

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