QUASICONFORMAL MAPPINGS
IN AHLFORS REGULAR SPACES

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The problem on continuous or homeomorphic extension to boundary of the quasi-conformal mappings in Euclidean \( n \)-space of dimension at least two is well-known. It is natural to ask whether the boundary extension theorems are valid in more abstract setting, i.e., the Ahlfors \( Q \)-regular metric measure spaces \((Q \geq 1)\). Those the three (metric, geometric and analytic) definitions of quasiconformality in Euclidean spaces can be formulated in this setting but they are not equivalent. Our interest in this talk is in the notion of geometric quasiconformality. We deal with the extension to boundary for geometrically quasiconformal mappings on domains in Ahlfors \( Q \)-regular spaces. For this, we define a number of concepts allowing us to describe the properties of a domain at a boundary point. We give several boundary extension theorems for geometrically quasiconformal mappings.

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We first recall some definitions and make some remarks of a general character which will be used in this paper.

Let \( X \) be a topological space. The space \( X \) is said to be path-connected (or pathwise connected or 0-connected) if for any two points \( x \) and \( y \) in \( X \) there exists a continuous function \( f \) from the unit interval \([0,1]\) to \( X \) with \( f(0) = x \) and \( f(1) = y \). (This function is called a path from \( x \) to \( y \).) The space \( X \) is said to be arc-connected (or arcwise connected) if any two distinct points can be joined by an arc, that is a path \( \gamma \) which is a homeomorphism between the unit interval \([0,1]\) and its image \( \gamma([0,1]) \). It can be shown that any Hausdorff space which is path-connected is also arc-connected. The space \( X \) is said to be rectifiable connected if any two points can be joined by a rectifiable path. A domain \( D \) in \( X \) is an open connected non-empty set in \( X \). An arc domain \( D \) in \( X \) is an open arc-connected set in \( X \). By a continuum we mean a compact connected set which contains at least two points. A neighbourhood of a set \( E \) is an open set containing \( E \).
The purpose of this paper is to give some results concerning the local and global boundary behaviour of quasiconformal mappings on domains in Ahlfors regular metric measure spaces. These will be done in terms of the properties possessed by the domains in question.

We therefore recall and introduce the concepts associated with the boundary of a domain in a metric space. Also, we give some relationships between these concepts. The standard references for Euclidean case are in [1], [2], [5], [7] and for metric case in [4]. Throughout the paper we shall consider only metric spaces. Further, all metric spaces are assumed to be rectifiable connected and all measures are assumed to be locally finite and Borel regular with dense support.

We shall denote by \( X \) or \( Y \) a metric measure space that satisfies the standard assumptions stated above.

Let us consider a metric measure space \((X, d, \mu)\), a domain \(D\) in \(X\) and a boundary point \(b\) of \(D\).

**Definition 1.** \(D\) is locally (arc) connected at \(b\) if \(b\) has arbitrarily small neighbourhoods \(U\) such that \(U \cap D\) is (arc) connected. This means that for each neighbourhood \(V\) of \(b\) there exists a neighbourhood \(U\) of \(b\) \(U \subset V\) such that \(U \cap D\) is (arc) connected.

**Definition 2.** \(D\) is finitely (arc) connected at \(b\) if \(b\) has arbitrarily small neighbourhoods \(U\) such that \(U \cap D\) has a finite number of (arc) components.

**Definition 3.** \(D\) is \(m\)-(arc) connected at \(b\), \(m = 1, 2, \ldots\), if \(b\) has arbitrarily small neighbourhoods \(U\) such that \(U \cap D\) consists of \(m\) (arc) components.

Obviously, a domain is locally (arc) connected at a boundary point if and only if it is 1-(arc) connected at the point.

Note that \(m\)-(arc) connectedness always implies finite (arc) connectedness. The converse is not true.

In the sequel, we consider a family \(\Gamma\) of (nonconstant) paths in \(X\). A Borel function \(\rho : X \rightarrow [0, \infty]\) is admissible for \(\Gamma\) if \(\int_{\gamma} \rho ds \geq 1\) for all locally rectifiable paths \(\gamma\) in \(\Gamma\). The set of all admissible functions for \(\Gamma\) is denoted by \(F(\Gamma)\).

**Definition 4.** The (conformal) \(p\)-modulus of \(\Gamma\), \(1 \leq p < \infty\), is defined as:

\[
M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho^p d\mu.
\]

Note, in particular, that the modulus of the collection of all non-locally rectifiable paths is zero.

**Definition 5.** A metric measure space \((X, d, \mu)\) is said to be (Ahlfors) \(Q\)-regular, \(Q > 0\), if there exists a constant \(c < \infty\) such that
\[ \frac{r^Q}{c} \leq \mu(B_r) \leq cr^Q \]

for every ball \( B_r \) in \( X \) with radius \( r < \text{diam } X \).

Note that if the above relation holds, then \( X \) has Hausdorff dimension \( Q \).

If \( E, F, D \) are subsets of \( X \) with \( E \subset \overline{D}, F \subset \overline{D} \), we denote by \( \Delta(E, F; D) \) the family of all paths which join \( E \) and \( F \) in \( D \).

Next, we only consider \( Q \)-regular metric measure spaces with \( 1 \leq Q < \infty \) and a domain \( D \) in \( X \). If \( \Gamma \) is a family of paths in \( X \), we write \( M(\Gamma) = M_Q(\Gamma) \).

We now introduce the

\textbf{Definition 6.} \( D \) is (arc) very weakly flat at a point \( b \in \partial D \) if for every number \( \delta > 0 \) there exist arbitrarily small neighbourhoods \( U \) of \( b \) such that \( M(\Delta(E, F; D)) \geq \delta \) for all (arc) connected sets \( E, F \subset D \) with \( E \cap \partial U \neq \emptyset, F \cap \partial U \neq \emptyset \).

More precisely, by the phrase “there exist arbitrarily small neighbourhoods \( U \) of \( b \)” in the above definition we understand that for every neighbourhood \( U \) of \( b \) there is a neighbourhood \( V \) of \( b \), \( V \subset U \), such that \( M(\Delta(E, F; D)) \geq \delta \) for all (arc) connected sets \( E, F \subset D \) with \( E \cap \partial V \neq \emptyset, F \cap \partial V \neq \emptyset \) and \( E \cap \partial U \neq \emptyset, F \cap \partial U \neq \emptyset \).

This definition is different from the definition of a domain that is weakly flat at a boundary point in [4, 13.3]. Moreover, if a domain is very weakly flat at a boundary point, then it is weakly flat at the point.

The property \( P_1 \) (see [7], Definition 17.5.(3)), the property \( P_2^* \) (see [1], Definition 5) and quasiconformal accessibility (see [5], 1.7(vii)) in the Euclidean space can be formulated in the same way in our framework.

\textbf{Definition 7.} We say that \( D \) has (arc) property \( P_1 \) (or is (arc) quasiconformally flat) at \( b \in \partial D \) if \( M(\Delta(E, F; D)) = \infty \) for all (arc) connected sets \( E, F \subset D \) with \( b \in \overline{E} \cap \overline{F} \).

\textbf{Definition 8.} We say that \( D \) has (arc) property \( P_2^* \) at \( b \in \partial D \) if for every point \( b_1 \in \partial D_1, b_1 \neq b \), there exist a continuum \( F \subset D \) and a number \( \delta > 0 \) such that \( M(\Delta(E, F; D)) \geq \delta \) for all (arc) connected sets \( E \subset D \) with \( b, b_1 \in \overline{E} \).

\textbf{Definition 9.} We say that \( D \) is (arc) quasiconformally accessible at \( b \in \partial D \) if for every neighbourhood \( U \) of \( b \) there exist a continuum \( F \subset D \) and a positive number \( \delta > 0 \) such that \( M(\Delta(E, F; D)) \geq \delta \) for all (arc) connected sets \( E \subset D \) with \( b \in \overline{E} \) and \( E \cap \partial U \neq \emptyset \).

Now, we introduce

\textbf{Definition 10.} We say that \( D \) is (arc) strictly quasiconformally accessible at \( b \in \partial D \) if there exist arbitrarily small neighbourhoods \( U \) of \( b \), a continuum
$F$ in $D$, and a real number $\delta > 0$ such that $M(\Delta(E, F; D)) \geq \delta$ for every $E \subset D$ (arc) connected, with $E \cap \partial U \neq \emptyset$, i.e., for every neighbourhood $U$ of $b$ there exist a neighbourhood $V$ of $b$ $V \subset U$, a continuum $F$ in $D$ and a real number $\delta > 0$ such that $M(\Delta(E, F; D)) \geq \delta$ for every $E \subset D$ (arc) connected with $E \cap \partial V \neq \emptyset$ and $E \cap \partial U \neq \emptyset$.

We mention that the definition of strictly quasiconformal accessibility is different from the definition of strict accessibility [4, 13.3]. Moreover, if a domain is strictly quasiconformal accessible at a boundary point, then it is strictly accessible at the point.

Note that (arc) strictly quasiconformal accessibility implies (arc) quasiconformal accessibility. This claim follows by Definitions 9, 10 and result below.

**Lemma 1.** Let $E$ be a connected subset of a topological space $X$. If $A \subset X$ and neither $E \cap A$ nor $E \cap (X \setminus A)$ is empty, then $E \cap \partial A \neq \emptyset$ [8, 11.26].

**Proof.** If $E \cap \partial A = \emptyset$ then $\emptyset = E \cap (\overline{A} \cap (X \setminus \overline{A})) = E \cap \overline{A} \cap (X \setminus \overline{A})$ and $E = E \cap \overline{X} = E \cap (A \cup (X \setminus \overline{A})) = E \cap (\overline{A} \cup (X \setminus \overline{A})) = (E \cap \overline{A}) \cup (E \cap (X \setminus \overline{A}))$ which contradict the connectedness of $E$. □

Next, we give two propositions which establish some relationships between the notions introduced above.

**Proposition 1.** Suppose that $D$ is an arc domain. If $D$ is (arc) very weakly flat at $b \in \partial D$, then $D$ is (arc) strictly quasiconformal accessible at $b$.

**Proof.** Let $U$ be a neighbourhood of $b$ and a number $\delta > 0$. We consider a ball $B(b, r) \subset U$, $0 < r < \sup d(b, x)$ and $0 < r_0 < r$. By the (arc) very weakly flatness of $D$, there exists a neighbourhood $V$ of $b$, $V \subset B(b, r_0)$ such that $M(\Delta(E, F; D)) \geq \delta$ for every (arc) connected sets $E, F \subset D$, with $E \cap \partial V \neq \emptyset$, $F \cap \partial V \neq \emptyset$, $E \cap \partial B(b, r_0) \neq \emptyset$, $F \cap \partial B(b, r_0) \neq \emptyset$. Since $D$ is an arc domain and $V$ an open set, there exist $x_1 \in D \cap \partial B(b, r_0)$ and $x_2 \in \partial V \cap D$. We choose as a continuum $F$ an arbitrary curve connecting the points $x_1$ and $x_2$ in $D$. Hence $M(\Delta(E, F; D)) \geq \delta$ for every (arc) connected set $E \subset D$, with $E \cap \partial V \neq \emptyset$ and $E \cap \partial B(b, r_0) \neq \emptyset$. On the other hand, every (arc) connected set $E$ in $D$ that intersects $\partial U$ and $\partial V$ will also intersect $\partial B(b, r_0)$. Consequently, $M(\Delta(E, F; D)) \geq \delta$ for every (arc) connected set $E \subset D$ with $E \cap \partial V \neq \emptyset$, $E \cap \partial U \neq \emptyset$. Thus, we get the desired conclusion. □

**Proposition 2.** Suppose that $D$ is an arc domain. If $D$ is very weakly flat at $b \in \partial D$, then $D$ is locally arc connected at $b$.

**Proof.** Since $D$ is very weakly flat at $b$, it is weakly flat at $b$. By Lemma 13.18 in [4], $D$ is locally arc connected at $b$ and the proof is complete. □
Since locally arc connectedness implies local connectedness, we deduce that if $D$ is very weakly flat at $b \in \partial D$, then $D$ is locally connected at $b$.

The following abbreviation will be used in this paper: if a domain has one of the properties in Definitions 1 through 3, 6 through 10 at each boundary point, then it is said to have the property in question on the boundary.

We consider two $Q$-regular metric measure spaces $(X, d, \mu)$ and $(Y, d', \mu')$ with $1 \leq Q < \infty$, and two domains $D \subset X$, $D' \subset Y$. Similarly to the geometric definition in $\mathbb{R}^n$, $n \geq 2$, by Väisälä [7, 13.1] we have:

**Definition 11.** A homeomorphism $f : D \to D'$ is called $K$-(geometrically) quasiconformal, $K \in [1, \infty]$, if

$$
\frac{M(\Gamma)}{K} \leq M(f(\Gamma)) \leq KM(\Gamma)
$$

for every family $\Gamma$ of paths in $D$. We also say that a homeomorphism $f : D \to D'$ is (geometrically) quasiconformal if $f$ is $K$-(geometrically) quasiconformal for some $K \in [1, \infty)$, i.e., if the distortion of moduli of path families under the mapping $f$ is bounded.

To simplify notation, we write quasiconformal mapping instead of geometrically quasiconformal mapping.

In the sequel the notation $f : D \to D'$ includes the assumption that $D \subset X$, $D' \subset Y$ and $D, D'$ are arc domains.

**Theorem 1.** Suppose that $f : D \to D'$ is a quasiconformal mapping, $\overline{D'}$ is compact and $D$ is (arc) very weakly flat at a point $b \in \partial D$. If $D'$ is finitely (arc) connected on the boundary, then there exists the limit of $f$ at $b$.

**Proof.** First, we prove that the cluster set $C(f, b)$ of $f$ at $b$ contains at most one point at which $D'$ is finitely connected. Note that $C(f, b) \neq \emptyset$ by the compactness of $\overline{D'}$, and that $C(f, b)$ belongs to the boundary of the set $D'$ since $f$ is a homeomorphism between two open sets. Suppose that $D'$ is finitely (arc) connected at two distinct points $b'_1, b'_2 \in C(f, b)$. We choose balls $B_1 = B(b'_1, r_1)$, $B_2 = B(b'_2, r_2)$ such that $\overline{B_1} \cap \overline{B_2} = \emptyset$.

By the definition of a cluster set, there exist two sequences $(x_j), (y_j)$, such that $x_j, y_j \in D$, $x_j \to b$, $y_j \to b$ and $f(x_j) \to b'_1$, $f(y_j) \to b'_2$. Since $D'$ is finitely arc connected at $b'_1$, there exists a neighbourhood $V_1$ of $b'_1$, $V_1 \subset B_1$, such that $V_1 \cap D'$ has a finite number of (arc) components. Therefore, there is a subsequence of $(f(x_j))$ which is contained in a single (arc) component of $V_1 \cap D'$, hence in a single (arc) component of $B_1 \cap D'$. Let $E_1$ be an (arc) component of $B_1 \cap D'$ which contains a subsequence of $(f(x_j))$. Similarly, $B_2 \cap D'$ has an (arc) component $E_2$ which contains a subsequence of $(f(y_j))$. The subsets $f^{-1}(E_1), f^{-1}(E_2)$ of $D$ are (arc) connected and $b \in f^{-1}(E_1) \cap f^{-1}(E_2)$. Set $\Gamma = \Delta(f^{-1}(E_1), f^{-1}(E_2) ; D)$ and $\Gamma' = \Delta(E_1, E_2; D')$. Let us
consider \( r_0 \in (0,d_0) \), \( d_0 = \sup_{x \in D} d(b,x) \) and \( B = B(b,r), 0 < r < r_0 \), such that 
\[
f^{-1}(E_1) \cap \partial B \neq \emptyset, f^{-1}(E_2) \cap \partial B \neq \emptyset.
\]
We can choose \( r \) such: take \( a_1 \in f^{-1}(E_1) \), \( a_2 \in f^{-1}(E_2) \). Set \( d_1 = d(b,a_1), d_2 = d(b,a_2) \). We take \( 0 < r < \min\{r_0, d_1, d_2\} \).

By the choice of \( r \) we have \( a_1, a_2 \notin B(b,r) \), \( b \in B(b,r) \) and it follows from Lemma 1 that 
\[
f^{-1}(E_1) \cap \partial B \neq \emptyset, f^{-1}(E_2) \cap \partial B \neq \emptyset.
\]

By the (arc) property of \( \Gamma \), for every \( \delta > 0 \) and \( U = B(b,r) \) there exists \( V = B(b,r_*) \) with \( 0 < r_* < r \) such that \( M(\Gamma) \geq \delta \) since 
\[
f^{-1}(E_1) \cap \partial D(b,r_*) \neq \emptyset, \quad f^{-1}(E_2) \cap \partial D(b,r_*) \neq \emptyset \quad \text{and} \quad f^{-1}(E_1) \cap \partial D \neq \emptyset, \quad f^{-1}(E_2) \cap \partial D \neq \emptyset.
\]

The quasiconformality of \( f \) implies that there exists a constant \( K < \infty \) such that
\[
M(\Gamma') \geq \frac{M(\Gamma)}{K} \geq \frac{\delta}{K}
\]
for every \( \delta > 0 \). Take \( h = d'(b_1', b_2') \). Since \( \overline{D'} \) is compact, we have \( d < \infty \) and, by Lemma 2.7 in [4],
\[
M(\Gamma') \leq \frac{\mu'(D')}{h_1^Q},
\]
where \( h_1 = h - r_1 - r_2 \).

Since \( \mu' \) is locally finite and \( \overline{D'} \) is compact, there exists a constant \( C > 0 \) such that \( \mu'(D') \leq C \) and by (2) we get
\[
M(\Gamma') \leq \frac{C}{h_1^Q}.
\]

Choosing \( \delta > \frac{KC}{h_1^Q} \), by (1) and (3) we reach a contradiction. Consequently, \( C(f,b) \) contains at most one point \( b' \) at which \( D' \) is finitely (arc) connected. On the other hand, \( D' \) is finitely (arc) connected on the boundary, hence 
\[
\lim_{x \to b} f(x) = b'.
\]
Thus, the proof is complete. \( \square \)

Using an argument similar to that employed in the proof of Theorem 1 one can show that Theorem 17.15 in [7] in the Euclidean \( n \)-space also holds in our framework.

**Theorem 2.** Suppose that \( f : D \to D' \) is a quasiconformal mapping, \( \overline{D'} \) is compact and \( D \) has (arc) property \( P_1 \) at a point \( b \in \partial D \). If \( D' \) is finitely (arc) connected on the boundary, then there exists the limit of \( f \) at \( b \).

**Proof.** Consider sets \( E_1, E_2, f^{-1}(E_1), f^{-1}(E_2), \Gamma \) and \( \Gamma' \) as in the proof of Theorem 1. Since \( b \in f^{-1}(E_1) \cap f^{-1}(E_2) \) and the sets \( f^{-1}(E_1), f^{-1}(E_2) \) are (arc) connected, by the (arc) property \( P_1 \) of \( D \) at \( b \) we have \( M(\Gamma) = \infty \). Quasiconformality of \( f \) implies that \( M(\Gamma') = \infty \). Taking \( h_1 \) as in the proof of
Theorem 1, we have \( M(\Gamma') \leq \frac{\varepsilon'(\mathcal{D})}{b_0^N} < \infty \), that contradicts the above equation. Consequently, we obtain the desired conclusion. \( \Box \)

Using Theorems 1 and 2 we obtain the important results below.

**Corollary 1.** If \( D \) is (arc) very weakly flat (or has (arc) property \( P_1 \)) on the boundary, \( D' \) is finitely (arc) connected on the boundary and \( \overline{D'} \) is compact, then every quasiconformal mapping \( f : D \to D' \) has a continuous extension \( f^* : \overline{D} \to \overline{D'} \).

**Corollary 2.** If \( D \) is (arc) very weakly flat (or has (arc) property \( P_1 \)) on the boundary, \( D' \) is locally (arc) connected on the boundary and \( \overline{D'} \) is compact, then every quasiconformal mapping \( f : D \to D' \) has a continuous extension \( f^* : \overline{D} \to \overline{D'} \).

**Corollary 3.** Suppose that \( D \) and \( D' \) are (arc) very weakly flat on the boundary and \( \overline{D}, \overline{D'} \) are compact. Then every quasiconformal mapping \( f : D \to D' \) has a homeomorphic extension \( f^* : \overline{D} \to \overline{D'} \).

**Proof.** The result is an immediate corollary of Proposition 2 and Corollary 2. \( \Box \)

In what follows we suppose that \( Q > 1 \).

The following theorem is an analogue of Theorem 17.15 in [7] in the Euclidean case. Remark that instead of property \( P_2 \) one considers (arc) property \( P_2^* \).

**Theorem 3.** Suppose that \( f : D \to D' \) is a quasiconformal mapping and \( \overline{D'} \) is compact. If \( D \) is locally (arc) connected at \( b \in \partial D \) and \( D' \) has (arc) property \( P_2^* \) at least one point of \( C(f, b) \), then there exists the limit of \( f \) at \( b \).

**Proof.** Assume that \( C(f, b) \) contains two distinct points \( b_1', b_2' \) and that \( D' \) has (arc) property \( P_2^* \) at \( b_1' \). By the definition of (arc) property \( P_2^* \), there exist a continuum \( F \subset \overline{D'} \) and a positive number \( \delta > 0 \) such that

\[
M(\Delta(E, F; D')) \geq \delta
\]

whenever \( E \) is (arc) connected in \( D' \) with \( b_1', b_2' \in \overline{E} \). We consider \( r_0 \in (0, d_0) \) with \( d_0 = \sup_{x \in D} d(x, b) \) and a natural number \( n_0 \) such that \( 0 < \frac{1}{n_0} < r_0 \). Since \( D \) is locally (arc) connected at \( b \), there is a neighbourhood \( V_{n_0} \) of \( b \) such that \( V_{n_0} \subset B_{n_0} = B(b, \frac{1}{n_0}) \) and \( V_{n_0} \cap D \) is (arc) connected. But there exists a ball \( B_{n_1} = B(b, \frac{1}{n_1}) \subset V_{n_0} \) with \( n_1 > n_0 \), where \( n_1 \) is a natural integer. Applying the same procedure we choose a sequence \( V_{n_0}, V_{n_1}, \ldots \) of neighbourhoods of \( b \) such that every \( C_{n_i} = V_{n_i} \cap D \) is (arc) connected, open and \( B_{n_{i+1}} \subset V_{n_i} \subset B_{n_i} \) for all \( i \geq 0 \). Set \( \Gamma_i = \Delta(C_{n_i}, f^{-1}(F); D) \) and \( \Gamma'_i = \Delta(f(C_{n_i}), F; D') \).
Moreover, \( \text{dist}(b, C_n) \to 0 \) as \( i \to \infty \). Since \( f \) is a homeomorphism, \( f(C_n) \) is (arc) connected, open and \( f^{-1}(F) \) is a continuum. Moreover, by the inclusion \( C(f, b) \subset \overline{f(V_n \cap D)} = \overline{f(C_n)} \) we have \( b_1', b_2' \in \overline{f(C_n)} \). Using (4) we get

\[
M(\Gamma_i') \geq \delta
\]

for any \( i \). By the quasiconformality of \( f \), there exists a constant \( K > 0 \) such that \( M(\Gamma_i') \leq K \cdot M(\Gamma_i) \) and therefore

\[
M(\Gamma_i) \geq \frac{\delta}{K}
\]

for any \( i \). Since \( f^{-1}(F) \) is a continuum, \( f^{-1}(F) \subset D, D \) is open, and \( b \in \partial D \), we have \( b \notin f^{-1}(F) \) and \( B_n \setminus f^{-1}(F) = \emptyset \) for \( i \) large, hence \( C_n \setminus f^{-1}(F) = \emptyset \). Fix such an \( i \) and denote \( r_i = \frac{1}{n_i} \). Every path connecting \( f^{-1}(F) \) and \( C_{n+j} \) intersects \( \partial B_n \) and \( \partial B_{n+j} \) for \( j \geq 1 \). But \( r_k \to 0 \) as \( k \to \infty \), hence \( 0 < 2r_{i+j} < r_i < \infty \) for \( j \) large.

By Lemma 3.14 in [3], there exists a constant \( C_0 \) only depending on \( Q \) and a constant \( C \) (by the \( Q \)-regularity of \( X \)) such that

\[
M(\Gamma_{i+j}) \leq C_0 \left( \log \frac{r_i}{r_{j+i}} \right)^{1-Q}.
\]

Since \( \left( \log \frac{r_i}{r_{j+i}} \right)^{1-Q} \to 0 \) as \( j \to \infty \), inequality (7) contradicts (6). Consequently, \( C(f, b) \) has at most one point for which \( D' \) has (arc) property \( P_2 \), hence we get the desired conclusion. \( \square \)

In the same manner we can prove the next result.

**Theorem 4.** Suppose that \( f : D \to D' \) is a quasiconformal mapping and \( \overline{D'} \) is compact. If \( D \) is locally (arc) connected at \( b \in \partial D \) and \( D' \) is (arc) strictly quasiconformally accessible at least one point of \( C(f, b) \), then there exists the limit of \( f \) at \( b \).

**Proof.** Suppose that \( C(f, b) \) contains two distinct points \( b_1', b_2' \) and that \( D' \) is (arc) strictly quasiconformally accessible at \( b_1' \). Let \( U \) be a neighbourhood of \( b_1' \) such that \( b_2' \notin U \). By the definition of (arc) strict quasiconformal accessibility of \( D' \) at \( b_1' \) there exist a neighbourhood \( V \) of \( b_1', V \subset U \), a continuum \( F \subset D' \) and a number \( \delta > 0 \) such that

\[
M(\Delta(E, F; D')) \geq \delta
\]

whenever \( E \subset D' \) is (arc) connected, with \( E \cap \partial V \neq \emptyset, E \cap \partial U \neq \emptyset \). As in the proof of Theorem 3, we consider the sets \( \Gamma_i \) and \( \Gamma_i' \). Since \( b_1', b_2' \in f(C_n) \), \( f(C_n) \) is (arc) connected, and \( b_2' \notin U \), by Lemma 1 we have \( f(C_n) \cap \partial V \neq \emptyset \), and \( f(C_n) \cap \partial U \neq \emptyset \). Inequality (8) implies

\[
M(\Gamma_i') \geq \delta
\]
for any \( i \). Finally, we use the arguments from the proof of Theorem 3 and obtain the desired conclusion. \( \square \)

**Corollary 4.** Suppose that \( f : D \to D' \) is a quasiconformal mapping and \( \overline{D} \) is compact. If \( D \) is locally (arc) connected at \( b \in \partial D \) and \( D' \) is (arc) very weakly flat at least one point of \( C(f,b) \), then there exists the limit of \( f \) at \( b \).

**Proof.** It follows from Theorem 4 and Proposition 1. \( \square \)

**Corollary 5.** If \( D \) is locally (arc) connected on the boundary, \( \overline{D'} \) is compact and \( D' \) is (arc) strictly quasiconformally accessible \([\text{or has (arc) property } P^*_2]\) on the boundary, then every quasiconformal mapping \( f : D \to D' \) has a continuous extension \( f^* : \overline{D} \to \overline{D'} \).

**Proof.** It follows from Theorems 3 and 4. \( \square \)

**Corollary 6.** If \( D \) is (arc) very weakly flat on the boundary, \( \overline{D'} \) is compact and \( D' \) is (arc) strictly quasiconformally accessible \([\text{or has (arc) property } P^*_2]\) on the boundary, then every quasiconformal mapping \( f : D \to D' \) has a continuous extension \( f^* : \overline{D} \to \overline{D'} \).

**Proof.** It is an immediate corollary of Proposition 2 and Corollary 5. \( \square \)

**Corollary 7.** Suppose that \( f : D \to D' \) is a quasiconformal mapping and \( \overline{D}, \overline{D'} \) are compact sets. If \( D, D' \) are locally (arc) connected on the boundary and (arc) strictly quasiconformally accessible \([\text{or has (arc) property } P^*_2]\) on the boundary, then \( f \) can be extended to a homeomorphism \( f^* : \overline{D} \to \overline{D'} \).

**Proof.** It follows from Theorem 4 and Corollary 4. \( \square \)

In order to prove the next theorem we use the result below that for the Euclidean case is proved in Theorem 1.10 in [5].

**Proposition 3.** Suppose that \( D \) is \( m(\text{arc}) \) connected at \( b \in \partial D \) and \((b_{1,k}), \ldots, (b_{m+1,k})\) are \( m+1 \) sequences of points in \( D \) with limit \( b \). If \( U \) is a neighbourhood of \( b \), then there exists a component of \( U \cap D \) which contains subsequences of two different sequences.

**Proof.** Let \((b_{1,k}), \ldots, (b_{m+1,k})\) be \( m+1 \) sequences of points of \( D \) with limit \( b \), and let \( U \) be a neighbourhood of \( b \).

By \( m \)-connectedness of \( D \) at \( b \), there exists a neighbourhood \( V \) of \( b \), \( V \subset U \), and \( V \cap D = E_1 \cup \ldots \cup E_m \), where \( E_i \) is a component of \( V \cap D \) for any \( i \in \{1, \ldots, m\} \). Since \( b_{j,k} \to b \) as \( k \to \infty \), for any \( j \in \{1, \ldots, m+1\} \) there exists a natural number \( k_0 \) such that \( b_{j,k} \in V \cap D \) whenever \( k \geq k_0 \). Therefore, there exists \( E_{j1} \) which contains a subsequence \((b'_{1,k})\) of \((b_{1,k})\). Using the same reasoning for the sequences \((b_{2,k}), \ldots, (b_{m+1,k})\) we find \( E_{j2}, \ldots, E_{jm+1} \), each of
them containing one subsequence \((b_{2,k}', \ldots, b_{m+1,k}')\). Since \(\{j_1, \ldots, j_{m+1}\} = \{1, \ldots, m\}\), at least one \(E_i, i \in \{1, \ldots, m\}\), contains subsequences of two different sequences mentioned above. These subsequences are contained in a single component of \(U \cap D\). The proof is complete. \(\Box\)

We shall prove that Theorem 2 in [2] established in the Euclidean case holds in our framework.

**Theorem 5.** Suppose that \(f : D \rightarrow D'\) is a quasiconformal mapping and that \(\overline{D}, \overline{D}'\) are compact sets. If \(D\) is \(m\)-(arc) connected at \(b \in \partial D\), then \(C(f, b)\) either contains at most \(m - 1\) points at which \(D'\) has (arc) property \(P^*_2\) or consists of \(m\) points.

**Proof.** Suppose, contrary to the assertion, that \(C(f, b)\) contains at least \(m + 1\) distinct points \(b_1', \ldots, b_{m+1}'\) and that \(D'\) has (arc) property \(P^*_2\) at \(b_1', \ldots, b_m'\). Since \(D'\) has (arc) property \(P^*_2\) at \(b_i'\) for each \(i \in \{1, \ldots, m\}\), for \(b_j', j \neq i, j \in \{1, \ldots, m + 1\}\), there exist a continuum \(A_{ij}'\) in \(D'\) and a real number \(\delta_{ij} > 0\) such that

\[
M(\Delta(F', A_{ij}'; D')) \geq \delta_{ij}
\]

whenever \(F'\) is an (arc) connected set in \(D'\) with \(b_i', b_j' \in \overline{F'}\).

Let \(\delta = \min\{\delta_{ij}, 1 \leq i \leq m, 1 \leq j \leq m + 1, i \neq j\}\), \(A_{ij} = f^{-1}(A_{ij}')\), \(d_{ij} = d(A_{ij}, \partial D)\), \(d = \min\{d_{ij}, i \neq j, 1 \leq i \leq m, 1 \leq j \leq m + 1\}\).

Fix \(\varepsilon, 0 < 2\varepsilon < d\). Since \(b_j' \in C(f, b)\) for each \(j \in \{1, \ldots, m + 1\}\), we can choose a sequence \((b_{jk})\) in \(D, b_{jk} \rightarrow b\) as \(k \rightarrow \infty\) and \(f(b_{jk}) \rightarrow b_j'\). Since \(D\) is \(m\)-(arc) connected at \(b\), by Proposition 3, there exist an (arc) component \(F\) of \(B(b, \varepsilon) \cap D\) and \(1 \leq i < j \leq m + 1\) such that \(F\) contains two subsequences of \((b_{ik})\) and \((b_{jk})\). The set \(F' = f(F)\) is (arc) connected and \(b_i', b_j' \in \overline{F'}\).

Taking \(\Gamma_{ij} = \Delta(A_{ij}, F; D)\), \(\Gamma_{ij}' = \Delta(A_{ij}', F'; D')\) and using (10) we get

\[
M(\Gamma_{ij}) \geq \delta.
\]

On the other hand, every path \(\gamma \in \Gamma_{ij}\) intersects both \(\partial B(b, d)\) and \(\partial B(b, \varepsilon)\) and by Lemma 3.14 in [3] there exists a constant \(C_0\) only depending on \(Q\) and a constant \(C\) (by \(Q\)-regularity of \(X\)) such that

\[
M(\Gamma_{ij}) \leq C_0 \left(\log \frac{d}{\varepsilon}\right)^{1-Q}.
\]

Since (12) holds for all \(0 < 2\varepsilon < d\), letting \(\varepsilon \rightarrow 0\), using (11) and the fact that \(f\) is a quasiconformal mapping, we reach a contradiction. The proof is complete. \(\Box\)

The next result offers an analogue of Theorem 5 in the case of (arc) strictly quasiconformal accessibility.
THEOREM 6. Suppose that \( f : D \to D' \) is a quasiconformal mapping and that \( \overline{D}, \overline{D'} \) are compact sets. If \( D \) is \( m \)-\( (arc) \) connected at \( b \in \partial D \), then \( C(f, b) \) either contains at most \( m - 1 \) points at which \( D' \) is \( (arc) \) strictly quasiconformally accessible or consists of \( m \) points.

Proof. Suppose that \( C(f, b) \) contains at least \( m + 1 \) distinct points \( b_1', \ldots, b_{m+1}' \) and that \( D' \) is \( (arc) \) strictly quasiconformally accessible at \( b_1', \ldots, b_m' \). Since \( D' \) is \( (arc) \) strictly quasiconformally accessible at \( b_i' \) for each \( i \in \{1, \ldots, m\} \), for any neighbourhood \( U_i \) of \( b_i' \) there exist a neighbourhood \( V_i \) of \( b_i' \), \( V_i \subset U_i \), a continuum \( A_i' \subset D' \) and a real number \( \delta_i > 0 \) such that

\[
M(\Delta(F', A_i'; D')) \geq \delta_i
\]

whenever \( F' \) is \( (arc) \) connected set in \( D' \) with \( F' \cap \partial U_i \neq \emptyset \). We can choose \( U_i \) such that \( b_j' \notin \overline{U_j} \) for all \( j \neq i, j \in \{1, \ldots, m + 1\} \). Set \( \delta = \min_{1 \leq i \leq m} \delta_i, \ A_i = f^{-1}(A_i'), \ d_i = d(A_i, \partial D), \ d = \min_{1 \leq i \leq m} d_i \). Let \( \varepsilon \) be such that \( 0 < 2\varepsilon < d \). Since \( b_j' \in C(f, b) \) for each \( j \in \{1, \ldots, m + 1\} \), we can choose a sequence \( (b_{jk}) \in D, \ b_{jk} \to b \) as \( k \to \infty \) and \( f(b_{jk}) \to b_{j}' \). By Proposition 3 there exist a component \( F \) of \( B(b, \varepsilon) \cap D \) and \( 1 \leq i < j \leq m + 1 \) such that \( F \) contains two subsequences of \( (b_{hk}) \) and \( (b_{jk}) \).

The set \( F' = f(F) \) is connected and \( b_i', b_j' \in \overline{F'} \). Since \( b_i' \in V_i, b_j' \notin V_i \) and \( b_i' \notin U_i, b_j' \notin U_j \) for \( j \neq i \), we have \( F' \cap \partial V_i \neq \emptyset \) and \( F' \cap \partial U_j \neq \emptyset \). Taking \( \Gamma_i = \Delta(A_i, F'; D), \ \Gamma_i' = \Delta(A_i', F'; D') \) and using (13) we obtain

\[
M(\Gamma_i') \geq \delta.
\]

But, every path \( \gamma \in \Gamma_i \) meets both \( \partial B(b, d) \) and \( \partial B(b, \varepsilon) \) and by Lemma 3.14 in [3] there exists a constant \( C_0 \) such that

\[
M(\Gamma_i) \leq C_0 \left( \log \frac{d}{\varepsilon} \right)^{1-Q}.
\]

Letting \( \varepsilon \to 0 \) and using (14) and the fact that \( f \) is a quasiconformal mapping, we reach a contradiction, which completes the proof. \( \Box \)

Finally, we show that Theorem 3 in [2] holds in our setting, too.

THEOREM 7. Let \( f : D \to D' \) be a quasiconformal mapping and suppose that \( D \) is \( m \)-\( (arc) \) connected at \( b \in \partial D \). If \( \overline{D'} \) is compact and \( D' \) has \( (arc) \) property \( P \) [or \( D' \) is \( (arc) \) strictly quasiconformally accessible] at each point of \( C(f, b) \) and \( C(f, b) \) is a connected set, then there exists the limit of \( f \) at \( b \).

Proof. If \( m = 1 \) then \( D \) is \( 1-(arc) \) connected at \( b \) and, by Theorem 5 [or Theorem 6], \( C(f, b) \) either contains no point at which \( D' \) has \( (arc) \) property \( P \) [or it is \( (arc) \) strictly quasiconformally accessible], or \( C(f, b) \) has a single
point. Since $D'$ has (arc) property $P^*_2$ [or it is (arc) strictly quasiconformally accessible] at each point of $C(f,b)$, the latter has a single point.

Assume that $m \geq 2$. By Theorem 5 [or Theorem 6], $C(f,b)$ either contains at most $m - 1$ points at which $D'$ has (arc) property $P^*_2$ [or is (arc) strictly quasiconformally accessible] or consists of $m$ points. Since $C(f,b)$ is connected, $C(f,b)$ has at most $m - 1$ points at which $D'$ has (arc) property $P^*_2$ [or is (arc) strictly quasiconformally accessible]. If $C(f,b)$ does not have a single point, since $C(f,b)$ is a connected set, there exists an infinity of points in $C(f,b)$. By hypothesis, $D'$ has (arc) property $P^*_2$ [or it is (arc) strictly quasiconformally accessible] at these points. This contradicts the fact that $C(f,b)$ has at most $m - 1$ points at which $D'$ has (arc) property $P^*_2$ [or it is (arc) strictly quasiconformally accessible]. □

Corollary 8. Let $f : D \to D'$ be a quasiconformal mapping and suppose that $D$ is $m$-(arc) connected at $b \in \partial D$. If $D'$ is compact and $D'$ is (arc) very weakly flat at each point of $C(f,b)$ and $C(f,b)$ is a connected set, then there exists the limit of $f$ at $b$.

Proof. It follows from Proposition 1 and Theorem 7. □

According to the discussion above, if one considers the metric spaces which have a base of topology consisting of path-connected sets (arc) locally path-connected, then instead of arc domains we can consider domains.

Note that if $X$ and $Y$ are proper $Q$-regular, $Q$-Loewner spaces, $Q > 1$, then we can consider quasiconformal mappings in any sense (metric, geometric or analytic) since the three definitions of quasiconformality are equivalent (see [6]).

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