PRODUCTS OF METRIC SPACES, COVERING NUMBERS, PACKING NUMBERS, AND CHARACTERIZATIONS OF ULTRAMETRIC SPACES

OLEKSIY DOVGOSHEY and OLLI MARTIO

We describe some Cartesian products of metric spaces and find conditions under which products of ultrametric spaces are ultrametric.

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1. INTRODUCTION

Let (X, d) be a metric space. The closed ball with center $c \in X$ and radius r, $0 < r < \infty$, is denoted by

$$B(c, r) = B_d(c, r) = \{x \in X : d(x, c) \leq r\}.$$

Let W be a subset of X and let $\varepsilon > 0$. A set $C \subseteq X$ is an *ε-net* for W if

$$W \subseteq \bigcup_{c \in C} B(c, \varepsilon).$$

A set $W \subseteq X$ is called *totally bounded* (*or precompact*) if for every $\varepsilon > 0$ there is a finite ε-net for W. The *covering number* of a totally bounded set $W \subseteq X$ is the smallest cardinality of subsets of W which are ε-nets for W. A set $A \subseteq X$ is called *ε-distinguishable* if $d(x, y) > \varepsilon$ for every distinct points $x, y \in A$ (see [7]). The *packing number* of a precompact set $W \subseteq X$ is the maximal cardinality of the ε-distinguishable sets $A \subseteq W$.

We denote by $\mathcal{N}_\varepsilon(W)$ and by $\mathcal{M}_\varepsilon(W)$ the covering number and, respectively, the packing number of a totally bounded set $W \subseteq X$. These quantities have been invented by Kolmogorov [6] in order to classify compact metric sets. Note that the function $\log_2 \mathcal{N}_\varepsilon(W)$ is the so-called *metric entropy* and it has been widely applied in approximation theory, geometric functional analysis, probability theory and complexity theory, see, for example, [7, 8, 2, 5].

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A main general fact about packing and covering numbers is the simple double inequality

$$\mathcal{M}_{2\varepsilon}(W) \le \mathcal{N}_{\varepsilon}(W) \le \mathcal{M}_{\varepsilon}(W). \tag{1.1}$$

In Section 2 of this paper we consider some transfinite generalizations of covering numbers and packing ones and obtain a more exact version of inequality (1.1), see Lemma 2.6. It implies the characterization of ultrametric spaces as spaces for which packing numbers equal covering numbers. In Sections 3 and 4 we introduce some "natural" metrics on the products of metric spaces and discuss conditions under which the products of ultrametric spaces are ultrametric.

2. THE EQUALITY BETWEEN COVERING NUMBERS AND PACKING NUMBERS

Let (X, d) be a metric space. Denote by $t_0 = t_0(d)$ the supremum of positive numbers t for which the function $(x, y) \mapsto (d(x, y))^t$ is a metric on X. This quantity has the following characterization, see [3].

LEMMA 2.1. *Let x, y and z be points in a metric space (X, d). If the inequality*

$$\max\{d(x, z), d(z, y)\} < d(x, y) \tag{2.1}$$

holds, then there exists a unique solution $s_0 \in [1, \infty[$ of the equation

$$(d(x, y))^s = (d(x, z))^s + (d(z, y))^s. \tag{2.2}$$

For points x, y and z in X write

$$s(x, y, z) := \begin{cases} s_0 & \text{if (2.1) holds,} \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3}$$

where s_0 is the unique root of equation (2.2).

PROPOSITION 2.2. *The equation*

$$t_0(d) = \inf\{s(x, y, z) : x, y, z \in X\}$$

holds in every metric space (X, d).

Remark 2.3. A point z in a metric space (X, d) *lies between* two distinct points x and y if $d(x, z) + d(z, y) = d(x, y)$ and $x \ne z \ne y$, see [9, p. 55]. Now, $t_0 = t_0(d)$ can be called the *betweenness exponent* of the space (X, d).

Recall that a metric space (X, d) is *ultrametric* if the metric d satisfies the *ultra-triangle inequality* $d(x, y) \le \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$. In this case d is called an *ultrametric*. Since (2.1) never holds in an ultrametric space (X,d) we have $t_0(d)=\infty$ in this case. In fact, (X,d) is ultrametric if and only if $t_0(d)=\infty$.

Lemma 2.4. *Let $B(a,r)$ be a closed ball in a metric space (X,d). Then*

$$\operatorname{diam}(B(a,r)) \le 2^{\frac{1}{t_0}} r, \tag{2.4}$$

where $t_0 = t_0(d)$ is the betweenness exponent of (X,d).

Proof. If $t_0(d)=\infty$ then (X,d) is ultrametric and $\operatorname{diam} B(a,r) \le r$ for every ball $B(a,r)$, see, for example, [4, p. 43]. In the case $t_0(d)<\infty$, the function $(x,y) \mapsto (d(x,y))^{t_0}$ is a metric. Hence, by the triangle inequality, we have

$$d^{t_0}(x,y) \le d^{t_0}(x,a) + d^{t_0}(y,a) \le 2r^{t_0}$$

for all $x,y \in B(a,r)$. The last inequality implies (2.4). □

There is the possibility of more refined classification of nonprecompact metric spaces by means of an extension of the range of values of the functions $\mathcal{N}_\varepsilon$ and $\mathcal{M}_\varepsilon$ to transfinite cardinal numbers.

Let W and A be subsets of X. Define

$$\widehat{\mathcal{N}}_\varepsilon^A(W) := \min\{\operatorname{card}(C) : C \text{ is an } \varepsilon\text{-net for } W \text{ and } C \subseteq A\}. \tag{2.5}$$

Moreover, for the sake of simplicity, write $\widehat{\mathcal{N}}_\varepsilon(W) := \widehat{\mathcal{N}}_\varepsilon^W(W)$.

For convenience we introduce

Definition 2.5. A set A is *maximal ε-distinguishable* with respect to W if A is ε-distinguishable, $A \subseteq W$ and for every ε-distinguishable $B \subseteq W$ the inclusion $A \subseteq B$ implies the equality $A = B$.

Write $\widehat{\mathcal{M}}_\varepsilon(W)$ for the smallest power of maximal ε-distinguishable sets $A \subseteq W$ and define the quantity $\mathcal{M}_\varepsilon^*(W)$ as the smallest cardinal number which is greater than or equal to $\operatorname{card}(A)$ for every ε-distinguishable $A \subseteq W$. It is clear that

$$\mathcal{M}_\varepsilon^*(W) = \mathcal{M}_\varepsilon(W) \quad \text{and} \quad \widehat{\mathcal{N}}_\varepsilon(W) = \mathcal{N}_\varepsilon(W)$$

for every precompact W.

Lemma 2.6. *Let W be a set in a metric space (X,d). Then for every $\varepsilon > 0$ the inequalities*

$$\mathcal{M}^*_{2^{\frac{1}{t_0}}\varepsilon}(W) \le \widehat{\mathcal{N}}_\varepsilon^X(W) \le \widehat{\mathcal{N}}_\varepsilon(W) \le \widehat{\mathcal{M}}_\varepsilon(W) \le \mathcal{M}_\varepsilon^*(W) \tag{2.6}$$

hold, where t_0 is the betweenness exponent of X.

Proof. The first inequality from the right is immediate. For the proof of the second one note that every maximal ε-distinguishable set $A \subseteq W$ is an ε-net for W. The inequality $\widehat{\mathcal{N}}_\varepsilon^X(W) \le \widehat{\mathcal{N}}_\varepsilon(W)$ is clear from the definitions. To prove the first inequality from the left it, suffices to show $\operatorname{card}(A) \le \widehat{\mathcal{N}}_\varepsilon^X(W)$

for every $2^{\frac{1}{t_0}}\varepsilon$-distinguishable set $A \subseteq W$. Let $\{x_i : i \in I\}$ be an ε-net for W with $\mathrm{card}(I) = \widehat{\mathcal{N}}_\varepsilon^X(W)$. Suppose that there exists a $2^{\frac{1}{t_0}}\varepsilon$-distinguishable set $A_0 \subseteq W$ for which $\mathrm{card}(A_0) > \mathrm{card}(I)$. This inequality and the inclusion

$$A_0 \subseteq \bigcup_{i\in I} B(x_i, \varepsilon)$$

imply that there exists a ball $B(x_i, \varepsilon)$ which contains at least two distinct points $y_i, z_i \in A_0$. (In the opposite case, A_0 and some subset of I have the same cardinality.) Lemma 2.4 implies that

$$d(y_i, z_i) \le 2^{\frac{1}{t_0}}\varepsilon.$$

This contradicts the assumption that A_0 is $2^{\frac{1}{t_0}}\varepsilon$-distinguishable. □

COROLLARY 2.7. *Let X be a nonprecompact metric space. Then for some $\varepsilon_0 > 0$ there is an ε_0-distinguishable, countable infinite set $A \subseteq X$.*

Example 2.8. Let X be a set of a power $\alpha > 2$ and let a be an element of X. For every two distinct $x, y \in X$ write

$$d(x,y) = \begin{cases} 2^{\frac{1}{t}} & \text{if } x \neq a \neq y, \\ 1 & \text{otherwise,} \end{cases}$$

where $t \in [1, \infty[$ and put $d(x, y) = 0$ if $x = y$. Proposition 2.2 implies that the metric space (X, d) has the betweenness exponent $t_0(d) = t$. If we define a set W as $W = X \setminus \{a\}$, then $\widehat{\mathcal{N}}_\varepsilon(W) = \widehat{\mathcal{M}}_\varepsilon(W) = \mathcal{M}^*_\varepsilon(W) = \mathrm{card}(W)$ but $\widehat{\mathcal{N}}_\varepsilon^X(W) = \mathcal{M}^*_{2^{\frac{1}{t}}\varepsilon}(W) = 1 = \widehat{\mathcal{M}}_\varepsilon(X)$ for every $\varepsilon \in]1, 2^{\frac{1}{t}}[$.

THEOREM 2.9. *Let (X, d) be a metric space. The following statements are equivalent.*

(i) *The space X is ultrametric.*

(ii) *For every $W \subseteq X$ the equations*

$$\mathcal{M}^*_\varepsilon(W) = \widehat{\mathcal{N}}_\varepsilon^X(W) = \widehat{\mathcal{N}}_\varepsilon(W) = \widehat{\mathcal{M}}_\varepsilon(W) \tag{2.7}$$

hold for all $\varepsilon > 0$.

(iii) *For every compact $W \subseteq X$ and every $\varepsilon > 0$ we have $\mathcal{N}_\varepsilon(W) = \mathcal{M}_\varepsilon(W)$.*

Proof. Since $t_0(d) = \infty$ holds if d is an ultrametric, inequalities (2.6) impliy (2.7) for ultrametric spaces. The implication (ii)⇒(iii) is trivial. If (X, d) is not an ultrametric space, then there are points $a, b, c \in X$ such that

$$d(a, b) > \max\{d(a, c), d(b, c)\}. \tag{2.8}$$

Write $\varepsilon := \max\{d(a,c), d(b,c)\}$. It follows from (2.8) that $\mathcal{M}_\varepsilon(\{a,b,c\}) \geq 2$. Moreover, since $B(c,\varepsilon) \supseteq \{a,b,c\}$, we have $\mathcal{N}_\varepsilon(\{a,b,c\}) \leq 1$. Hence $\mathcal{N}_\varepsilon(\{a,b,c\}) \neq \mathcal{M}_\varepsilon(\{a,b,c\})$. □

Consider now equations (2.7) for non ultrametric spaces.

Recall that a cardinal number α is the *density* of a metric space X if

$$\alpha = \min_A(\operatorname{card}(A)),$$

where the minimum is taken over the family of all dense sets $A \subseteq X$. For the density of X we use the notation $\operatorname{den}X$. For convenience, we repeat some definitions related to the confinality of the cardinals, see, for example, [10]. We understand the ordinal numbers as some special well-ordered sets $\alpha, \beta, \ldots$ for which the statements:

– α is similar to an initial segment of β and $\alpha \neq \beta$, $\alpha \prec \beta$;

– α is proper subset of β, $\alpha \subsetneq \beta$;

– α belongs to β, $\alpha \in \beta$,

are equivalent. An ordinal number β is an *initial* ordinal if for all ordinals α the implication $(\alpha \prec \beta) \Rightarrow (|\alpha| \lneqq |\beta|)$ holds, where $|\alpha|$ and $|\beta|$ are the corresponding cardinality of α and β. By cardinal numbers we mean initial ordinals. An ordinal number α is *confinal* in an ordinal β if there is an one-to-one increasing mapping $f : \alpha \to \beta$ such that for every ordinal $\gamma \in \beta$ there exists an ordinal $\delta \in \alpha$ with $\gamma \prec f(\delta)$ or $\gamma = f(\delta)$. The *confinality* of an ordinal β is the least ordinal α with α confinal in β. We write $\operatorname{cf}(\beta)$ for the confinality of β. If α is the confinality for some β, then α is a cardinal (see [10, p. 91]).

Theorem 2.10. *Let W be a subset of a metric space X. Suppose that* $\operatorname{den}(W)$ *is a cardinal of an uncountable confinality. Then there is $\varepsilon_0 > 0$ such that the equations*

$$\widehat{\mathcal{N}}_\varepsilon^X(W) = \widehat{\mathcal{N}}_\varepsilon(W) = \widehat{\mathcal{M}}_\varepsilon(W) = \widehat{\mathcal{M}}_\varepsilon^*(W) = \operatorname{den}(W) \tag{2.9}$$

hold for all $\varepsilon \in]0, \varepsilon_0[$.

Write, as usual, $\aleph_0$ for $\operatorname{card}(\mathbb{N})$ and $\mathfrak{c} = 2^{\aleph_0} = \operatorname{card}(\mathbb{R})$.

Corollary 2.11. *Let W be a subset of a metric space X. If* $\operatorname{den}(W) = \mathfrak{c}$, *then there is $\varepsilon_0 > 0$ such that the equations*

$$\widehat{\mathcal{N}}_\varepsilon^X(W) = \widehat{\mathcal{N}}_\varepsilon(W) = \widehat{\mathcal{M}}_\varepsilon(W) = \widehat{\mathcal{M}}_\varepsilon^*(W) = \mathfrak{c} \tag{2.10}$$

hold for all $\varepsilon \in]0, \varepsilon_0[$.

Proof. Since for each infinite cardinal γ we have $\gamma \prec \operatorname{cf}(2^\gamma)$, see [10, Theorem 44, p. 93], $\mathfrak{c}$ has an uncountable confinality. □

Corollary 2.12. *Let (X,τ) be a metrizable topological space, let $W \subseteq X$ be a set such that* $\operatorname{den}(W)$ *is a cardinal of an uncountable confinality, and let D be a finite family of metrics d each of which induces the topology τ on X. Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$ the values $\widehat{\mathcal{N}}_\varepsilon(W)$, $\widehat{\mathcal{N}}_\varepsilon^X(W)$, $\widehat{\mathcal{M}}_\varepsilon(W)$ and $\mathcal{M}_\varepsilon^*(W)$ do not depend on the choice of $d \in D$.*

Proof of Theorem 2.10. The definitions of cardinal numbers $\widehat{\mathcal{N}}_\varepsilon(W)$ and $\operatorname{den}(W)$ imply that the inequality $\widehat{\mathcal{N}}_\varepsilon^X(W) \le \widehat{\mathcal{N}}_\varepsilon(W) \le \operatorname{den}(W)$ holds for all $\varepsilon > 0$. Hence, by (2.6), we have $\widehat{\mathcal{M}}_\varepsilon(W) \le \mathcal{M}_\varepsilon^*(W) \le \operatorname{den}(W)$ if $\varepsilon > 0$. Moreover, if there is $\varepsilon_0 > 0$ such that

$$\operatorname{den}(W) \le \widehat{\mathcal{N}}_{\varepsilon_0}(W), \tag{2.11}$$

then the last inequality and (2.6) imply

$$\widehat{\mathcal{N}}_{\varepsilon_0}^X(W) \ge \mathcal{M}^*_{2^{-\frac{1}{t_0}}\varepsilon_0}(W) \ge \widehat{\mathcal{M}}_{2^{-\frac{1}{t_0}}\varepsilon_0}(W) \ge \widehat{\mathcal{N}}_{2^{-\frac{1}{t_0}}\varepsilon_0}(W) \ge \operatorname{den}(W).$$

Therefore, it is sufficient to show (2.11) with some $\varepsilon_0 > 0$.

If D is a dense subset of W, then for every $k \in]0,1[$ and all $\varepsilon > 0$ we have the double inequality

$$\widehat{\mathcal{N}}_{k\varepsilon}(W) \ge \widehat{\mathcal{N}}_\varepsilon(D) \ge \widehat{\mathcal{N}}_{\frac{\varepsilon}{k}}(W). \tag{2.12}$$

Indeed, if $C = \{c_i : i \in I\}$ is a $k\varepsilon$-net for W with $\operatorname{card}(C) = \widehat{\mathcal{N}}_{k\varepsilon}(W)$, then the density of D in W implies that for every $c_i \in C$ there is $b_i \in D$ such that $B(b_i,\varepsilon) \supseteq B(c_i, k\varepsilon)$. Hence

$$D \subseteq W \subseteq \bigcup_{i\in I} B(c_i, k\varepsilon) \subseteq \bigcup_{i\in I} B(b_i, \varepsilon),$$

i.e., $\{b_i : i \in I\}$ is an ε-net for D, so the first inequality in (2.12) is proved. Similarly, if $P = \{p_i : i \in I\}$ is an ε-*net* for D with $\operatorname{card}(P) = \widehat{\mathcal{N}}_\varepsilon(D)$, then for every $x \in W$ there is $p_i \in P$ such that $x \in B(p_i, \frac{\varepsilon}{k})$. Hence P is an $\frac{\varepsilon}{k}$-*net* for W, that implies the second inequality in (2.12).

Let D be a dense subset of W such that

$$\operatorname{card}(D) = \operatorname{den}(W). \tag{2.13}$$

Consider a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \dots$ with $\lim\limits_{i\to\infty} \varepsilon_i = 0$. Suppose that a set D_i is an ε_i-*net* for D with $D_i \subseteq D$ and with

$$\operatorname{card}(D_i) = \widehat{\mathcal{N}}_{\varepsilon_i}(D) \tag{2.14}$$

for every $i \in \mathbb{N}$. The set

$$\widetilde{D} := \bigcup_{i=1}^{\infty} D_i \tag{2.15}$$

is a dense subset of W and $\widetilde{D} \subseteq D$. Hence, by (2.13), $\mathrm{card}(\widetilde{D}) = \mathrm{den}(W)$. Suppose also that the inequality

$$\mathrm{card}(D_i) \lneq \mathrm{den}(W) \tag{2.16}$$

holds for each D_i. Let γ be an initial ordinal such that $|\gamma| = \mathrm{card}(\widetilde{D})$ and let $f : \gamma \to \widetilde{D}$ be a bijection. Inequality (2.16) implies that for every ordinal $\alpha_i := f^{-1}(D_i)$ there is an ordinal $\beta_i \in \gamma$ such that α_i is similar to an initial segment of β_i and $\alpha_i \neq \beta_i$.

It follows from this and (2.15) that $\aleph_0$ is confinal in the ordinal number $\mathrm{den}(W)$, contrary to the supposition of the theorem. Thus there is $\varepsilon_{i_0} > 0$ such that $\mathrm{card}(D_{i_0}) = \mathrm{den}(W)$. This equality and (2.12) imply (2.11) with $\varepsilon_0 = k\varepsilon_{i_0}$. $\square$

3. METRICS ON PRODUCTS OF METRIC SPACES

Let (X, d_X) and (Y, d_Y) be two metric spaces.

Definition 3.1. A metric d on the product $X \times Y$ is said to be *distance-increasing* if

$$d((x_1, y_1), (x_2, y_2)) \leq d((x_3, y_3), (x_4, y_4)) \tag{3.1}$$

whenever

$$d_X(x_1, x_2) \leq d_X(x_3, x_4) \quad \text{and} \quad d_Y(y_1, y_2) \leq d_Y(y_3, y_4); \tag{3.2}$$

d is *partial distance-preserving* if

$$d((x_1, y), (x_2, y)) = d_X(x_1, x_2) \quad \text{and} \quad d((x, y_1), (x, y_2)) = d_Y(y_1, y_2) \tag{3.3}$$

for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$.

Remark 3.2. When

$$d_X(x_1, x_2) = d_X(x_3, x_4), \quad d_Y(y_1, y_2) = d_Y(y_3, y_4), \tag{3.4}$$

from (3.1) and (3.2) we obtain

$$d((x_1, y_1), (x_2, y_2)) = d((x_3, y_3), (x_4, y_4)), \tag{3.5}$$

i.e., the distance-function $d : (X \times Y) \times (X \times Y) \to \mathbb{R}^+$ depends only on "partial" distance-functions d_X and d_Y. Consequently, there is a mapping $F : D_X \times D_Y \to \mathbb{R}^+$ with

$$D_X := \{d_X(x, y) : x, y \in X\}, \quad D_Y := \{d_Y(x, y) : x, y \in Y\} \tag{3.6}$$

such that the diagram

$$\begin{array}{ccc} (X \times Y) \times (X \times Y) & \xrightarrow{d} & \mathbb{R}^+ \\ \text{Id}\big\downarrow & & \big\uparrow F \\ (X \times X) \times (Y \times Y) & \xrightarrow{d_X \otimes d_Y} & D_X \times D_Y \end{array} \tag{3.7}$$

is commutative. Here Id is the identity mapping

$$\text{Id}((x_1, y_1), (x_2, y_2)) = ((x_1, x_2), (y_1, y_2))$$

and $d_X \otimes d_y$ is the direct product of the partial distance functions d_X and d_Y, namely,

$$d_X \otimes d_Y((x_1, x_2), (y_1, y_2)) = (d_X(x_1, x_2), d_Y(y_1, y_2)).$$

Diagram (3.7) shows that we can derive the metric properties of the product $X \times Y$ using the corresponding ones of the function F. This approach to the study of metric products was originated by the paper of Bernig, Foertsch and Schroeder [1].

Example 3.3. For every $p \in [1, \infty]$ let d_p be a metric on $X \times Y$ defined as

$$d_p((x_1, y_1), (x_2, y_2)) = ((d_X(x_1, x_2))^p + (d_Y(y_1, y_2))^p)^{\frac{1}{p}} \tag{3.8}$$

if $1 \leq p < \infty$ and

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \tag{3.9}$$

if $p = \infty$. It is clear that the metrics d_p are distance-increasing and partial distance-preserving for every $p \in [1, \infty]$.

Proposition 3.4. *Let (X, d_X) and (Y, d_Y) be two metric spaces and let d be a distance-increasing, partial distance-preserving metric on the product $X \times Y$. Then the double inequality*

$$d_\infty((x_1, y_1), (x_2, y_2)) \leq d((x_1, y_1), (x_2, y_2)) \leq d_1((x_1, y_1), (x_2, y_2)) \tag{3.10}$$

holds for all $(x_i, y_i) \in X \times Y$, $i = 1, 2$, where the metrics d_∞ and d_1 are defined by (3.9) *and* (3.8), *respectively.*

Proof. To prove the first inequality in (3.10) we may assume that

$$d_\infty((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2). \tag{3.11}$$

Since $d_Y(y_1, y_2) \geq 0 = d_Y(y_1, y_1)$ and d is distance-increasing, we have

$$d((x_1, y_1), (x_2, y_2)) \geq d((x_1, y_1), (x_2, y_1)).$$

This inequality, the first equality in (3.3) and (3.11) imply that

$$d((x_1, y_1), (x_2, y_2)) \geq d_X(x_1, x_2) = d_\infty((x_1, y_1), (x_2, y_2)),$$

i.e., the first inequality in (3.10) holds.

To prove the right hand side of (3.10), consider the triangle inequality

$$d((x_1,y_1),(x_2,y_2)) \le d((x_1,y_1),(x_1,y_2)) + d((x_1,y_2),(x_2,y_2)) \tag{3.12}$$

for the metric d. From this and (3.3) we obtain

$$d((x_1,y_2),(x_2,y_2)) \le d_X(x_1,x_2) + d_Y(y_1,y_2) = d_1((x_1,y_1),(x_2,y_2)),$$

as required. □

Recall that there is a *natural topology* on the product space. It is the coarsest topology for which the canonical projections to the factors are continuous.

COROLLARY 3.5. *Let (X, d_X) and (Y, d_Y) be two metric spaces. All distance-increasing, partial distance-preserving metrics on the product $X \times Y$ induce the natural topology on this product.*

Proof. Let d be a partial distance-preserving, distance-increasing metric on $X \times Y$. Inequality (3.10) implies that $d_\infty \le d \le 2d_\infty$. Hence the spaces $(X \times Y, d_\infty)$ and $(X \times Y, d)$ have the coinciding sets of convergent sequences. Consequently, these spaces have the same topology. Moreover, it is well-known that d_∞ induces the natural topology on $X \times Y$. Therefore, the topology of the space $(X \times Y, d)$ also is natural. □

Proposition 3.4 admits a partial converse.

PROPOSITION 3.6. *Let (X, d_X) and (Y, d_Y) be two metric spaces. If d is a metric on $X \times Y$ such that the double inequality* (3.10) *holds, then d is partial distance-preserving and*

$$d((x_1,y_1),(x_2,y_2)) \le 2d((x_3,y_3),(x_4,y_4)) \tag{3.13}$$

whenever inequalities (3.2) *hold.*

Proof. The first part of the proposition directly follows from (3.10), because d_∞ and d_1 is partial distance-preserving. To prove the second part we may use the elementary inequality

$$a + b \le 2\max\{a, b\}$$

which holds for all $a, b \in \mathbb{R}$. □

Example 3.7. Let (X, d_X) and (Y, d_Y) be two three-point metric spaces such that

$$d_X(x_i, x_j) = d_Y(y_i, y_j) = |i - j|$$

for all $x_i, x_j \in X$ and all $y_i, y_j \in Y$, $i, j \in \{1, 2, 3\}$. Consider the metric space $(X \times Y, d)$ for which the metric d is defined by the distance-matrix from

Figure 1. Then the double inequality (3.10) holds for all $(x_i, y_i) \in X \times Y,\ i = 1, 2$, and moreover we have

$$1 = d((x_1, y_1), (x_2, y_2)) = \frac{1}{2} d((x_2, y_2), (x_3, y_3)).$$

Consequently, d is not distance-increasing and 2 is the best possible constant in inequality (3.13).

	$x_1\ y_1$	$x_1\ y_2$	$x_1\ y_3$	$x_2\ y_1$	$x_2\ y_2$	$x_2\ y_3$	$x_3\ y_1$	$x_3\ y_2$	$x_3\ y_3$
$x_1\ y_1$	**0**	1	2	**1**	1	2	**2**	2	2
$x_1\ y_2$	1	**0**	1	1	**1**	2	2	**2**	2
$x_1\ y_3$	2	1	**0**	2	2	**1**	2	2	**2**
$x_2\ y_1$	**1**	1	2	**0**	1	2	**1**	2	2
$x_2\ y_2$	1	**1**	2	1	**0**	1	2	**1**	2
$x_2\ y_3$	2	2	**1**	2	1	**0**	2	2	**1**
$x_3\ y_1$	**2**	2	2	**1**	2	2	**0**	1	2
$x_3\ y_2$	2	**2**	2	2	**1**	2	1	**0**	1
$x_3\ y_3$	2	2	**2**	2	2	**1**	2	1	**0**

Fig. 1. The distance-matrix of a space $(X \times Y, d)$
for $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$.

The product space $(X \times Y, d)$ inherits many useful properties of the factors if d is distance-increasing and partial distance-preserving. Recall that a metric space (X, d) is *proper* if each closed and bounded set $A \subseteq X$ is compact.

Proposition 3.8. *Let (X, d_X) and (Y, d_Y) be two metric spaces. If d is a metric on $X \times Y$ such that* (3.10) *holds, then the statements below hold.*

(i) *$(X \times Y, d)$ is bounded if and only if (X, d_X) and (Y, d_Y) are bounded.*

(ii) *$(X \times Y, d)$ is complete if and only if (X, d_X) and (Y, d_Y) are complete.*

(iii) *$(X \times Y, d)$ is proper if and only if (X, d_X) and (Y, d_Y) are proper.*

Proof. Propositions (i) and (ii) can be obtained by standard arguments.

For the proof of (iii), observe that a metric space (Z, ρ) is proper if and only if every closed ball $B_\rho(a, r) := \{x \in Z : \rho(x, a) \le r\}$ is compact. Suppose that (X,d_X) and (Y,d_Y) are proper. It follows from the first inequality in (3.10) that

$$B_d((x_1,y_1),r) \subseteq B_{d_\infty}((x_1,y_1),r) = B_{d_X}(x_1,r) \times B_{d_Y}(y_1,r).$$

The last direct product is compact because the balls $B_{d_X}(x_1,r)$ and $B_{d_Y}(y_1,r)$ are compact. Hence $B_d((x_1,y_1),r)$ is compact as a closed subset of a compact set.

Suppose that $(X \times Y, d)$ is proper. By Proposition (3.6), d is partial distance-preserving. Hence for every closed ball $B_d((x_1,y_1),r)$ the sets

$$(X \times \{y_1\}) \cap B_d((x_1,y_1),r) \quad \text{and} \quad (\{x_1\} \times Y) \cap B_d((x_1,y_1),r) \tag{3.14}$$

are isometric to the balls $B_{d_X}(x_1,r)$ and $B_{d_Y}(y_1,r)$, respectively. Since the sets $X \times \{y_1\}$ and $\{x_1\} \times Y$ are closed, the sets in (3.14), hence the closed balls $B_{d_X}(x_1,r)$ and $B_{d_Y}(y_1,r)$, are compact. □

THEOREM 3.9. *Let (X,d_X) and (Y,d_Y) be metric spaces and let d be a partial distance-preserving metric on $X \times Y$ such that the inequality*

$$d_\infty((x_1,y_1),(x_2,y_2)) \le d((x_1,y_1),(x_2,y_2)) \tag{3.15}$$

holds for all $(x_i,y_i) \in X \times Y$, $i=1,2$. Then d is an ultrametric if and only if d_X and d_Y are ultrametrics and $d = d_\infty$.

Proof. Suppose that d_X and d_Y are ultrametrics. Then for all $(x_i,y_i) \in X \times Y$, $i=1,2,3$, we have

$$\begin{aligned} &\max\{d_\infty((x_1,y_1),(x_2,y_2)), d_\infty((x_2,y_2),(x_3,y_3))\} \\ &= \max\{\max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}, \max\{d_X(x_2,x_3), d_Y(y_2,y_3)\}\} \\ &= \max\{\max\{d_X(x_1,x_2), d_X(x_2,x_3)\}, \max\{d_Y(y_1,y_2) d_Y(y_2,y_3)\}\} \\ &\ge \max\{d_X(x_1,x_3), d_Y(y_1,y_3)\} = d_\infty((x_1,y_1),(x_3,y_3)), \end{aligned}$$

i.e., $(X \times Y, d_\infty)$ is an ultrametric space if (X,d_X) and (Y,d_Y) are ultrametric.

Conversely, let $(X \times Y, d)$ be an ultrametric space. Since d is partial distance-preserving, we have

$$\begin{aligned} d_X(x_1,x_3) = d((x_1,y),(x_3,y)) &\le \max\{d((x_1,y),(x_2,y)), d((x_2,y),(x_3,y))\} \\ &= \max\{d_X(x_1,x_2), d_X(x_2,x_3)\} \end{aligned}$$

for every $y \in Y$ and $x_1,x_2,x_3 \in X$. Hence d_X is an ultrametric. A similar argument yields that d_Y is an ultrametric if d is an ultrametric. To prove that $d = d_\infty$, it is sufficient to show that the inequality

$$d((x_1,y_1),(x_2,y_2)) \le d_\infty((x_1,y_1),(x_2,y_2)) \tag{3.16}$$

holds for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Since d is a partial distance-preserving ultrametric, we have

$$\begin{aligned} d((x_1,y_1),(x_2,y_2)) &\le \max\{d((x_1,y_1),(x_1,y_2)), d((x_1,y_2),(x_2,y_2))\} \\ &= \max\{d_Y(y_1,y_2), d_X(x_1,x_2)\}, \end{aligned}$$

i.e., (3.16) holds. □

Remark 3.10. Let (X, d_X), (Y, d_Y) and $(X \times Y, d)$ be metric spaces such that $d_\infty \le d$. It follows from Proposition 3.6 and inequality (3.12) that a metric d is partial distance-preserving if and only if $d \le d_1$.

Corollary 3.11. *Let (X, d_X) and (Y, d_Y) be metric spaces and let d be a distance-increasing and partial distance-preserving metric on the product $X \times Y$. Then d is an ultrametric if and only if d_X and d_Y are ultrametrics and $d = d_\infty$.*

Proof. It follows from Theorem 3.9 and Proposition 3.4. □

4. PRODUCTS OF PACKING NUMBERS AND PRODUCTS OF ULTRAMETRIC SPACES

In this section we give some conditions under which a product of metric spaces is ultrametric.

Theorem 4.1. *Let (X, d_X) and (Y, d_Y) be ultrametric spaces and let d be a partial distance-preserving metric on $(X \times Y)$ such that the inequality*

$$d_\infty((x_1,y_1),(x_2,y_2)) \le d((x_1,y_1),(x_2,y_2)) \tag{4.1}$$

holds for all $((x_1,y_1),(x_2,y_2)) \in (X \times Y) \times (X \times Y)$. Then the statements below are equivalent.

(i) *d is an ultrametric on $X \times Y$.*

(ii) *The equation*

$$\mathcal{M}_\varepsilon(W \times Z) = \mathcal{M}_\varepsilon(W) \cdot \mathcal{M}_\varepsilon(Z) \tag{4.2}$$

holds for all compact sets $W \subseteq X$ and $Z \subseteq Y$ and every $\varepsilon > 0$.

Lemma 4.2. *Let (X, d_X), (Y, d_Y) and $(X \times Y, d)$ be ultrametric spaces. Suppose that d is partial distance-preserving and (4.1) holds for all $((x_1,y_1), (x_2,y_2)) \in (X \times Y) \times (X \times Y)$. Then the equations (4.2) and*

$$\mathcal{N}_\varepsilon(W \times Z) = \mathcal{N}_\varepsilon(W) \cdot \mathcal{N}_\varepsilon(Z) \tag{4.3}$$

and

$$\mathcal{M}_\varepsilon(W \times Z) = \mathcal{N}_\varepsilon(W \times Z) \tag{4.4}$$

hold for all compact sets $W \subseteq X$, $Z \subseteq Y$ and every $\varepsilon > 0$.

Proof. Let W and Z be compact sets $W \subseteq X,\ Z \subseteq Y$ and let $\varepsilon > 0$. Theorem 3.9 implies that $d = d_\infty$ if the conditions of the lemma hold. It follows from the definition of the covering numbers that

$$\mathcal{N}_\varepsilon(W \times Z) \le \mathcal{N}_\varepsilon(W) \cdot \mathcal{N}_\varepsilon(Z). \tag{4.5}$$

Indeed, if C_W and C_Z are finite ε-nets for W and Z, respectively, then the direct product $C_W \times C_Z$ is a finite ε-net for $W \times Z$ in the space $(X \times Y, d_\infty)$. Consequently, $\mathcal{N}_\varepsilon(W \times Z) \le \operatorname{card}(C_W) \cdot \operatorname{card}(C_Z)$. Using this inequality for C_W and C_Z with $\operatorname{card}(C_W) = \mathcal{N}_\varepsilon(W)$ and $\operatorname{card}(C_Z) = \mathcal{N}_\varepsilon(Z)$, we obtain (4.5). Similarly, the definition of the packing numbers implies the inequality

$$\mathcal{M}_\varepsilon(W \times Z) \ge \mathcal{M}_\varepsilon(W) \cdot \mathcal{M}_\varepsilon(Z). \tag{4.6}$$

for the subspace $W \times Z$ of the space $(X \times Y, d_\infty)$.

Statement (iii) of Theorem 2.9 gives $\mathcal{M}_\varepsilon(W) = \mathcal{N}_\varepsilon(W)$ and $\mathcal{M}_\varepsilon(Z) = \mathcal{N}_\varepsilon(Z)$ because (X, d_X) and (Y, d_Y) are ultrametric spaces. The metric $d = d_\infty$ induces the natural topology on $X \times Y$. Thus, $W \times Z$ is compact in $(X \times Y, d_\infty)$, so Theorem 2.9 (iii) also implies the equality $\mathcal{M}_\varepsilon(W \times Z) = \mathcal{N}_\varepsilon(W \times Z)$. Consequently, from (4.5) and (4.6) we obtain

$$\mathcal{N}_\varepsilon(W)\mathcal{N}_\varepsilon(Z) = \mathcal{M}_\varepsilon(W)\mathcal{M}_\varepsilon(Z) \le \mathcal{M}_\varepsilon(W \times Z) = \mathcal{N}_\varepsilon(W \times Z) \le \mathcal{N}_\varepsilon(W)\mathcal{N}_\varepsilon(Z).$$

Equations (4.2)–(4.4) are thus proved. □

Proof of Theorem 4.1. It was shown in Lemma 4.2 that (i)⇒(ii). To prove (ii)⇒(i) suppose that (4.2) holds for every $\varepsilon > 0$ and all compacts $W \subseteq X,\ Z \subset Y$ but $(X \times Y, d)$ is not ultrametric. Then, by Theorem 3.9, there are points $(x_i, y_i) \in X \times Y,\ i = 1, 2$, such that

$$\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} < d((x_1, y_1), (x_2, y_2)). \tag{4.7}$$

Write

$$W := \{x_1, x_2\}, \quad Z := \{y_1, y_2\} \tag{4.8}$$

and

$$\varepsilon := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}. \tag{4.9}$$

Then we obviously have

$$\mathcal{M}_\varepsilon(W) = \mathcal{M}_\varepsilon(Z) = 1. \tag{4.10}$$

Note also that inequality (4.7) implies that the set $\{(x_1, y_1), (x_2, y_2)\}$ is an ε-distinguishable subset of $W \times Z$ in the space $(X \times Y, d)$. Hence, the inequality $\mathcal{M}_\varepsilon(W \times Z) \ge 2$ holds. This inequality and (4.10) contradict (4.2). So, the implication (ii)⇒(i) holds. □

If d is partial distance-preserving and $d_\infty \le d$ but only one from the spaces (X, d_X) and (Y, d_Y) is ultrametric, then, generally, the metric space

$(X \times Y, d)$ may be nonultrametric even if (4.2) holds for all compact sets $W \subseteq X, Z \subseteq Y$ and every $\varepsilon > 0$.

Example 4.3. Let $X = \{x\}$ be an one-point metric space. Then X is ultrametric and for every (Y, d_Y) there is a unique partial distance-preserving metric $d = d_\infty$ on $X \times Y$, i.e., the function $(X \times Y, d) \ni (x, y) \mapsto y \in (Y, d_Y)$ is an isometry if d is partial distance-preserving. Furthermore, it is clear that every $W \subseteq X$ is either empty or one-point and

$$\mathcal{M}_\varepsilon(\emptyset) = \mathcal{M}_\varepsilon(X) - 1 = 0.$$

Hence (4.2) holds for all compact sets $W \subseteq X,\ Z \subseteq Y$ and every $\varepsilon > 0$ but $(X \times Y, d)$ is ultrametric if and only if (Y, d_Y) is ultrametric.

PROPOSITION 4.4. *Let (X, d_X) and (Y, d_Y) be metric spaces and let d be a partial distance-preserving metric on $X \times Y$ such that $d_\infty \leq d$. Then the space $(X \times Y, d)$ is ultrametric if and only if the equations*

$$\mathcal{N}_\varepsilon(W \times Z) = \mathcal{M}_\varepsilon(W \times Z) \tag{4.11}$$

and (4.2) *hold for all compact sets $W \subseteq X,\ Z \subseteq Y$ and every $\varepsilon > 0$.*

The following fact is included in the proof of Theorem 3.9.

LEMMA 4.5. *Let (X, d_X) and (Y, d_Y) be metric spaces and let d be a partial distance-preserving ultrametric on $X \times Y$. Then (X, d_X) and (Y, d_Y) are ultrametric spaces.*

Proof of Proposition 4.4. If $(X \times Y, d)$ is ultrametric, then, by Lemma 4.5, (X, d_X) and (Y, d_Y) are ultrametric. Consequently, (4.2) follows from Theorem 4.1. The set $W \times Z$ is compact if W and Z are compact. Hence, (4.11) follows from Theorem 2.9 (iii).

Now, suppose that (4.11) and (4.2) hold for all compact sets $W \subseteq Z$, $Z \subseteq Y$ and every $\varepsilon > 0$. To prove that $(X \times Y, d)$ is ultrametric, it is sufficient, by Theorem 4.1, to show that (X, d_X) and (Y, d_Y) are ultrametric spaces. By (4.11) with an one-point set W we have $\mathcal{N}_\varepsilon(Z) = \mathcal{M}_\varepsilon(Z)$ for every compact set $Z \subseteq Y$ and every $\varepsilon > 0$, because d is partial distance-preserving. Hence, by Theorem 2.9, Y is an ultrametric space. Similarly, X is an ultrametric space. □

Example 4.6 below shows that in Theorem 4.1 the packing numbers cannot be replaced by covering numbers.

Example 4.6. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be two-point metric spaces with metrics d_X, d_Y such that

$$d_X(x_1, x_2) = d_Y(y_1, y_2) = 1.$$

Let $(X \times Y, d)$ be a product of the spaces (X, d_X) and (Y, d_Y) such that d is generated by the distance-matrix from Figure 2. Then d is a partial distance-preserving and $d_\infty \le d$. Moreover, a computation shows that (4.3) holds for all $W \subseteq X$, $Z \subseteq Y$ and every $\varepsilon > 0$. Specifically, we have $\mathcal{N}_1(X \times Y) = \mathcal{N}_1(X) \cdot \mathcal{N}_1(Y)$ because $B_d((x_1, y_2), 1) \supseteq X \times Y$. Note that $(X \times Y, d)$ is not an ultrametric space if $1 < a \le 2$.

	$x_1\ y_1$	$x_1\ y_2$	$x_2\ y_1$	$x_2\ y_2$
x_1 y_1	0	1	1	a
x_1 y_2	1	0	1	1
x_2 y_1	1	1	0	1
x_2 y_2	a	1	1	0

Fig. 2. The distance-matrix of a metric space $(X \times Y, d)$ for $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Here a is an arbitrary real number from $[1, 2]$.

PROPOSITION 4.7. *Let (X, d_X) and (Y, d_Y) be ultrametric spaces and let d be a partial distance-preserving metric on $X \times Y$ such that $d_\infty \le d$. Suppose that* (4.3) *holds for all compact sets $W \subseteq Z$, $Z \subseteq Y$ and every $\varepsilon > 0$. Then*

$$\min\{d((x_1, y_1), (x_2, y_2)), d((x_2, y_1), (x_1, y_2))\} = d_\infty\{(x_1, y_1), (x_2, y_2)\} \tag{4.12}$$

for all $\{x_1, x_2\} \subseteq X$ and $\{y_1, y_2\} \subseteq Y$.

Proof. Suppose that (4.12) does not hold for some $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then by the inequality $d_\infty \le d$ we have

$$d((x_1, y_1), (x_2, y_2)) > \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \tag{4.13}$$

and

$$d((x_2, y_1), (x_1, y_2)) > \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}. \tag{4.14}$$

Write $W := \{x_1, x_2\}$, $Z := \{y_1, y_2\}$, $\varepsilon := d_\infty((x_1, x_2), (y_1, y_2))$. Then it is clear that

$$\mathcal{N}_\varepsilon(W) = \mathcal{N}_\varepsilon(Z) = 1. \tag{4.15}$$

Moreover, inequalities (4.13) and (4.14) imply that $(W \times Z) \setminus B_d((x, y), \varepsilon) \neq \emptyset$ for every $(x, y) \in X \times Y$. Consequently, we have $\mathcal{N}_\varepsilon(W \times Z) > 1$. To complete the proof, it suffices to observe that the last inequality and (4.15) contradict (4.3). $\square$

Corollary 4.8. *Let (X, d_X) and (Y, d_Y) be ultrametric spaces and let d be a partial distance-preserving metric on $X \times Y$ such that $d_\infty \leq d$. Suppose that the equation*

$$d((x_1, y_1), (x_2, y_2)) = d((x_2, y_1), (x_1, y_2)) \tag{4.16}$$

holds for all $\{x_1, x_2\} \subseteq X$ and all $\{y_1, y_2\} \subseteq Y$. Then $(X \times Y, d)$ is ultrametric if and only if (4.3) holds for all compact sets $W \subseteq X$, $Z \subseteq Y$ and every $\varepsilon > 0$.

Proof. Suppose that $(X \times Y, d)$ is ultrametric. Then (4.3) holds, see Lemma 4.2. Conversely, if (4.3) holds for all compact $W \subseteq X$, $Z \subseteq Y$ and every $\varepsilon > 0$ then, by Proposition 4.7, we have (4.12), Note that (4.12) and (4.16) imply the equality $d = d_\infty$. By Theorem 3.9, $(X \times Y, d)$ is an ultrametric space. □

Remark 4.9. If the distance function $d : (X \times Y) \times (X \times Y) \to \mathbb{R}^+$ depends only on "partial" distances d_X and d_Y, see diagram (3.7), then (4.16) evidently holds. Note that (4.16) holds for all points from the space $(X \times Y, d)$ in Example 3.7 but, in this case, there is no function F for which diagram (3.7) is commutative.

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Institute of Applied Mathematics
and Mechanics of NASU
R. Luxemburg str. 74
Donetsk 83114, Ukraine
aleksdov@mail.ru

and

University of Helsinki
Department of Mathematics and Statistics
P.O. Box 68 FI-00014, Finland
olli.martio@helsinki.fi