

RELATIVISTIC PROBABILITY WAVES

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A canonical structure compatible with the action of the Lorentz group can be obtained considering the energy and time as conjugate variables of an extended phase space. Scalar probability waves, describing free relativistic particles, are associated with functional coherent states on this space for an extended Liouville equation. Relativistic action waves are provided by distributions localized in momentum, evolving according to the continuity and Hamilton-Jacobi equations. Presuming the existence of minimum space and time intervals, the action distributions take the form of relativistic Wigner functions. Evidence for the existence of such intervals is extracted from the particle data. The nonrelativistic quantum dynamics is retrieved approximating the time distribution function by a Gaussian wave packet.

AMS 2010 Subject Classification: 70Hxx, 81Sxx, 83A05.

Key words: extended phase-space, discretization, relativistic Wigner functions.

1. INTRODUCTION

The first evidence on the granular structure of the phase-space [11] emerged from the study of the relativistic system represented by thermal radiation, within classical statistical mechanics. However, the usual Hamiltonian dynamics, of time-dependent coordinates and momenta, is not appropriate as a starting point for a Lorentz-covariant theory.

A canonical structure compatible with the action of the Lorentz group can be obtained extending the nonrelativistic phase space by energy and time, as conjugate variables [8]. This extension was found to be consistent with electromagnetism, as well as with algebraic quantization [4]. An eight-dimensional phase space also provides the framework to derive coupled Vlasov-Maxwell equations [3]. In this work, probability waves in space-time are related to specific coherent solutions of a relativistic Liouville equation.

The extended Hamiltonian dynamics of a classical particle is reviewed in Section 2. As the observable time becomes a canonical coordinate, the evolution is described in terms of a parameter called universal time. The relativistic Liouville equation for the distribution function is presented in Section 3. Similarly to the nonrelativistic treatment [5], “action waves” are provided by

coherent functionals localized in the momentum space. For such probability distributions the Liouville equation reduces to the coupled continuity and Hamilton-Jacobi equations.

The transition from action distributions on the extended phase space, to Wigner functions, when the configuration space is discretized [5], is discussed in Section 4. Unlike the fundamental length at the Planck scale ($\sim 10^{-35}\text{m}$) used in string theory [17], here the discretization length ℓ is presumed comparable to the Compton wavelength. It is shown that evidence for the relevance of such an ℓ arises from the particle data. The quantum “wave function” obtained after discretization belongs to an extended Hilbert space ([6]-Appendix), and it evolves with respect to the universal time according to a relativistic Schrödinger equation. In the “stationary” case this reduces to the Klein-Gordon equation, while the nonrelativistic limit is essentially “nonstationary”. Within the present approach, the Wigner transform for relativistic quantum systems is a quasiprobability on the extended phase-space, rather than over trajectories [10]. Conclusions are summarized in Section 5.

2. CLASSICAL DYNAMICS IN THE EXTENDED PHASE SPACE

The phase-space M of a classical system can be extended to a phase-space M^e , which includes the energy and time as conjugate variables [8]. The canonical coordinates on M^e consist of the canonical coordinates on M , denoted $\{(q_i, p_i), i = 1, n\}$, and (q_0, p_0) , supposed to be linear functions of time and energy, $q_0 = ct$, respectively $p_0 = -E/c$, where c is a dimensional constant, identified with the speed of light in vacuum [4].

Let u be the “universal time” parameter along the trajectories on M^e , $d_u \equiv d/du$ the derivative with respect to u , and

$$(1) \quad \mathcal{L}_{X_{H^e}} = \sum_{i=1}^n (d_u q_i) \partial_{q_i} + (d_u p_i) \partial_{p_i} + (d_u t) \partial_t + (d_u E) \partial_E$$

the Lie derivative $\mathcal{L}_{X_{H^e}} f \equiv -\{H^e, f\}^e$. Here $\{*, *\}^e$ and H^e are the extended Poisson bracket and Hamilton function, respectively, on M^e . In the case of a nonrelativistic system H^e can be taken of the form $H_N^e = H + cp_0$, where H is the usual Hamilton function defined on M [4]. For this expression, the corresponding equations of motion in the extended phase-space are

$$\begin{aligned} d_u q_i &= \frac{\partial H}{\partial p_i}, & d_u p_i &= -\frac{\partial H}{\partial q_i}, & i &= 1, n, \\ d_u t &= 1, & d_u E &= \frac{\partial H}{\partial t}. \end{aligned}$$

The first group of equations reproduces the usual Hamilton equations on M . The second group shows that the choice of H_N^e corresponds to $u = t$, and ensures the conservation of the energy when H is independent of time.

In the extended phase-space, the transition from Newtonian to relativistic mechanics (recalled in Appendix 1) consists essentially in a change of Hamiltonian. A free relativistic particle having $q^e \equiv (q_0, \mathbf{q})$ and $p^e \equiv (p_0, \mathbf{p})$ as phase-space coordinates on $M^e \equiv \mathbb{R}^8$, can be described by¹

$$H_0^e = -c\sqrt{p_0^2 - \mathbf{p}^2}.$$

When $p_0^2 \approx m_0^2 c^2 \gg \mathbf{p}^2$ this extended Hamiltonian reduces to the nonrelativistic expression $H_{N0}^e = \mathbf{p}^2/2m_0 + cp_0$, while in general it provides the equations of motion

$$(2) \quad d_u q_0 = -c \frac{p_0}{\sqrt{p_0^2 - \mathbf{p}^2}}, \quad d_u p_0 = 0$$

$$(3) \quad d_u \mathbf{q} = c \frac{\mathbf{p}}{\sqrt{p_0^2 - \mathbf{p}^2}}, \quad d_u \mathbf{p} = 0.$$

With respect to a particular inertial frame, the usual velocity² $\mathbf{V} = d\mathbf{q}/dt$ is the ratio $\mathbf{V} = cd_u \mathbf{q}/d_u q_0 = -c\mathbf{p}/p_0$.

The inertial parameters I_μ for H_0^e , defined by

$$\frac{1}{I_\mu} \equiv \frac{1}{p_\mu} \frac{\partial H_0^e}{\partial p_\mu}, \quad \mu = 0, 1, 2, 3$$

take the values

$$(4) \quad I_1 = I_2 = I_3 = -I_0 = -\frac{H_0^e}{c^2} = m_0,$$

as provided by the invariant value of H_0^e , denoted $-m_0 c^2$.

3. RELATIVISTIC ACTION WAVES

The statistical properties of classical systems composed of N identical relativistic particles can be described by a distribution function $f^e \geq 0$, depending on u , defined on the one-particle extended phase-space M^e . If $d\Omega_{m^e}$ denotes the volume element around the point $m^e \in M^e$, then $f^e(m^e, u)$ is normalized using the integrality condition

$$(5) \quad \int d\Omega_{m^e} f^e(m^e, u) = N, \quad N \geq 1.$$

¹In the external potential $V(\mathbf{q})$ we may consider $H^e = -c\sqrt{(p_0 + V/c)^2 - \mathbf{p}^2}$ [7].

²The standard Lagrangian of the free relativistic particle can be found in [1], p. 237.

For macroscopic systems, the probability to find a particle localized in $d\Omega_{m^e}$, proportional to $f^e(m^e, u)d\Omega_{m^e}$, is given essentially by the particle density in $d\Omega_{m^e}$, while at small N a definition in terms of the average universal time interval of localization in $d\Omega_{m^e}$ should be expected.

Let us consider a system containing a single relativistic particle ($N = 1$), with $d\Omega_{m^e} \equiv d^4q d^4p$, ($d^4p \equiv dp_0 d^3p$, $d^4q \equiv dq_0 d^3q$), and Hamiltonian $H_0^e(p^e) = -c\sqrt{p_0^2 - \mathbf{p}^2}$. The distribution function $f^e(q^e, p^e, u)$ evolves according to the relativistic Liouville equation

$$(6) \quad \partial_u f^e + \mathcal{L}_{X_{H_0^e}} f^e = 0,$$

where $\mathcal{L}_{X_{H_0^e}}$ provided by (1–3) has the form

$$\mathcal{L}_{X_{H_0^e}} = \frac{c\mathbf{p} \cdot \nabla}{\sqrt{p_0^2 - \mathbf{p}^2}} - \frac{cp_0 \partial_0}{\sqrt{p_0^2 - \mathbf{p}^2}},$$

with $\partial_0 \equiv \partial/\partial q_0$ and $\nabla \equiv \partial/\partial \mathbf{q}$. Thus, (6) becomes

$$(7) \quad \sqrt{p_0^2 - \mathbf{p}^2} \partial_u f^e - cp_0 \partial_0 f^e + c\mathbf{p} \cdot \nabla f^e = 0.$$

To find coherent solutions of this equation it is convenient to use the Fourier transform³

$$(8) \quad \tilde{f}^e(q^e, k^e, u) \equiv \int d^4p \, e^{ik_0 p_0 + i\mathbf{k} \cdot \mathbf{p}} f^e(q^e, p^e, u).$$

If $\sqrt{p_0^2 - \mathbf{p}^2}$ can be expressed in the form $m_0 c + \delta m_0 c$, where δm_0 as a function of p_0^2 and $\mathbf{p}^2$ is a power series, then by Fourier transform (7) becomes

$$(9) \quad \tilde{H}_0^e \partial_u \tilde{f}^e - ic^2 \partial_{k_0} \partial_0 \tilde{f}^e + ic^2 \nabla_{\mathbf{k}} \cdot \nabla \tilde{f}^e = 0,$$

where $\partial_{k_0} \equiv \partial/\partial k_0$, $\nabla_{\mathbf{k}} \equiv \partial/\partial \mathbf{k}$, and formally

$$(10) \quad \tilde{H}_0^e = -c\sqrt{(-i\partial_{k_0})^2 - (-i\nabla_{\mathbf{k}})^2}.$$

Various densities in space-time, such as the localization probability n^e , or current J_μ , can be expressed directly in terms of $\tilde{f}$ and its partial derivatives $\partial_{k_\mu} \equiv \partial/\partial k_\mu$, $\mu = 0, 1, 2, 3$, at $k^e = 0$ by

$$(11) \quad n^e(q^e, u) \equiv \int d^4p \, f^e(q^e, p^e, u) = \tilde{f}^e(q^e, 0, u),$$

$$(12) \quad J_\mu(q^e, u) \equiv \frac{1}{m_0} \int d^4p \, p_\mu f^e(q^e, p^e, u) = -\frac{i}{m_0} \partial_{k_\mu} \tilde{f}^e(q^e, 0, u).$$

³By the Plancherel's theorem $\tilde{f}^e$ provides the spectrum of the “momentum frequencies”, $k^e/2\pi$, $k^e \equiv (k_0, \mathbf{k})$.

In general, the mean value of an observable $\mathcal{O}(q^e, p^e)$, which is a polynomial as a function of the momentum components, has the expression

$$\langle \mathcal{O} \rangle(u) \equiv \int d^4q d^4p \mathcal{O} f^e(u) = \int d^4q \mathcal{O}(q^e, -i\partial_{k^e}) \tilde{f}^e(q^e, 0, u).$$

A particular class of coherent solutions for the relativistic Liouville equation (7) consists of the “action distributions”

$$(13) \quad f_0^e(q^e, p^e, u) = n^e(q^e, u) \delta(p_0 - \partial_0 S) \delta(\mathbf{p} - \nabla S),$$

where n^e is the localization probability density in space-time, and $S(q^e, u)$ is the generating function of the Hamilton-Jacobi theory. By Fourier transform (13) becomes

$$(14) \quad \tilde{f}_0^e = n^e e^{ik_0 \partial_0 S + i\mathbf{k} \cdot \nabla S}$$

while (9) reduces to the system of equations

$$(15) \quad \partial_u [n^e \sqrt{(\partial_0 S)^2 - (\nabla S)^2}] = c \partial_0 (n^e \partial_0 S) - c \nabla \cdot (n^e \nabla S)$$

and

$$n^e \partial_\mu \mathcal{J} = 0, \quad \mathcal{J} = \partial_u S - c \sqrt{(\partial_0 S)^2 - (\nabla S)^2},$$

where $\partial_\mu \equiv \partial/\partial q_\mu$, $\mu = 0, 1, 2, 3$. Thus, the solution $\mathcal{J} = 0$ is nothing but the Hamilton-Jacobi equation in the extended phase-space,

$$(16) \quad \partial_u S + H_0^e(\partial_0 S, \nabla S) = 0.$$

Considering $\partial_u S = m_0 c^2$, (16) takes the form

$$(\partial_0 S)^2 - (\nabla S)^2 = m_0^2 c^2,$$

and (15) becomes the continuity equation

$$(17) \quad m_0 \partial_u n^e = \partial_0 (n^e \partial_0 S) - \nabla \cdot (n^e \nabla S).$$

In terms of the density (11), the mean value of the time coordinate is

$$\langle t \rangle = \frac{1}{c} \int d^4q \, q_0 \, n^e,$$

so that

$$(18) \quad d_u \langle t \rangle = \frac{1}{c} \int d^4q \, q_0 \, \partial_u n^e.$$

Let us presume that $m_0 \neq 0$ and n^e is limited in time, confined to a finite volume $\mathcal{V}$ in space, ∇S vanishing along the normal to the boundary of $\mathcal{V}$. In this case (17, 18) yield

$$(19) \quad d_u \langle t \rangle = -\frac{1}{m_0 c} \int d^4q \int_{\mathcal{V}} d^3q \, n^e \partial_0 S = \frac{\langle E \rangle}{m_0 c^2},$$

where $\langle E \rangle$ is the mean value of the energy. Because $\langle E \rangle$ is a positive constant⁴, (19) shows that $\langle t \rangle$ and u are in the linear relationship

$$(20) \quad \langle t \rangle = \frac{\langle E \rangle}{m_0 c^2} u.$$

The localization to a finite domain in space-time makes possible to define (up to a translation) an “intrinsic frame” (IF), as the frame selected by the condition $\langle \mathbf{p} \rangle_{\text{IF}} = 0$. Expressed in terms of the intrinsic expectation values, (20) shows that the universal time u corresponds up to the factor $m_0 c^2 / \langle E \rangle_{\text{IF}}$ to the mean time in the intrinsic frame, $\langle t \rangle_{\text{IF}}$. Thus, if m_0 can be defined as $\langle E \rangle_{\text{IF}} / c^2$, then $u = \langle t \rangle_{\text{IF}}$. For a density $n^e(q^e, u) = \delta(q_0 - cu)n(\mathbf{q}, u)$, localized in time, (17) reduces in the nonrelativistic limit to the usual continuity equation $\delta(t - u)[m_0 \partial_u n + \nabla \cdot (n \nabla S)] = 0$. The result shows that the usual density n , integrable over $\mathbb{R}^3$, depending on time as a parameter, is recovered when the extended density n^e is concentrated on a specific “leaf of present” from a regular foliation $\{\Lambda_u \subset \mathbb{R}^4, T_{q^e} \Lambda_u = \mathbb{R}^3 / q^e \equiv (q_0 = cu, \mathbf{q} \in \mathbb{R}^3)\}_{u \in \mathbb{R}}$ parameterized by u of the space-time manifold.

4. THE RELATIVISTIC SCHRÖDINGER EQUATION

For the free particle $S(q^e, u) \equiv \sum_{\mu=0}^3 S_{\mu}(q_{\mu}, u)$ is separable, and if the partial derivatives $\partial_{\mu} S$ in (14) are written as finite differences $[S_{\mu}(q_{\mu} + \ell_{\mu}/2, u) - S_{\mu}(q_{\mu} - \ell_{\mu}/2, u)]/\ell_{\mu}$, then at $k^e \neq 0$ the action distribution becomes

$$(21) \quad \tilde{f}_0^e(q^e, k^e, u) = \lim_{\sigma_{\mu} \rightarrow 0} \Psi(q_+^e, u) \Psi^*(q_-^e, u), \quad (q_{\pm}^e)_{\mu} = q_{\mu} \pm \frac{\sigma_{\mu} k_{\mu}}{2},$$

where σ_{μ} denotes the ratio $\sigma_{\mu} \equiv \ell_{\mu}/k_{\mu}$ and $\Psi(q^e, u)$ is the complex function $\Psi = \sqrt{n^e} \exp(i \sum_{\mu=0}^3 S_{\mu}/\sigma_{\mu})$. However, when $k^e \rightarrow 0$ (21) and the related densities (11), (12) are either undefined, or depend critically on the manner in which ℓ_{μ} and k_{μ} are correlated. If the limit of σ_{μ} when both ℓ_{μ} and k_{μ} decrease to zero is finite, having the same value σ for all components, then we may consider also for nonseparable S the “quantum distributions”

$$(22) \quad \tilde{f}_{\Psi}^e(q^e, k^e, u) \equiv \Psi(q_+^e, u) \Psi^*(q_-^e, u),$$

with $\Psi = \sqrt{n^e} \exp(iS/\sigma)$, as possible functional coherent states for (9). In this case, the normalization condition (5) for the corresponding “quasi-probability” distribution f_{Ψ}^e takes the form

$$\int d^4 q d^4 p f_{\Psi}^e(q^e, p^e, u) = \int d^4 q |\Psi(q^e, u)|^2 = 1,$$

⁴ As required to ensure the Lorentz action (44) [4].

and the phase-space overlap between two distributions $f_{\Psi_1}^e, f_{\Psi_2}^e$ is

$$\langle f_{\Psi_1}^e, f_{\Psi_2}^e \rangle \equiv \int d^4q d^4p f_{\Psi_1}^e f_{\Psi_2}^e = \frac{|\langle \Psi_1 | \Psi_2 \rangle|^2}{(2\pi\sigma)^4},$$

where

$$(23) \quad \langle \Psi_1(u_1) | \Psi_2(u_2) \rangle \equiv \int d^4q \Psi_1^*(q^e, u_1) \Psi_2(q^e, u_2).$$

A linear relationship $\ell_\mu = \sigma k_\mu$ with a finite, isotropic, Lorentz-invariant, universal phase-space element σ could be related in principle to the existence of minimum space and time intervals for a certain energy domain, but is more difficult to justify than in the nonrelativistic case [5]. In electrodynamics we can find limits such as the classical electron radius $r_e = \alpha \hbar / m_e c = 2.8$ fm, and the related cutoff energy $e^2 / 4\pi\epsilon_0 r_e = m_e c^2$. In general, we can note that in the intrinsic frame, after a cutoff at $\approx 3m_0 c^2$ of the energy range (Appendix 2), f_Ψ^e still remains unchanged in 94% of the velocity domain $[0, c)$. With this approximation, the inverse of (8) can be replaced by a multiple Fourier series in which the components of k^e from the factor $\exp(-ik_0 p_0 - i\mathbf{k} \cdot \mathbf{p})$ take an infinite set of discrete values separated by $\kappa = \pi / 3m_0 c$. Also, if the time distribution has the variance δt_0^2 , then the intervals of ordered, physical time (e.g., between measurements [12], or the lifetime $\tau_L = \hbar / \Gamma$ for unstable particles) should be greater than δt_0 , and the length between any two fixed endpoints along a trajectory parametrized by $\langle t \rangle$, greater than $\ell = c \delta t_0$. Thus, a finite value $\sigma = \ell / \kappa \sim m_0 c^2 \delta t_0$ should be expected. For a quantum particle, with $\sigma = \hbar$ we get $\delta t_0 \sim \hbar / m_0 c^2$ and $\ell \sim \hbar / m_0 c$, proportional to the inverse of the mass.

It is interesting to remark that beside the formal arguments, evidence for the physical relevance of the interval $\hbar / m_0 c^2$ arises from the particle data. The values obtained for the ratio $m_0 c^2 / \Gamma \sim \tau_L / \delta t_0$ between the mass (in MeV) and decay width (Γ), using the experimental data [16] for meson and baryon resonances are represented in Figure 1, (A) and (B), respectively. These values are well interpolated by functions of the form $C_1 + C_2 / \Gamma$, where (C_1, C_2) are (2.7, 1147 MeV) for mesons and (2.8, 1427 MeV) for baryons. Thus, τ_L appears to be limited below by $2\hbar / m_0 c^2$.

Denoting $\tilde{f}_\Psi^e \equiv (\hat{U}\Psi)(\hat{U}^{-1}\Psi^*)$, with $\hat{U} = \exp[\sigma(k_0\partial_0 + \mathbf{k} \cdot \nabla)/2]$, (9) becomes

$$(24) \quad \tilde{H}_0^e \partial_u \tilde{f}_\Psi^e = i \frac{\sigma c^2}{2} [(\hat{U}\square\Psi)(\hat{U}^{-1}\Psi^*) - (\hat{U}\Psi)(\hat{U}^{-1}\square\Psi^*)],$$

where $\square \equiv \partial_0^2 - \nabla^2$.

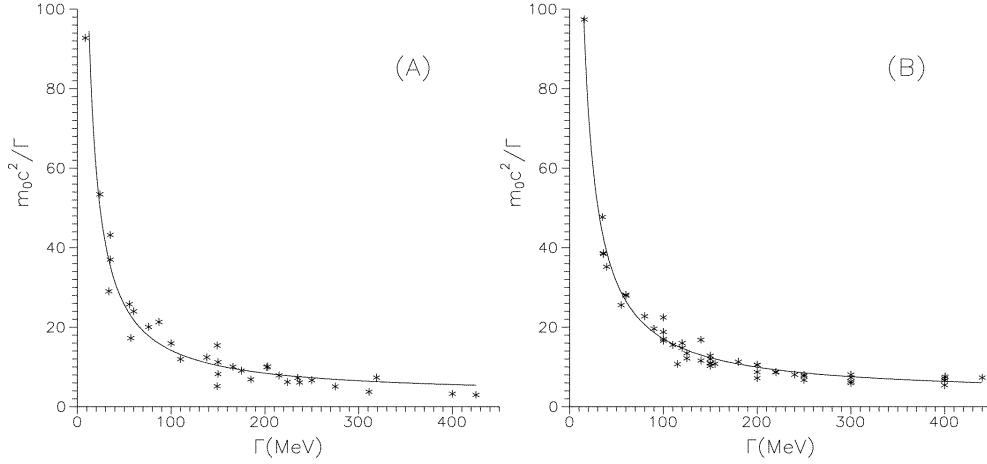


Fig. 1. $m_0 c^2 / \Gamma$ from experimental data (*), for 32 light unflavored meson resonances ($\omega, \eta, \phi, \pi, \rho, a, b, f$) with $\Gamma \geq 8.43$ MeV (A) and 48 baryon resonances ($N, \Delta, \Lambda, \Sigma$) with $\Gamma \geq 15.6$ MeV (B). The interpolation functions $C_1 + C_2/\Gamma$ are represented by solid lines.

A “static” distribution $\partial_u \tilde{f}_\Psi^e = 0$ is obtained if $\square \Psi = a \Psi$, where a is a real constant. This constant can be estimated by using the expectation values of H_0^e or $(H_0^e)^2$. For simplicity, $\langle (H_0^e)^2 \rangle = m_0^2 c^4$ means

$$(25) \quad \int d^4 q d^4 p (p_0^2 - \mathbf{p}^2 - m_0^2 c^2) f_\Psi^e(q^e, p^e, u) = 0$$

or

$$(26) \quad \int d^4 q [(\hat{\mathcal{K}} \Psi) \Psi^* + \Psi (\hat{\mathcal{K}} \Psi^*)] = 0,$$

where $\hat{\mathcal{K}} = -\sigma^2 \square - m_0^2 c^2$. When $\square \Psi = a \Psi$, (26) yields $a \sigma^2 = -m_0^2 c^2$, so that $\hat{\mathcal{K}} \Psi = 0$, or

$$(27) \quad -\sigma^2 \square \Psi = m_0^2 c^2 \Psi.$$

In the quantum theory this represents the Klein-Gordon equation [2]. Although all particles described by (27) are unstable, closer to stability are quark-antiquark systems like the $\pi^\pm$ and $K^\pm$ mesons⁵ with a lifetime ~ 10 ns, much larger than $\hbar/m_0 c^2 \sim 10^{-24}$ s.

In the nonstationary case (24) becomes

$$\begin{aligned} \tilde{H}_0^e [(\hat{U} i \partial_u \Psi)(\hat{U}^{-1} \Psi^*) - (\hat{U} \Psi) \hat{U}^{-1} (i \partial_u \Psi)^*] = \\ = -\frac{\sigma c^2}{2} [(\hat{U} \square \Psi)(\hat{U}^{-1} \Psi^*) - (\hat{U} \Psi)(\hat{U}^{-1} \square \Psi^*)], \end{aligned}$$

⁵The singlet (triplet) states of $e^- e^+$ positronium have a lifetime of 1.2 ns (140 ns).

or

$$(28) \quad \tilde{H}_0^e \Psi_-^* i \partial_u \Psi_+ = -\frac{\sigma c^2}{2} \Psi_-^* (\square - a) \Psi_+,$$

$$(29) \quad \tilde{H}_0^e \Psi_+ i \partial_u \Psi_-^* = \frac{\sigma c^2}{2} \Psi_+ (\square - a) \Psi_-^*,$$

with $\Psi_+ \equiv \hat{U} \Psi$ and $\Psi_-^* \equiv \hat{U}^{-1} \Psi^*$. By contrast to the nonrelativistic case [5], in general (28–29) cannot be reduced to separate equations for Ψ_+ and Ψ_-^* due to $\tilde{H}_0^e$, which acts on both functions. However, Ψ_+ and Ψ_-^* become complex conjugate at $k^e = 0$, so that when $\sigma = \hbar$, the limit

$$(30) \quad \lim_{k^e \rightarrow 0} (\Psi_-^*)^{-1} \tilde{H}_0^e \Psi_-^* \hat{U} i \partial_u \Psi = -\frac{\hbar c^2}{2} \left(\square + \frac{m_0^2 c^2}{\hbar^2} \right) \Psi,$$

can be formally considered as a relativistic Schrödinger equation for the wave function Ψ . Nonstationary solutions of this equation correspond for instance to wave-packets of the form

$$(31) \quad \Psi(q^e, u) = \chi(q_0, u) \psi(\mathbf{q}, u)$$

where $\chi(q_0, u) \equiv \chi_{\mathcal{Q}_0, \mathcal{P}_0}(q_0, u)$ is a Glauber coherent state [6]

$$(32) \quad \chi(q_0, u) = \sqrt{\frac{\Omega}{c\sqrt{\pi}}} e^{-\Omega^2(q_0 - \mathcal{Q}_0)^2/2c^2 + i\mathcal{P}_0(q_0 - \mathcal{Q}_0/2)/\sigma},$$

with the centroid at $\mathcal{Q}_0 = -u\mathcal{P}_0/m_0$ and variance $c^2/2\Omega^2$. The parameters $\mathcal{P}_0, \mathcal{Q}_0$ are related to the energy and time expectation values $\langle E \rangle, \langle t \rangle$ by the relations $\mathcal{P}_0 \equiv \langle p_0 \rangle = -\langle E \rangle/c$, respectively $\mathcal{Q}_0 \equiv \langle q_0 \rangle = c\langle t \rangle$.

The function $f_\Psi^e(q^e, p^e, u)$ defined by $\tilde{f}_\Psi^e(q^e, k^e, u)$, inverting (8), is⁶

$$f_\Psi^e(q^e, p^e, u) = \frac{1}{\pi\sigma} e^{-\Omega^2(q_0 - \mathcal{Q}_0)^2/c^2 - c^2(p_0 - \mathcal{P}_0)^2/\Omega^2\sigma^2} f_\psi(\mathbf{q}, \mathbf{p}, u),$$

where $f_\psi(\mathbf{q}, \mathbf{p}, u)$ is the usual Wigner transform of $\psi(\mathbf{q}, u)$. Thus, the time variance $\delta t^2 \equiv \langle t^2 \rangle - \langle t \rangle^2 = 1/2\Omega^2$, as well as the energy variance $\delta E^2 = c^2 \delta p_0^2$, $\delta p_0^2 \equiv \langle p_0^2 \rangle - \langle p_0 \rangle^2 = \sigma^2 \Omega^2/2c^2$, are both finite and satisfy the uncertainty relation $\delta E \delta t = \sigma/2$.

With (32), the dependence on k_0 in $\tilde{f}_\Psi^e(q^e, k^e, u)$ can be separated in

$$\tilde{g}_0(k_0) = e^{-\delta p_0^2 k_0^2/2 + i\mathcal{P}_0 k_0},$$

so that

$$\tilde{f}_\Psi^e(q^e, k^e, u) = \tilde{g}_0(k_0) |\chi(q_0, u)|^2 (\hat{U}_{\mathbf{k}} \psi)(\hat{U}_{\mathbf{k}}^{-1} \psi^*),$$

⁶ Because f_Ψ^e remains finite for $|p_0| < m_0 c$, (32) should be regarded as an approximation.

and (28) becomes

$$(33) \quad \tilde{H}_0^e \mathcal{F}_k \chi^* \hat{U}_{\mathbf{k}} i \partial_u \Psi = -\frac{\sigma c^2}{2} \mathcal{F}_k \chi^* \hat{U}_{\mathbf{k}} (\square - a) \Psi.$$

Here $\hat{U}_{\mathbf{k}} = e^{\sigma \mathbf{k} \cdot \nabla / 2}$, while $\mathcal{F}_k$ denotes the function $\mathcal{F}_k(\mathbf{q}, u) = \tilde{g}_0(k_0) \hat{U}_{\mathbf{k}}^{-1} \psi^*(\mathbf{q}, u)$.

For nonrelativistic particles we can expect that ψ evolves over a time-scale much larger than $1/\Omega$, and approximate solutions of (33) can be obtained by taking the average over q_0 . Using the equalities $\partial_u \Psi = (\partial_u \chi) \psi + \chi(\partial_u \psi)$,

$$\int dq_0 \chi^*(q_0, u) \partial_u \chi(q_0, u) = -\frac{i \mathcal{P}_0}{2\sigma} d_u \mathcal{Q}_0 = \frac{i \mathcal{P}_0^2}{2\sigma m_0},$$

and

$$(34) \quad \langle p_0^2 \rangle = \int dq_0 \chi^*(q_0, u) (-\sigma^2 \partial_0^2) \chi(q_0, u) = \mathcal{P}_0^2 + \delta p_0^2,$$

the integration over q_0 in both sides of (33) yields

$$(35) \quad \tilde{H}_0^e \mathcal{F}_k \hat{U}_{\mathbf{k}} \left(i\sigma \partial_u \psi - \frac{\mathcal{P}_0^2}{2m_0} \psi \right) = \frac{c^2}{2} \mathcal{F}_k \hat{U}_{\mathbf{k}} (\langle p_0^2 \rangle + \sigma^2 \nabla^2 - m_0^2 c^2) \psi.$$

In general, $\tilde{H}_0^e \mathcal{F}_k \hat{U}_{\mathbf{k}}$ is a complicated operator because $\tilde{H}_0^e$ of (10) introduces mixed partial derivatives, acting both on $\tilde{g}_0(k_0)$ and $\hat{U}_{\mathbf{k}}$. However, in the limit $k^e \rightarrow 0$

$$\tilde{H}_0^e \tilde{g}_0(k_0) \approx -c \sqrt{\langle p_0^2 \rangle + \nabla_{\mathbf{k}}^2} \tilde{g}_0(k_0),$$

where, according to the constraint (25), $\langle p_0^2 \rangle = m_0^2 c^2 + \langle \mathbf{p}^2 \rangle$. Moreover, because

$$\lim_{k^e \rightarrow 0} \int d^3 q (\langle \mathbf{p}^2 \rangle + \nabla_{\mathbf{k}}^2) \mathcal{F}_k \hat{U}_{\mathbf{k}} \psi = \langle \mathbf{p}^2 \rangle + \sigma^2 \int d^3 q \psi^* \nabla^2 \psi = 0,$$

we approximate $\tilde{H}_0^e \mathcal{F}_k \hat{U}_{\mathbf{k}} \approx -m_0 c^2 \mathcal{F}_k \hat{U}_{\mathbf{k}}$, so that as $k^e \rightarrow 0$ (35) becomes

$$(36) \quad i\sigma \partial_u \psi - \frac{\mathcal{P}_0^2}{2m_0} \psi = \frac{1}{2m_0} (m_0^2 c^2 - \langle p_0^2 \rangle - \sigma^2 \nabla^2) \psi.$$

Using (34), this reduces further to

$$i\sigma \partial_u \psi = \frac{1}{2m_0} (m_0^2 c^2 - \delta p_0^2 - \sigma^2 \nabla^2) \psi.$$

The constant $(m_0^2 c^2 - \delta p_0^2)/2m_0$ can be included in a global u -dependent phase-factor of ψ , while by changing the parametrization to the mean time $\langle t \rangle = \mathcal{Q}_0/c$, one obtains

$$(37) \quad i\sigma \partial_{\langle t \rangle} \psi = \frac{c}{2\mathcal{P}_0} \sigma^2 \nabla^2 \psi.$$

According to (25) and (34), $\mathcal{P}_0 = -\sqrt{m_x^2 c^2 + \langle \mathbf{p}^2 \rangle}$ with $m_x = \sqrt{m_0^2 - \delta p_0^2/c^2}$ the effective mass, and for $\langle \mathbf{p}^2 \rangle \ll m_x^2 c^2$, (37) is well approximated by

$$i\sigma\partial_{\langle t \rangle}\psi = -\frac{\sigma^2\nabla^2}{2m_x}\left(1 - \frac{\langle \mathbf{p}^2 \rangle}{2m_x^2 c^2}\right)\psi.$$

In the case of an atomic electron ($\sigma = \hbar$, $m_x = m_e$, 1 a.u. = $\alpha^2 m_e c^2$), the correction term $\langle H_c \rangle = \hbar^2 \langle \mathbf{p}^2 \rangle \langle \nabla^2 \rangle / 4m_e^3 c^2$ can be compared with the usual contribution due to the variation of mass with velocity, $\langle H'_1 \rangle = -\langle p^4 \rangle / 8m_e^3 c^2$ [9]. For the ground state of hydrogen, $\langle H_c \rangle = -\alpha^2/4$ a.u., while $\langle H'_1 \rangle = -5\alpha^2/8$ a.u.. The whole correction to this order found by expanding in powers of α the exact solution of the Dirac equation is $-\alpha^2/8$ a.u. [9].

The interval $2\delta t = \sqrt{2}/\Omega \equiv \hbar/\delta E$ is a measure of the time shift $|\mathcal{Q}'_0 - \mathcal{Q}_0|/c$ for which the overlap $|\langle \chi_{\mathcal{Q}'_0, \mathcal{P}_0} | \chi_{\mathcal{Q}_0, \mathcal{P}_0} \rangle|^2$ between two states (32), and the corresponding transition amplitude (23), remain significant. In general, this is much larger than $\delta t_0 \sim \hbar/m_0 c^2$. For instance, if we take $\delta E \approx \epsilon_r m_x c^2$, where $\epsilon_r \equiv \delta m_x/m_x = 0.3 \cdot 10^{-6}$ is the relative standard deviation at the measurement of the electron and proton mass, then $\delta t \sim 1.6 \cdot 10^6 \delta t_0$. In the case of the electron, $\delta t_0 = \ell/c \sim 10^{-21}$ s is comparable to the estimates of the "jump time" $\tau_J \sim 10^{-20}$ s for atomic transitions [13]. These change however the electron wave function over a distance larger than $10^3 \ell$, so that δt could be a reasonable upper limit for τ_J .

5. SUMMARY AND CONCLUSIONS

The phase-space description of the physical states provides the geometrical framework for the nonrelativistic many-body theory, statistical mechanics and canonical quantization [1]. The asymmetry between time and the usual phase-space coordinates requires though "the second quantization", to obtain a consistent Lorentz-covariant quantum theory.

For classical relativistic systems, we may also extended the usual phase-space by energy and time, as canonical variables. In this work, scalar probability waves, describing free particles, are associated with functional coherent states for the Liouville equation (7) in such an extended phase-space.

The canonical equations of motion for a relativistic particle are recalled in Section 2. As the usual time becomes a coordinate, the trajectories are parameterized by a variable u called universal time. Action waves $n^e(q^e, u)^{[S]}$, associated with the functional coherent solutions (13) of (7), have been discussed in Section 3. These are localized in the momentum space, and propagate according to the continuity and Hamilton-Jacobi relativistic equations (15, 16). It is shown that in a finite system the universal time is proportional to the expectation value of the time coordinate in the intrinsic frame.

The transition from $n^e(q^e, u)^{[S]}$ to the quantum waves $\Psi(q^e, u)$ was studied in Section 4. Thus, presuming the existence of minimum space and time intervals, the action distribution (14) takes the form of the relativistic quantum distribution (22). In such a case, the corresponding functional coherent solutions of the Liouville equation arise by an extended Wigner transform of the wave functions provided by the relativistic Schrödinger equation (30). For an ideal “stationary” system, this reduces to the Klein-Gordon equation (27). Most physical situations are though nonstationary, as all mesons undergo irreversible decay, while in the nonrelativistic quantum theory time is the same as in classical mechanics. When time is described quasiclassically, by a Gaussian wave-packet, then over large intervals compared to the width, the extended formalism reduces to the usual nonrelativistic quantum dynamics.

6. APPENDIX 1: GALILEI AND LORENTZ ACTIONS

Let us consider a particle of mass m , described by the Cartesian phase-space coordinates $(\mathbf{q}, \mathbf{p})$. An infinitesimal Galilei transformation $\Gamma_Q : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$, acting both on the coordinate space ($Q = \mathbb{R}^3$) and time ($\mathbb{R}$), is defined by $[\mathbf{q}', t'] = [\mathbf{q}, t] + \gamma_0(\xi, \mathbf{d}, \mathbf{v}, \tau)[\mathbf{q}, t]$, where

$$(38) \quad \gamma_0(\xi, \mathbf{d}, \mathbf{v}, \tau)[\mathbf{q}, t] = [\xi \mathbf{q} - \mathbf{d} - t\mathbf{v}, -\tau].$$

The algebra $\mathfrak{g}$ of the Galilei group is isomorphic to $\mathfrak{so}(3) + \mathbb{R}^7$, and $\gamma_0 \in \mathfrak{g}$ is specified by $\xi \in \mathfrak{so}(3)$, $\mathbf{d} \in \mathbb{R}^3$, $\mathbf{v} \in \mathbb{R}^3$ and $\tau \in \mathbb{R}$. The elements ξ , $\mathbf{d}$ and $\mathbf{v}$ correspond to static rotations, translations and boost, respectively, of the space coordinates, while τ describes translations along the time axis.

The action Γ_Q of the Galilei group can be lifted to an action Γ_M on the phase-space $M = T^*\mathbb{R}^3$, by assuming that at the transformation specified by (38), the momentum also changes to

$$(39) \quad \mathbf{p}' = \mathbf{p} + \xi \mathbf{p} - m\mathbf{v}.$$

However, as the boost transformations depend on time explicitly, and

$$(40) \quad p'_0 = p_0 + \mathbf{v} \cdot \mathbf{p}/c,$$

($p_0 = -E/c$), Γ_Q can be lifted directly to an action Γ_{M^e} on the extended phase-space M^e of Section 2. If the coordinates on M^e are represented as column vectors

$$\tilde{q} = \begin{bmatrix} \mathbf{q} \\ q_0 \end{bmatrix}, \quad \tilde{p} = \begin{bmatrix} \mathbf{p} \\ p_0 \end{bmatrix}$$

then the infinitesimal transformation Γ_{M^e} defined by (38), (39) and (40) takes the form

$$(41) \quad \begin{bmatrix} \tilde{q}' \\ \tilde{p}' \end{bmatrix} = \begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix} + \begin{bmatrix} -\tilde{Y} \\ -\tilde{X} \end{bmatrix} + \begin{bmatrix} -\hat{a}^T & \hat{c} \\ -\hat{b} & \hat{a} \end{bmatrix} \begin{bmatrix} \tilde{q} \\ \tilde{p} \end{bmatrix},$$

where

$$(42) \quad \tilde{X} = \begin{bmatrix} m\mathbf{v} \\ 0 \end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix} \mathbf{d} \\ \tau \end{bmatrix}, \quad \hat{a} = \begin{bmatrix} \xi & \mathbf{0} \\ \mathbf{v}/c & 0 \end{bmatrix},$$

and $\hat{b}, \hat{c}$ are 4×4 zero matrices.

The mass m , introduced with the lift (39) is the positive, isotropic, inertial parameter for a Hamiltonian H defined on M . In the case of a Hamiltonian H^e defined on M^e , there is also an inertial parameter specified by the dependence of H^e on the additional momentum component (p_0). Following [4], this new inertial parameter is taken for simplicity as $\pm m$, with $+$ or $-$ sign in the isotropic, respectively quasi-isotropic case. This yields a relationship of the form $p_0 = \pm mc$, or $E = \mp mc^2$, which shows that the lift (41) of Γ_Q should be obtained by placing the velocity $\mathbf{v}$ from (39) in the matrix $\hat{a}$, instead of $\tilde{X}$. Thus, (42) is replaced by

$$(43) \quad \tilde{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix} \mathbf{d} \\ \tau \end{bmatrix}, \quad \hat{a} = \begin{bmatrix} \xi & \mp \mathbf{v}/c \\ \mathbf{v}/c & 0 \end{bmatrix}.$$

According to (41), the new element in $\hat{a}$ changes the Galilei action (38) of the inertial equivalence group, and in the case $E = mc^2$ (the Einstein formula), it provides the Lorentz action

$$(44) \quad \gamma_L(\xi, \mathbf{d}, \mathbf{v}, \tau)[\mathbf{q}, t] = [\xi \mathbf{q} - \mathbf{d} - t\mathbf{v}, -\mathbf{v} \cdot \mathbf{q}/c^2 - \tau].$$

To find the action of the Lorentz group, (44) should be integrated to finite transformations. Let us presume that $\xi = 0$, $\mathbf{d} = 0$, $\tau = 0$, and decompose $\mathbf{q}$, $\mathbf{p}$ with respect to the versor $\mathbf{n}$ of the boost velocity as $\mathbf{q} = \mathbf{q}_\perp + q_\parallel \mathbf{n}$, $\mathbf{p} = \mathbf{p}_\perp + p_\parallel \mathbf{n}$. In the representation

$$\tilde{q} = \begin{bmatrix} \mathbf{q}_\perp \\ q_\parallel \\ q_0 \end{bmatrix}, \quad \tilde{p} = \begin{bmatrix} \mathbf{p}_\perp \\ p_\parallel \\ p_0 \end{bmatrix},$$

we get $\hat{a} = \rho \hat{a}_0$, where $\rho \equiv |\mathbf{v}|/c$,

$$\hat{a}_0 = \begin{bmatrix} \hat{0}_\perp & \hat{0} \\ \hat{0} & \hat{\sigma}_x \end{bmatrix},$$

$\hat{0}_\perp = \hat{0}$ are 2×2 zero matrices, and

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Because

$$e^{\rho \hat{\sigma}_x} = \cosh \rho \hat{1} + \sinh \rho \hat{\sigma}_x,$$

for a boost transformation with the finite velocity $\mathbf{V} = V\mathbf{n}$ the equations

$$\frac{d\tilde{q}}{d\rho} = -\hat{a}_0^T \tilde{q}, \quad \frac{d\tilde{p}}{d\rho} = \hat{a}_0 \tilde{p}$$

can be integrated to $\mathbf{q}'_{\perp} = \mathbf{q}_{\perp}$, $\mathbf{p}'_{\perp} = \mathbf{p}_{\perp}$, and

$$\begin{aligned} q'_{\parallel} &= \cosh \rho q_{\parallel} - \sinh \rho q_0, & q'_0 &= \cosh \rho q_0 - \sinh \rho q_{\parallel} \\ p'_{\parallel} &= \cosh \rho p_{\parallel} + \sinh \rho p_0, & p'_0 &= \cosh \rho p_0 + \sinh \rho p_{\parallel}. \end{aligned}$$

These expressions show clearly the invariance of the Poisson bracket in the extended phase-space, because if $\{q_{\mu}, q_{\nu}\}^e = \{p_{\mu}, p_{\nu}\}^e = 0$, $\{q_{\mu}, p_{\nu}\}^e = \delta_{\mu\nu}$, then also $\{q'_{\mu}, q'_{\nu}\}^e = \{p'_{\mu}, p'_{\nu}\}^e = 0$, $\{q'_{\mu}, p'_{\nu}\}^e = \delta_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$.

The parameter ρ is related to the finite boost velocity V by physical considerations, such as $V = cdq_{\parallel}/dq_0$ when $dq'_{\parallel} = 0$. The result $V = c \tanh \rho$ provides the standard Lorentz transformations

$$\begin{aligned} \mathbf{q}'_{\parallel} &= \frac{\mathbf{q}_{\parallel} - \mathbf{V}t}{\sqrt{1 - V^2/c^2}}, & t' &= \frac{t - \mathbf{V} \cdot \mathbf{q}/c^2}{\sqrt{1 - V^2/c^2}}, \\ \mathbf{p}'_{\parallel} &= \frac{\mathbf{p}_{\parallel} - \mathbf{V}E/c^2}{\sqrt{1 - V^2/c^2}}, & E' &= \frac{E - \mathbf{V} \cdot \mathbf{p}}{\sqrt{1 - V^2/c^2}}. \end{aligned}$$

For states with negative energy ($E = -mc^2$), $\mp \mathbf{v}$ in (43) takes the $-$ sign, the hyperbolic functions become trigonometric functions, and the restricted Lorentz group $\text{SO}^*(1, 3)$ is replaced by the rotation group in space-time $\text{SO}(4)$ [4], isomorphic to $\text{SU}(2) \times \text{SU}(2)$.

APPENDIX 2: THE RELATIVISTIC PERFECT GAS

For a nondegenerate gas of fermions with energy $\epsilon_p = \sqrt{\mathbf{p}^2 c^2 + m_0^2 c^4}$ at equilibrium, the usual distribution function has the form [15]

$$(45) \quad f_{\mu_c, T}(\mathbf{p}) = \frac{2}{h^3} e^{(\mu_c - \epsilon_p)/k_B T},$$

so that if $\mathcal{V}$ denotes the confinement volume, then

$$N = \mathcal{V} \int d^3p f_{\mu_c, T}(\mathbf{p}), \quad E = \mathcal{V} \int d^3p \epsilon_p f_{\mu_c, T}(\mathbf{p})$$

are the number of particles, and the total energy, respectively. The function (45) is also a stationary solution of the classical Fokker-Planck equation

$$\partial_t f + \frac{1}{m} \mathbf{p} \cdot \nabla f = \gamma \nabla_p \cdot \left(\frac{\mathbf{p}}{m} + k_B T \nabla_p \right) f,$$

with $m = \epsilon_p/c^2$. The energy can be expressed in the form

$$E = \frac{8\pi\mathcal{V}}{h^3c^3} e^{\mu_c/k_B T} \int_{m_0c^2}^{\infty} d\epsilon g_T(\epsilon),$$

where $g_T(\epsilon) = \epsilon^2 \sqrt{\epsilon^2 - m_0^2 c^4} \exp(-\epsilon/k_B T)$. Here the upper integration limit ϵ_M was presumed infinite, although in most physical situations particles with high enough energy can escape the system before thermalization. Moreover, the opening of pair creation reaction channels [14] at $\epsilon = 3m_0c^2, 5m_0c^2, \dots$ also affects the distribution. Therefore, a reasonable limit of the energy range for a stationary distribution with a well-defined number of particles is $\epsilon_M \approx 3m_0c^2$, which corresponds to the maximum of $g_T(\epsilon)$ at $T = m_0c^2/k_B$, when the old sound velocity formula $v_s = \sqrt{k_B T/m_0}$ yields $v_s = c$.

As a real function $f(X)$ defined on the finite domain $[-\Delta, \Delta]$ can be represented in the form

$$f(X) = \frac{1}{2\Delta} \sum_{n=-\infty}^{\infty} e^{-in\pi X/\Delta} \tilde{f}_n,$$

with

$$\tilde{f}_n = \int_{-\Delta}^{\Delta} dX e^{in\pi X/\Delta} f(X),$$

by the simple limitation of the energy range, the inverse of (8) can be expressed in terms of a multiple Fourier series.

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Received 26 August 2009

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