DIFFERENTIAL SANDWICH THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH SRIVASTAVA-ATTIYA OPERATOR

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The main purpose of the present paper is to investigate some subordination-preserving and superordination results involving Hadamard product for certain analytic functions preserving properties of a certain family of integral operators. Relevant connections of the results are remarked and the new results are also pointed out.

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1. INTRODUCTION

Let $\mathbb{H}$ be the class of analytic functions in $\mathbb{U} := \{z : |z| < 1\}$ and $\mathbb{H}\,[a, n]$ be the subclass of $\mathbb{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathbb{A}$ be the subclass of $\mathbb{H}$ consisting of functions of the form

$$f(z) = z + a_2 z^2 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For two functions $f(z) \in \mathbb{A}$ and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.2}$$

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We recall here a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined in [32] by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$(a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\};\ s \in \mathbb{C},\ \text{when } |z|<1;\ \Re(s)>1 \text{ and } |z|=1),$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\})$; $\mathbb{N} := \{1,2,3,\ldots\}$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [10], Ferreira and Lopez [12], Garg et al. [14], Lin and Srivastava [16], Lin et al. [17], and others. Srivastava and Attiya [31] introduced and investigated the linear operator: $\mathcal{J}_{s,b} : \mathcal{A} \to \mathcal{A}$ defined in terms of the Hadamard product by

$$\mathcal{J}_{s,b} f(z) = \mathcal{G}_{s,b} * f(z) \tag{1.4}$$

$$(z \in U;\ b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\};\ s \in \mathbb{C};\ f \in \mathcal{A}),\ \text{where, for convenience,}$$

$$G_{s,b}(z) := (1+b)^s[\Phi(z,s,b) - b^{-s}] \quad (z \in U). \tag{1.5}$$

We recall here the following relationships which follow easily by using (1.4) and (1.5)

$$\mathcal{J}_b^s f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^s a_n z^n. \tag{1.6}$$

Motivated essentially by the Srivastava-Attiya operator, Al-Shaqsi and Darus [1] introduced and investigated the integral operator

$$\mathcal{J}_{s,b}^{\lambda,\mu} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^s \frac{\lambda!(n+\mu-2)!}{(\mu-2)!(n+\lambda-1)!} a_n z^n \quad (z \in \mathbb{U}), \tag{1.7}$$

where (and throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}$; $s \in \mathbb{C}$, $\mu \geq 0$ and $\lambda > -1$. Further note that $J_{s,b}^{1,2}$ is the Srivastava-Attiya operator, and $J_{0,b}^{\lambda,\mu}$ is the well-known Choi-Saigo-Srivastava operator [9].

Assuming $\lambda = 1$ and $\mu = 2$, we state the following integral operators by specializing s and b.

(1) For $s = 0$

$$\mathcal{J}_b^0(f)(z) := f(z). \tag{1.8}$$

(2) For $s = 1$ and $b = 0$

$$\mathcal{J}_0^1(f)(z) := \int_0^z \frac{f(t)}{t} \mathrm{d}t := \mathcal{L}f(z). \tag{1.9}$$

(3) For $s=1$ and $b=\nu$ $(\nu>-1)$

$$\mathcal{J}_{\nu}^{1}(f)(z):=\mathcal{F}_{\nu}(f)(z)=\frac{1+\nu}{z^{\nu}}\int_{0}^{z}t^{\nu-1}f(t)\mathrm{d}t:=z+\sum_{n=2}^{\infty}\left(\frac{1+\nu}{n+\nu}\right)a_n z^n. \tag{1.10}$$

(4) For $s=\sigma$ $(\sigma>0)$ and $b=1$

$$\mathcal{J}_{1}^{\sigma}(f)(z):=z+\sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^{\sigma}a_n z^n=\mathcal{I}^{\sigma}(f)(z), \tag{1.11}$$

where $\mathcal{L}(f)$ and $\mathcal{F}_{\nu}$ are the familiar Alexander and Bernardi integral operators respectively, and $\mathcal{I}^{\sigma}(f)$ is the Jung-Kim-Srivastava integral operator [15] closely related to some multiplier transformation studied by Flett [13]. It is easy to verify from (1.7),

$$z\left(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)\right)'(z)=(\lambda+1)\mathcal{J}_{s,b}^{\lambda,\mu}f(z)-\lambda\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z), \tag{1.12}$$

$$z\left(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)\right)'(z)=(b+1)\mathcal{J}_{s,b}^{\lambda,\mu}f(z)-b\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z), \tag{1.13}$$

$$z\left(\mathcal{J}_{s,b}^{\lambda,\mu}f(z)\right)'(z)=\mu\mathcal{J}_{s,b}^{\lambda,\mu+1}f(z)-(\mu-1)\mathcal{J}_{s,b}^{\lambda,\mu}f(z). \tag{1.14}$$

Let $p,h\in\mathbb{H}$ and let $\phi(r,s,t;z):\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$. If p and $\phi\big(p(z),zp'(z),z^2p''(z);z\big)$ are univalent and if p satisfies the second order superordination

$$h(z)\prec\phi(p(z),zp'(z),z^2p''(z);z), \tag{1.15}$$

then p is a solution of the differential superordination (1.15). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q\prec p$ for all p satisfying (1.15). A univalent subordinant $\widetilde{q}$ that satisfies $q\prec\widetilde{q}$ for all subordinants q of (1.15) is said to be the best subordinant. Recently, Miller and Mocanu [21] obtained conditions on h, q and ϕ for which the following implication holds

$$h(z)\prec\phi(p(z),zp'(z),z^2p''(z);z)\Rightarrow q(z)\prec p(z).$$

In [5] Bulboacă (see also [4]) considered certain classes of first order differential superordinations as well as superordination-preserving integral operators by using the results of Miller and Mocanu [21]. Further, using the results in [4] and [21], Aouf and Mostafa [2], Magesh et al. [18, 19], Mostafa and Aouf [22], Murugusundaramoorthy and Magesh [23, 24, 25], and Selvaraj and Karthikeyan [30] have obtained sandwich results for certain classes of analytic functions.

Making use of the operator $\mathcal{J}_{s,b}^{\lambda,\mu}$, in this paper we find sufficient condition for certain normalized analytic functions $f(z)$ in $\mathbb{U}$ such that $(f*\Psi)(z)\neq 0$

and f to satisfy

$$q_1(z) \prec \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z} \right)^{\eta} \prec q_2(z), \tag{1.16}$$

where q_1, q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = 1$, $q_2(0) = 1$ and $\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ are analytic functions in $\mathbb{U}$ with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$. Also, we obtain the number of known results as their special cases.

2. BASIC DEFINITIONS AND LEMMAS

For our present investigation, we shall need the following:

Lemma 2.1 ([27, p. 159, Theorem 6.2]). *The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = +\infty$, is a subordination chain if*

$$\operatorname{Re}\left\{ z \frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0, \quad z \in \mathbb{U},\ t \geq 0.$$

Definition 2.1 ([21, p. 817, Definition 2]). Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - \mathbb{E}(f)$, where

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} - \mathbb{E}(f)$.

Lemma 2.2 ([20, p. 132, Theorem 3.4h]). *Let q be univalent in the unit disk $\mathbb{U}$ and θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + Q(z).$$

Suppose that

(1) $Q(z)$ is starlike univalent in $\mathbb{U}$ and

(2) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \mathbb{U}$.

If p is analytic with $p(0) = q(0)$, $p(\mathbb{U}) \subseteq \mathbb{D}$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

LEMMA 2.3 ([5, p. 289, Corollary 3.2]). *Let q be convex univalent in the unit disk $\mathbb{U}$ and ϑ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$. Suppose that*

(1) $\operatorname{Re}\{\vartheta'(q(z))/\varphi(q(z))\} > 0$ *for* $z \in \mathbb{U}$ *and*

(2) $\psi(z) = zq'(z)\varphi(q(z))$ *is starlike univalent in* $\mathbb{U}$.

If $p(z) \in \mathbb{H}[q(0), 1] \cap Q$, *with* $p(\mathbb{U}) \subseteq \mathbb{D}$, *and* $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathbb{U}$ *and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

3. SUBORDINATION RESULTS

Using Lemma 2.2, we first prove the following theorem.

THEOREM 3.1. *Let* $\Phi, \Psi \in \mathbb{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ *such that* $\eta \neq 0$ *and* $\alpha + \beta \neq 0$, *and* q *be convex univalent with* $q(0) = 1$, *and assume that*

$$\operatorname{Re}\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{3.1}$$

If $f \in \mathbb{A}$ *satisfies*

(3.2)

$$\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) = \Delta(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)},$$

where

(3.3)

$$\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) :=$$

$$\begin{cases} \gamma_1 + \gamma_2 \left(\frac{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta\mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{2\eta} \\ +\gamma_3 \left(\frac{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta\mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\eta} \\ +\gamma_4\eta \left(\frac{\alpha(\mu+1)[\mathcal{J}_{s,b}^{\lambda,\mu+2}(f*\Phi)(z) - \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z)] + \beta\mu[\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Psi)(z) - \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)]}{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta\mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}\right), \end{cases}$$

then

$$\left(\frac{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f * \Phi)(z) + \beta\mathcal{J}_{s,b}^{\lambda,\mu}(f * \Psi)(z)}{(\alpha + \beta)z}\right)^{\eta} \prec q(z)$$

and q *is the best dominant.*

Proof. Define the function p by

$$p(z) := \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f * \Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f * \Psi)(z)}{(\alpha+\beta)z} \right)^{\eta} \quad (z \in \mathbb{U}). \tag{3.4}$$

Then, the function p is analytic in $\mathbb{U}$ and $p(0) = 1$. Therefore, by making use of (3.4) and (1.12), we obtain

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)} = \tag{3.5}$$

$$\begin{cases} \gamma_1 + \gamma_2 \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z} \right)^{2\eta} + \gamma_3 \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z} \right)^{\eta} \\ + \gamma_4 \eta \left(\frac{\alpha(\mu+1)[\mathcal{J}_{s,b}^{\lambda,\mu+2}(f*\Phi)(z) - \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z)] + \beta\mu[\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Psi)(z) - \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)]}{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)} \right). \end{cases}$$

By using (3.5) in (3.2), we have

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)} \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}. \tag{3.6}$$

By setting

$$\theta(w) := \gamma_1 + \gamma_2 \omega^2(z) + \gamma_3 \omega \quad \text{and} \quad \phi(\omega) := \frac{\gamma_4}{w},$$

it can be easily observed that $\theta(w)$, $\phi(w)$ are analytic in $\mathbb{C} - \{0\}$ and $\phi(w) \neq 0$. Also, we see that

$$Q(z) := zq'(z)\phi(q(z)) = \gamma_4 \frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z)) + Q(z) = \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $\mathbb{U}$ and

$$\operatorname{Re}\left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re}\left\{ \frac{\gamma_3}{\gamma_4} q(z) + \frac{2\gamma_2}{\gamma_4} q^2(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)} \right\} > 0.$$

By the hypothesis of Theorem 3.1, the result now follows by an application of Lemma 2.2. □

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let* $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ *such that* $\eta \neq 0$ *and* $\alpha + \beta \neq 0$, *and* q *be convex univalent with* $q(0) = 1$, *and* (3.1) *holds* *true. If $f \in \mathbb{A}$ satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z} \right)^{2\eta} + \gamma_3 \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z} \right)^{\eta}$$
$$+\gamma_4 \eta \left(\frac{\alpha(\mu+1)[\mathcal{J}_{s,b}^{\lambda,\mu+2} f(z) - \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z)] + \beta\mu[\mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) - \mathcal{J}_{s,b}^{\lambda,\mu} f(z)]}{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)} \right)$$
$$\prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z} \right)^{\eta} \prec q(z)$$

and q is the best dominant.

By taking $s = 0$ in Corollary 3.1, we state the following corollary.

Corollary 3.2. *Let $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ such that $\eta \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and* (3.1) *holds true. If $f \in \mathbb{A}$ satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{\alpha \mathcal{I}_{\lambda}^{\mu+1} f(z) + \beta \mathcal{I}_{\lambda}^{\mu} f(z)}{(\alpha+\beta)z} \right)^{2\mu} + \gamma_3 \left(\frac{\alpha \mathcal{I}_{\lambda}^{\mu+1} f(z) + \beta \mathcal{I}_{\lambda}^{\mu} f(z)}{(\alpha+\beta)z} \right)^{\eta}$$
$$+\gamma_4 \eta \left(\frac{\alpha(\mu+1)[\mathcal{I}_{\lambda}^{\mu+2} f(z) - \mathcal{I}_{\lambda}^{\mu+1} f(z)] + \beta\mu[\mathcal{I}_{\lambda}^{\mu+1} f(z) - \mathcal{I}_{\lambda}^{\mu} f(z)]}{\alpha \mathcal{I}_{\lambda}^{\mu+1} f(z) + \beta \mathcal{I}_{\lambda}^{\mu} f(z)} \right)$$
$$\prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{\alpha \mathcal{I}_{\lambda}^{\mu+1} f(z) + \beta \mathcal{I}_{\lambda}^{\mu} f(z)}{(\alpha+\beta)z} \right)^{\eta} \prec q(z),$$

where $\mathcal{I}_{\lambda}^{\mu} f(z)$ is the Choi-Saigo-Srivastava operator [9] *and q is the best dominant.*

By taking $s = \alpha = 0$, $\lambda = 1$, $\mu = 2$ and $\beta = 1$ in Theorem 3.1, we state the following corollary.

Corollary 3.3. *Let $\Psi \in \mathbb{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)$ $(\gamma_4 \neq 0)$, $\eta \in \mathcal{C}$ such that $\eta \neq 0$ and q be convex univalent with $q(0) = 1$, and* (3.1) *holds true. If*

$f \in \mathbb{A}$ *satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{(f * \Psi)(z)}{z}\right)^{2\eta} + \gamma_3 \left(\frac{(f * \Psi)(z)}{z}\right)^{\eta} + \gamma_4 \eta \left(\frac{z(f * \Psi)'(z)}{(f * \Psi)(z)} - 1\right)$$
$$\prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$\left(\frac{(f * \Psi)(z)}{z}\right)^{\eta} \prec q(z)$$

and q *is the best dominant.*

By fixing $\Psi(z) = \frac{z}{1-z}$ in Corollary 3.3, we obtain the following corollary.

Corollary 3.4. *Let* $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)$ $(\gamma_4 \neq 0)$, $\eta \in \mathcal{C}$ *such that* $\eta \neq 0$ *and* q *be convex univalent with* $q(0) = 1$, *and* (3.1) *holds true. If* $f \in \mathbb{A}$ *satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{f(z)}{z}\right)^{2\eta} + \gamma_3 \left(\frac{f(z)}{z}\right)^{\eta} + \gamma_4 \eta \left(\frac{zf'(z)}{f(z)} - 1\right)$$
$$\prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$\left(\frac{f(z)}{z}\right)^{\eta} \prec q(z)$$

and q *is the best dominant.*

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.1, we have the following corollary.

Corollary 3.5. *Let* $\Phi, \Psi \in \mathbb{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ *such that* $\eta \neq 0$ *and* $\alpha + \beta \neq 0$, *and* q *be convex univalent with* $q(0) = 1$. *Assume that*

$$\mathrm{Re}\left\{\frac{\gamma_3}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right) + \frac{2\gamma_2}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{1-ABz^2}{(1+Az)(1+Bz)}\right\} > 0.$$

If $f \in \mathbb{A}$ *and*

$$\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) \prec \gamma_1 + \gamma_2 \left(\frac{1+Az}{1+Bz}\right)^2 + \gamma_3 \frac{1+Az}{1+Bz} + \gamma_4 \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f * \Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f * \Psi)(z)}{(\alpha+\beta)z}\right)^{\eta} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

By putting $\gamma_1 = \gamma_4 = \beta = 1$, $\alpha = \gamma_2 = \gamma_3 = 0$, $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ and $q(z) = (1 + Bz)^{\eta(A-B)/B}$ in Theorem 3.1, we state the following corollary.

Corollary 3.6. *If $f \in \mathbb{A}$ satisfies*

$$1 + \eta \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1 + Az}{1 + Bz},$$

then

$$\left(\frac{f(z)}{z} \right)^{\eta} \prec (1 + Bz)^{\eta(A-B)/B}$$

and $(1 + Bz)^{\eta(A-B)/B}$ is the best dominant.

4. SUPERORDINATION RESULTS

Now, by applying Lemma 2.3, we prove the following theorem.

Theorem 4.1. *Let $\Phi, \Psi \in \mathbb{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ such that $\eta \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and assume that*

$$\operatorname{Re}\left\{ \frac{\gamma_3}{\gamma_4} q(z) + \frac{2\gamma_2}{\gamma_4} q^2(z) \right\} \geq 0. \tag{4.1}$$

If $f \in \mathbb{A}$, $\left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z} \right)^{\eta} \in \mathbb{H}[q(0), 1] \cap Q$. Let $\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi)$ be univalent in $\mathbb{U}$ and

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)} \prec \Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi), \tag{4.2}$$

where $\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi)$ is given by (3.3), then

$$q(z) \prec \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f * \Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f * \Psi)(z)}{(\alpha + \beta)z} \right)^{\eta}$$

and q is the best subordinant.

Proof. Define the function p by

$$p(z) := \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f * \Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f * \Psi)(z)}{(\alpha + \beta)z} \right)^{\eta}. \tag{4.3}$$

Simple computation from (4.3), we get,

$$\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) = \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)}.$$

By setting $\vartheta(w) = \gamma_1 + \gamma_2 w^2 + \gamma_3 w$ and $\phi(w) = \frac{\gamma_4}{w}$, it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\phi(w)$ is analytic in $\mathbb{C} - \{0\}$ and $\phi(w) \neq 0$.

If we let

$$\begin{aligned} L(z,t) &= \vartheta(q(z)) + \phi(q(z))tzq'(z) = \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 t \frac{zq'(z)}{q(z)} \\ &= a_1(t)z + \cdots . \end{aligned} \tag{4.4}$$

Differentiating (4.4) with respect to z and t, we have

$$\begin{aligned} \frac{\partial L(z,t)}{\partial z} &= 2\gamma_2 q(z)q'(z) + \gamma_3 q'(z) + t\gamma_4 \left[\frac{zq''(z)}{q(z)} + \frac{q'(z)}{q(z)} - z\left(\frac{q'(z)}{q(z)}\right)^2 \right] \\ &= a_1(t) + \dots \end{aligned}$$

and

$$\frac{\partial L(z,t)}{\partial t} = \gamma_4 \frac{zq'(z)}{q(z)}.$$

Also,

$$\frac{\partial L(0,t)}{\partial z} = \gamma_4 q'(0) \left[\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4} q(0) + t\frac{1}{q(0)} \right].$$

From the univalence of q we have $q'(0) \neq 0$ and $q(0) = 1$, it follows that $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = +\infty$.

A simple computation yields,

$$\operatorname{Re}\left\{ z \frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} = \operatorname{Re}\left\{ \frac{\gamma_3}{\gamma_4} q(z) + \frac{2\gamma_2}{\gamma_4} q^2(z) + t\left(1 + \frac{zq''(z)}{q'(z)} - zq'(z)\right) \right\}.$$

Using the fact that q is convex univalent function in $\mathbb{U}$ and $\gamma_4 \neq 0$, we have,

$$\operatorname{Re}\left\{ z \frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0 \text{ if } \Re\left\{ \frac{\gamma_3}{\gamma_4} q(z) + \frac{2\gamma_2}{\gamma_4} q^2(z) \right\} > 0, \quad z \in \mathbb{U},\ t \geq 0.$$

Now, Theorem 4.1 follows by applying Lemma 2.3. $\square$

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 4.1, we obtain the following corollary.

Corollary 4.1. *Let* $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ *such that* $\eta \neq 0$ *and* $\alpha + \beta \neq 0$, *and* q *be convex univalent with* $q(0) = 1$, *and* (4.1) *hold true. If* $f \in \mathbb{A}$, $\left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z}\right)^{\eta} \in \mathbb{H}[q(0), 1] \cap Q$. *Let*

$$\gamma_1 + \gamma_2 \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z}\right)^{2\mu} + \gamma_3 \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z}\right)^{\eta}$$
$$+\gamma_4 \eta \left(\frac{\alpha(\mu+1)[\mathcal{J}_{s,b}^{\lambda,\mu+2} f(z) - \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z)] + \beta\mu[\mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) - \mathcal{J}_{s,b}^{\lambda,\mu} f(z)]}{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}\right)$$

be univalent in $\mathbb{U}$ *and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}$$
$$\prec \gamma_1 + \gamma_2 \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z}\right)^{2\mu} + \gamma_3 \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z}\right)^{\eta}$$
$$+\gamma_4 \eta \left(\frac{\alpha(\mu+1)[\mathcal{J}_{s,b}^{\lambda,\mu+2} f(z) - \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z)] + \beta\mu[\mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) - \mathcal{J}_{s,b}^{\lambda,\mu} f(z)]}{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}\right),$$

then

$$q(z) \prec \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1} f(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu} f(z)}{(\alpha+\beta)z}\right)^{\eta}$$

and q *is the best subordinant.*

By taking $s = 0$ in Corollary 4.1, we state the following corollary.

Corollary 4.2. *Let* $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ *such that* $\eta \neq 0$ *and* $\alpha + \beta \neq 0$, *and* q *be convex univalent with* $q(0) = 1$, *and* (4.1) *hold true. If* $f \in \mathbb{A}$, $\left(\frac{\alpha \mathcal{I}_{\lambda}^{\mu+1} f(z) + \beta \mathcal{I}_{\lambda}^{\mu} f(z)}{(\alpha+\beta)z}\right)^{\eta} \in \mathbb{H}[q(0), 1] \cap Q$. *Let*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)} \gamma_1$$
$$+\gamma_2 \left(\frac{\alpha \mathcal{I}_{\lambda}^{\mu+1} f(z) + \beta \mathcal{I}_{\lambda}^{\mu} f(z)}{(\alpha+\beta)z}\right)^{2\mu} + \gamma_3 \left(\frac{\alpha \mathcal{I}_{\lambda}^{\mu+1} f(z) + \beta \mathcal{I}_{\lambda}^{\mu} f(z)}{(\alpha+\beta)z}\right)^{\eta}$$
$$+\gamma_4 \eta \left(\frac{\alpha(\mu+1)[\mathcal{I}_{\lambda}^{\mu+2} f(z) - \mathcal{I}_{\lambda}^{\mu+1} f(z)] + \beta\mu[\mathcal{I}_{\lambda}^{\mu+1} f(z) - \mathcal{I}_{\lambda}^{\mu} f(z)]}{\alpha \mathcal{I}_{\lambda}^{\mu+1} f(z) + \beta \mathcal{I}_{\lambda}^{\mu} f(z)}\right)$$

be univalent in $\mathbb{U}$ *and*

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4\frac{zq'(z)}{q(z)}$$
$$\prec \gamma_1+\gamma_2\left(\frac{\alpha\mathcal{I}_\lambda^{\mu+1}f(z)+\beta\mathcal{I}_\lambda^{\mu}f(z)}{(\alpha+\beta)z}\right)^{2\mu}+\gamma_3\left(\frac{\alpha\mathcal{I}_\lambda^{\mu+1}f(z)+\beta\mathcal{I}_\lambda^{\mu}f(z)}{(\alpha+\beta)z}\right)^{\eta}$$
$$+\gamma_4\eta\left(\frac{\alpha(\mu+1)[\mathcal{I}_\lambda^{\mu+2}f(z)-\mathcal{I}_\lambda^{\mu+1}f(z)]+\beta\mu[\mathcal{I}_\lambda^{\mu+1}f(z)-\mathcal{I}_\lambda^{\mu}f(z)]}{\alpha\mathcal{I}_\lambda^{\mu+1}f(z)+\beta\mathcal{I}_\lambda^{\mu}f(z)}\right),$$

then

$$q(z)\prec\left(\frac{\alpha\mathcal{I}_\lambda^{\mu+1}f(z)+\beta\mathcal{I}_\lambda^{\mu}f(z)}{(\alpha+\beta)z}\right)^{\eta}$$

where $\mathcal{I}_\lambda^{\eta}f(z)$ *is the Choi-Saigo-Srivastava operator* [9] *and* q *is the best subordinant.*

By setting $s=\alpha=0$, $\lambda=1$, $\mu=2$ and $\beta=1$ in Theorem 4.1, we derive the following corollary.

Corollary 4.3. *Let* $\Psi\in\mathbb{A}$, $\gamma_i\in\mathbb{C}$ $(i=1,\ldots,4)$ $(\gamma_4\neq 0)$, $\eta\in\mathcal{C}$ *such that* $\eta\neq 0$ *and* q *be convex univalent with* $q(0)=1$, *and* (4.1) *hold true. If* $f\in\mathbb{A}$, $\left(\frac{(f*\Psi)(z)}{z}\right)^{\eta}\in\mathbb{H}[q(0),1]\cap Q$. *Let* $\gamma_1+\gamma_2\left(\frac{(f*\Psi)(z)}{z}\right)^{2\eta}+\gamma_3\left(\frac{(f*\Psi)(z)}{z}\right)^{\eta}+\gamma_4\eta\left(\frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}-1\right)$ *be univalent in* $\mathbb{U}$ *and*

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4\frac{zq'(z)}{q(z)}$$
$$\prec \gamma_1+\gamma_2\left(\frac{(f*\Psi)(z)}{z}\right)^{2\eta}+\gamma_3\left(\frac{(f*\Psi)(z)}{z}\right)^{\eta}+\gamma_4\eta\left(\frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}-1\right)$$

then

$$q(z)\prec\left(\frac{(f*\Psi)(z)}{z}\right)^{\eta}$$

and q *is the best subordinant.*

By putting $\Psi(z)=\frac{z}{1-z}$, in Corollary 4.3, we obtain the following corollary.

Corollary 4.4. *Let* $\gamma_i\in\mathbb{C}$ $(i=1,\ldots,4)$ $(\gamma_4\neq 0)$, $0\neq\eta\in\mathcal{C}$ *and* q *be convex univalent with* $q(0)=1$, *and* (4.1) *hold true. If* $f\in\mathbb{A}$, $\left(\frac{f(z)}{z}\right)^{\eta}\in\mathbb{H}[q(0),1]\cap Q$. *Let* $\gamma_1+\gamma_2\left(\frac{f(z)}{z}\right)^{2\eta}+\gamma_3\left(\frac{f(z)}{z}\right)^{\eta}+\gamma_4\eta\left(\frac{zf'(z)}{f(z)}-1\right)$ *be univalent* *in* $\mathbb{U}$ *and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}$$
$$\prec \gamma_1 + \gamma_2 \left(\frac{f(z)}{z}\right)^{2\eta} + \gamma_3 \left(\frac{f(z)}{z}\right)^{\eta} + \gamma_4 \eta \left(\frac{zf'(z)}{f(z)} - 1\right)$$

then

$$q(z) \prec \left(\frac{f(z)}{z}\right)^{\eta}$$

and q *is the best subordinant.*

By taking $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 4.1, we obtain the following corollary.

Corollary 4.5. *Let* $\Phi, \Psi \in \mathbb{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ *such that* $\eta \neq 0$ *and* $\alpha + \beta \neq 0$, *and* q *be convex univalent with* $q(0) = 1$, *and* $\mathrm{Re}\left\{\frac{\gamma_3}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right) + \frac{2\gamma_2}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right)^2\right\} > 0$. *If* $f \in \mathbb{A}$, $\left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\eta}$ $\in \mathbb{H}[q(0), 1] \cap Q$. *Let* $\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi)$ *be univalent in* $\mathbb{U}$ *and*

$$\gamma_1 + \gamma_2 \left(\frac{1 + Az}{1 + Bz}\right)^2 + \gamma_3 \frac{1 + Az}{1 + Bz} + \gamma_4 \frac{(A - B)z}{(1 + Az)(1 + Bz)} \prec \Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f * \Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f * \Psi)(z)}{(\alpha + \beta)z}\right)^{\eta}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

5. SANDWICH RESULTS

There is a complete analog of Theorem 3.1 for differential subordination and Theorem 4.1 for differential superordination. We can combine the results of Theorem 3.1 with Theorem 4.1 and obtain the following sandwich theorem.

Theorem 5.1. *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{U}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 4)$ $(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ *such that* $\eta \neq 0$ *and* $\alpha + \beta \neq 0$, *and let* q_2 *satisfy* (3.1) *and* q_1 *satisfy* (4.1). *For* $f, \Phi, \Psi \in \mathbb{A}$, *let* $\left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\eta}$

$\in \mathbb{H}[1,1] \cap Q$ *and* $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *defined by* (3.3) *be univalent in* $\mathbb{U}$ *satisfying*

$$\gamma_1+\gamma_2 q_1^2(z)+\gamma_3 q_1(z)+\gamma_4\frac{zq_1'(z)}{q_1(z)}$$
$$\prec \Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)\prec \gamma_1+\gamma_2 q_2^2(z)+\gamma_3 q_2(z)+\gamma_4\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z)\prec\left(\frac{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z)+\beta\mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\eta}\prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By taking $q_1(z)=\frac{1+A_1z}{1+B_1z}$ $(-1\le B_1<A_1\le 1)$ and $q_2(z)=\frac{1+A_2z}{1+B_2z}$ $(-1\le B_2<A_2\le 1)$ in Theorem 5.1 we obtain the following result.

Corollary 5.1. *For* $f,\Phi,\Psi\in\mathbb{A}$, *let* $\left(\frac{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z)+\beta\mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\eta}\in$ $\mathbb{H}[1,1]\cap Q$ *and* $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *defined by* (3.3) *be univalent in* $\mathbb{U}$ *satisfying*

$$\gamma_1+\gamma_2\left(\frac{1+A_1z}{1+B_1z}\right)^2+\gamma_3\frac{1+A_1z}{1+B_1z}+\gamma_4\frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)}$$
$$\prec\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)\prec\gamma_1+\gamma_2\left(\frac{1+A_2z}{1+B_2z}\right)^2+\gamma_3\frac{1+A_2z}{1+B_2z}+\gamma_4\frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)}$$

then

$$\frac{1+A_1z}{1+B_1z}\prec\left(\frac{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z)+\beta\mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\eta}\prec\frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$, $\frac{1+A_2z}{1+B_2z}$ *are respectively the best subordinant and best dominant.*

We remark that, one can easily restated Theorem 5.1 for the different choices of $\Phi(z)$, $\Psi(z)$, λ, α, β, μ, s, b, γ_1, γ_2, γ_3 and γ_4.

Remark 5.2. (1) Setting $\mu=2$, $\beta=\lambda=\gamma_1=1$, $s=\gamma_2=\gamma_3=\alpha=0$, $\Phi(z)=\frac{z}{1-z}\Psi(z)=\frac{z}{(1-z)^2}$, $q(z)=\frac{1}{(1-z)^{2ab}}$ $(b\in\mathcal{C}-\{0\})$, $\eta=a$ and $\gamma_4=\frac{1}{b}$ in Theorem 3.1, we get the result obtained by Obradovič et al., [26, Theorem 1].

(2) Setting $\mu=2$, $\beta=\lambda=\gamma_1=1$, $s=\gamma_2=\gamma_3=\alpha=0$, $\Phi(z)=\Psi(z)=\frac{z}{1-z}$, $q(z)=\frac{1}{(1-z)^{2b}}$ $(b\in\mathcal{C}-\{0\})$, $\eta=1$ and $\gamma_4=\frac{1}{b}$ in Theorem 3.1 and then combining this together with Lemma 2.2, we obtain the result of Srivastava and Lashin [33, Theorem 3].

(3) Taking $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=0$, $\alpha=0$, $\beta=1$, $\Phi(z)=\frac{z}{1-z}$, $\gamma_4=\frac{e^{i\lambda}}{ab\cos\lambda}$ $(a,b\in\mathbb{C},|\lambda|<\frac{\pi}{2})$, $\eta=a$ and $q(z)=(1-z)^{-2ab\cos\lambda e^{-i\lambda}}$ in Corollary 3.3, we obtain the result of Aouf et al. [3, Theorem 1].

(4) For $\mu = 2$, $\beta = \lambda = 1$, $s = \alpha = 0$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty}[1 + (n - 1)\lambda]^m \frac{(b)_{n-1}}{(c)_{n-1}} a_n z^n$, all the results in [30] are special cases of our results.

(5) For $\mu = 2$, $\beta = \lambda = 1$, $s = \alpha = 0$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty}[1 + (n-1)\lambda]^m b_n z^n$, all the results in [2] are particular cases of our results.

6. **CONCLUSION**

We conclude this paper by remarking that in view of the function class defined by the subordination relation (1.16) and expressed in terms of the convolution (1.2) involving arbitrary coefficients, the main results would lead to additional new results. In fact, by appropriately selecting the arbitrary sequences ($\Phi(z)$ and $\Psi(z)$), and specializing the parameters η, λ, b, s, α, β, μ, $\gamma_1, \gamma_2, \gamma_3$ and γ_4 and the function $q(z)$ the results presented in this paper would find further applications for the classes which incorporate generalized forms of linear operators illustrated below:

(1)

$$\Phi(z) = z + \sum_{n=2}^{\infty} \frac{(\mu)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \frac{z^n}{(n-1)!}$$

and

$$\Psi(z) = z + \sum_{n=2}^{\infty} n \frac{(\mu)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \frac{z^n}{(n-1)!},$$

where $\mu, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ $(l, m \in \mathcal{N} = 1, 2, 3, \dots)$ are complex parameters $\beta_j \notin \{0, -1, -2, \dots\}$ for $j = 1, 2, \dots m$, $l \le m+1$ (results for Dziok-Srivastava operator [11]);

(2) $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \frac{z^n}{(n-1)!}$ and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} n \frac{(a)_{n-1}}{(c)_{n-1}} \frac{z^n}{(n-1)!}$, where $c \neq 0, -1, -2, \dots$ (results for Carlson and Shaffer operator [6]);

(3) $\Phi(z) = \frac{z}{(1-z)^{\lambda+1}}, \lambda \ge -1$, and $\Psi(z) = \frac{z}{(1-z)^{\lambda+2}}, \lambda \ge -1$ (results for Ruscheweyh derivative operator [28]);

(4) $\Phi(z) = z + \sum\limits_{n=2}^{\infty} n^k z^n$, and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} n^{k+1} z^n$, $k \ge 0$ (results for Salagean operator [29]);

(5) $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} n \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, where $\lambda \ge 0$, $k \in \mathcal{Z}$ (results for Multiplier transformation [7, 8]),

in Theorem 3.1, Theorem 4.1 and Theorem 5.1 would eventually lead further new results. These considerations can fruitfully be worked out and we skip the details in this regard.

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