COMPACT DIFFERENCE OF WEIGHTED COMPOSITION OPERATORS ON $\mathcal{N}_p$-SPACES IN THE BALL

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We obtain necessary and sufficient conditions for the compactness of differences of weighted composition operators acting from $\mathcal{N}_p$-spaces into the Beurling/Bergman-type spaces A^{-q} of holomorphic functions in the unit ball.

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1. INTRODUCTION

1.1. Notation and definitions

Let $\mathbb{B}$ be the unit ball in the complex vector space $\mathbb{C}^n$, $\mathcal{O}(\mathbb{B})$ denotes the space of functions that are holomorphic in $\mathbb{B}$, with the compact-open topology, and $H^\infty(\mathbb{B})$ denotes the Banach space of bounded holomorphic functions on $\mathbb{B}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$.

If $z = (z_1, z_2, \dots, z_n), \zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{C}^n$, then $\langle z, \zeta\rangle = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n$ and $|z| = (z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)^{1/2}$.

Let $p > 0$, the Beurling-type space (sometime also called the Bergman-type space) $A^{-p}(\mathbb{B})$ in the unit ball is defined as

$$A^{-p}(\mathbb{B}) := \left\{ f(z) \in \mathcal{O}(\mathbb{B}) : |f|_p = \sup_{z\in\mathbb{B}} |f(z)|(1-|z|^2)^p < \infty \right\}.$$

For a holomorphic self-mapping φ of $\mathbb{B}$ and a holomorphic function $u : \mathbb{B} \to \mathbb{C}$, the linear operator $W_{u,\varphi} : \mathcal{O}(\mathbb{B}) \to \mathcal{O}(\mathbb{B})$

$$W_{u,\varphi}(f)(z) = u(z) \cdot (f \circ \varphi(z)), f \in \mathcal{O}(\mathbb{B}), z \in \mathbb{B},$$

is called the *weighted composition operator* with symbols u and φ. Observe that $W_{u,\varphi}(f) = M_u C_\varphi(f)$, where $M_u(f) = uf$, is the *multiplication operator* with symbol u, and $C_\varphi(f) = f \circ \varphi$ is the *composition operator* with symbol φ. If u is identically 1, then $W_{u,\varphi} = C_\varphi$, and if φ is the identity, then $W_{u,\varphi} = M_u$.

Composition operators and weighted composition operators acting on spaces of holomorphic functions in the unit disk $\mathbb{D}$ of the complex plane have been studied quite well. We refer the readers to the monographs [2, 9] for detailed information. Composition operators on $A^{-p}(\mathbb{D})$ have also been intensively studied (see, *e.g.*, [5] and references therein).

1.2. $\mathcal{N}_p$-spaces in the unit ball

Given a point $a \in \mathbb{B}$, we can associate to it the automorphism $\Phi_a \in \mathrm{Aut}(\mathbb{B})$ (see, *e.g.*, [8], Section 2.2). In [7], we introduce the $\mathcal{N}_p$-spaces in $\mathbb{B}$ for $p > 0$, which is defined as follows:

$$\mathcal{N}_p(\mathbb{B}) := \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_p = \sup_{a \in \mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p \, dV(z) \right)^{1/2} < \infty \right\},$$

where dV is the Lebesgue normalized volume measure on $\mathbb{B}$ (*i.e.* $V(\mathbb{B}) = 1$).

Several important properties of the $\mathcal{N}_p$-spaces, and of the weighted composition operators from $\mathcal{N}_p$-spaces to the spaces A^{-q} have been characterized in [7]. We cite here main results from [7] for the reader's convenience.

THEOREM 1.1. *The following statements hold:*

(a) *For $p > q > 0$, we have $H^\infty(\mathbb{B}) \hookrightarrow \mathcal{N}_q(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B}) \hookrightarrow A^{-\frac{n+1}{2}}(\mathbb{B})$, where the last embedding is given by $|f|_{\frac{n+1}{2}} \le \frac{2^{p+n}}{3^{p/2}} \|f\|_p, \forall f \in \mathcal{N}_p(\mathbb{B})$.*

(b) *For $p > 0$, if $p > 2k - 1, k \in (0, \frac{n+1}{2}]$, then $A^{-k}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$. In particular, when $p > n$, all $\mathcal{N}_p(\mathbb{B}) = A^{-\frac{n+1}{2}}(\mathbb{B})$.*

(c) *$\mathcal{N}_p(\mathbb{B})$ is a functional Banach space with the norm $\|\cdot\|_p$, and moreover, its norm topology is stronger than the compact-open topology.*

(d) *For $0 < p < \infty$, $\mathcal{B}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$, where $\mathcal{B}(\mathbb{B})$ is the Bloch space in $\mathbb{B}$.*
Here the symbol $X \hookrightarrow Y$ means the continuous embedding of X into Y.

THEOREM 1.2. *Let $\varphi\colon \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping, $u\colon \mathbb{B} \to \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u,\varphi} : \mathcal{N}_p(\mathbb{B}) \to A^{-q}(\mathbb{B})$*

(1) *is bounded if and only if*

$$\sup_{z \in \mathbb{B}} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty;$$

(2) *is compact if and only if*

$$\lim_{r \to 1^-} \sup_{|\varphi(z)| > r} \frac{|u(z)|(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} = 0.$$

The aim of this paper is to obtain necessary and sufficient conditions for the compactness of differences of weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$-spaces into the spaces $A^{-q}(\mathbb{B})$. It should be noted that the problem of compact difference of composition operators and weighted composition operators on spaces of holomorphic functions in several variables (either polydisk or ball) has been treated in several papers, in which the methods of proof, in general, follow the standard lines, using the characterization of compact operators in the corresponding function spaces (see, *e.g.*, [1, 4, 6, 10]). In the present paper, for $\mathcal{N}_p$-spaces, we focus on different technical issues, beginning with giving an estimate the quantity $|f(z) - f(w)|$, where $f \in \mathcal{N}_p(\mathbb{B})$ and $z, w \in \mathbb{B}$; and then inspiring by the pseudohyperbolic metric in $\mathbb{B}$, we use the function $\rho_{\varphi,\phi}(z) = |\Phi_{\varphi(z)}(\phi(z))|$ to characterize the compact difference between $\mathcal{N}_p(\mathbb{B})$ and $A^{-q}(\mathbb{B})$. Although our approach is standard, the results have their own interest due to the novelty of $\mathcal{N}_p(\mathbb{B})$-spaces. Especially, the criterion of compactness of differences depends on n, the dimension of the complex space $\mathbb{C}^n$, while it is independent of $p > 0$.

2. COMPACT DIFFERENCE

Let φ_1, φ_2 be two holomorphic self-mappings on $\mathbb{B}$, $u_1, u_2 : \mathbb{B} \to \mathbb{C}$ two holomorphic mappings, and $p, q > 0$. Let further, W_{u_1,φ_1} and W_{u_2,φ_2} be two weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$.

We need some auxiliary results.

LEMMA 2.1. *The operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{N}_p(\mathbb{B}) \to A^{-q}(\mathbb{B})$ is compact if and only if $|(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})(f_m)|_q \to 0$ as $m \to \infty$ for any bounded sequence $\{f_m\}$ in $\mathcal{N}_p(\mathbb{B})$ such that $\{f_m\}$ converges to zero uniformly on every compact subset of $\mathbb{B}$.*

The proof of Lemma 2.1 follows by standard arguments, and hence, we omit the details.

Now recall that the *pseudohyperbolic metric* in the ball is defined as

$$\rho(z, w) = |\Phi_w(z)|, \quad z, w \in \mathbb{B}.$$

It is a true metric (see, *e.g.*, [3]). Also it is easy to verify, in particular, that $\rho(0, w) = |w|, \rho(\Phi_w(z), w) = |z|$.

The following lemmas play an important role in the proof of our main result. They also have their own interest.

LEMMA 2.2. *For $z, w \in \mathbb{B}$, if $\rho(z, w) \le \frac{1}{2}$, then*

$$\frac{1}{6} \le \frac{1 - |z|^2}{1 - |w|^2} \le 6.$$

Proof. Let $z, w \in \mathbb{B}$. For simplicity, denote $r = \rho(z, w)$. By ([3], Theorem 1(c)) we have

$$\frac{|\rho(\Phi_w(z), 0) - \rho(0, w)|}{1 - \rho(\Phi_w(z), 0)\rho(0, w)} \leq \rho(\Phi_w(z), w) \leq \frac{\rho(\Phi_w(z), 0) + \rho(0, w)}{1 + \rho(\Phi_w(z), 0)\rho(0, w)},$$

or equivalently,

$$\frac{|r - |w||}{1 - r|w|} \leq |z| \leq \frac{r + |w|}{1 + r|w|}.$$

Actually, for the left inequality, we need only a weaker version. Namely,

$$\frac{|w| - r}{1 - r|w|} \leq |z| \leq \frac{|w| + r}{1 + r|w|}. \tag{2.1}$$

Furthermore, since $r \in (0, \frac{1}{2}]$, from the left inequality of (2.1), it follows that

$$\frac{1 - |z|}{1 - |w|} \leq \frac{1 - \frac{|w| - r}{1 - r|w|}}{1 - |w|} = \frac{1 + r}{1 - r|w|} \leq 3,$$

while the right inequality of (2.1) gives

$$\frac{1 - |z|}{1 - |w|} \geq \frac{1 - \frac{|w| + r}{1 + r|w|}}{1 - |w|} = \frac{1 - |r|}{1 + r|w|} \geq \frac{1}{3}.$$

We also have

$$\frac{1}{2} \leq \frac{1}{1 + |w|} \leq \frac{1 + |z|}{1 + |w|} \leq \frac{2}{1 + |w|} \leq 2.$$

Therefore, we get

$$\frac{1}{6} \leq \frac{1 - |z|^2}{1 - |w|^2} = \frac{1 - |z|}{1 - |w|} \cdot \frac{1 + |z|}{1 + |w|} \leq 6. \quad \square$$

LEMMA 2.3. *For $f \in \mathcal{N}_p(\mathbb{B})$ and $z, w \in \mathbb{B}$, we have*

$$|f(z) - f(w)| \leq c\|f\|_p \max\left\{\frac{1}{(1 - |z|^2)^{\frac{n+1}{2}}}, \frac{1}{(1 - |w|^2)^{\frac{n+1}{2}}}\right\} \rho(z, w).$$

Here $c = \frac{6^{\frac{n+1}{2}} \cdot 2^{p+n+1}(3 + 2\sqrt{3})\sqrt{n}}{3^{p/2}}$.

Proof. We consider two cases:

Case 1: $\rho(z, w) \geq \frac{1}{4}$.
Since $|f(z) - f(w)| \leq |f(z)| + |f(w)|$, by Theorem 1.1(a), we have

$$\min\left\{(1 - |z|^2)^{\frac{n+1}{2}}, (1 - |w|^2)^{\frac{n+1}{2}}\right\} |f(z) - f(w)|$$
$$\leq (1 - |z|^2)^{\frac{n+1}{2}} |f(z)| + (1 - |w|^2)^{\frac{n+1}{2}} |f(w)|$$

$$\le 2|f|_{\frac{n+1}{2}} \le \frac{2^{p+n+1}}{3^{p/2}}\|f\|_p \le \frac{2^{p+n+3}}{3^{p/2}}\|f\|_p\rho(z,w),$$

which implies that

$$|f(z)-f(w)| \le \frac{2^{p+n+3}}{3^{p/2}}\|f\|_p \max\left\{\frac{1}{(1-|z|^2)^{\frac{n+1}{2}}}, \frac{1}{(1-|w|^2)^{\frac{n+1}{2}}}\right\}\rho(z,w).$$

Case 2: $\rho(z,w) < \frac{1}{4}$.

Take and fix $w \in \mathbb{B}$. From $\rho(\Phi_w(z),w) = |z|$ it follows that if $z \in \overline{\mathbb{B}_{1/2}}$, then $\rho(\Phi_w(z),w) \le \frac{1}{2}$. In this case, by Theorem 1.1(a) and Lemma 2.2, we have

$$\begin{aligned} |f(\Phi_w(z))| &\le \frac{|f|_{\frac{n+1}{2}}}{(1-|\Phi_w(z)|^2)^{\frac{n+1}{2}}} \le \frac{2^{p+n}\|f\|_p}{3^{p/2}(1-|\Phi_w(z)|^2)^{\frac{n+1}{2}}} \\ &= \frac{2^{p+n}\|f\|_p}{3^{p/2}(1-|w|^2)^{\frac{n+1}{2}}}\cdot\left[\frac{1-|w|^2}{1-|\Phi_w(z)|^2}\right]^{\frac{n+1}{2}} \le \frac{6^{\frac{n+1}{2}}\cdot 2^{p+n}\|f\|_p}{3^{p/2}(1-|w|^2)^{\frac{n+1}{2}}}. \end{aligned}$$

Now we follow the standard scheme to estimate a quantity $|f(z)-f(w)|$. Set $g_w = f\circ\Phi_w$, then

$$|f(z)-f(w)| = |f(\Phi_w(\Phi_w(z)) - f(\Phi_w(0))| = |g_w(\Phi_w(z)) - g_w(0)|.$$

For each $z \in \mathbb{B}$ with $\rho(z,w) = |\Phi_w(z)| < \frac{1}{4}$, we have

$$|f(z)-f(w)| = |g_w(\Phi_w(z)) - g_w(0)| \le |\nabla g_w(t)|\cdot|\Phi_w(z)| = |\nabla g_w(t)|\rho(z,w),$$

where $t=(t_1,t_2,\ldots,t_n)$ is some point in $\mathbb{B}$ with $|t| \le |\Phi_w(z)| \le \frac{1}{4}$.

Furthermore,

$$\begin{aligned} |\nabla g_w(t)|\rho(z,w) &\le \sqrt{n}\rho(z,w)\max_{1\le k\le n}\left|\frac{\partial g_w}{\partial z_k}(t)\right| \\ &\le \sqrt{n}\rho(z,w)\max_{1\le k\le n}\left|\frac{1}{2\pi i}\int_{|\xi_k|=\frac{\sqrt{3}}{4}}\frac{g_w(t_1,t_2,\ldots,\xi_k,\ldots,t_n)}{(\xi_k-t_k)^2}d\xi_k\right| \\ &\le \frac{\sqrt{n}\rho(z,w)}{2\pi}\max_{1\le k\le n}\int_{|\xi_k|=\frac{\sqrt{3}}{4}}\left|\frac{g_w(t_1,t_2,\ldots,\xi_k,\ldots,t_n)}{(\xi_k-t_k)^2}\right||d\xi_k|. \end{aligned}$$

Note that for $(t_1,t_2,\ldots,\xi_k,\ldots,t_n)$ with $|t|\le\frac{1}{4}$ and $|\xi_k| = \frac{\sqrt{3}}{4}$, we have

$$\rho\left(\Phi_w(t_1,t_2,\ldots,\xi_k,\ldots,t_n),w\right) = \rho((t_1,t_2,\ldots,\xi_k,\ldots,t_n),0)$$

$$= |(t_1,t_2,\ldots,\xi_k,\ldots,t_n)| \le \sqrt{|t|^2+|\xi_k|^2} \le \sqrt{\left(\frac{1}{4}\right)^2+\left(\frac{\sqrt{3}}{4}\right)^2} = \frac{1}{2},$$

and so

$$|g_w(t_1,t_2,\ldots,\xi_j,\ldots,t_n)| = |f\big(\Phi_w(t_1,t_2,\ldots,\xi_k,\ldots,t_n)\big)| \le \frac{6^{\frac{n+1}{2}}\cdot 2^{p+n}\|f\|_p}{3^{p/2}(1-|w|^2)^{\frac{n+1}{2}}}.$$

Also,

$$\max_{1\le k\le n}\int_{|\xi_k|=\frac{\sqrt{3}}{4}}\frac{|d\xi_k|}{|\xi_k-t_k|^2}\le\max_{1\le k\le n}\int_{|\xi_k|=\frac{\sqrt{3}}{4}}\frac{|d\xi_k|}{(\frac{\sqrt{3}}{4}-|t_k|)^2}$$

$$\le\max_{1\le k\le n}\int_{|\xi_k|=\frac{\sqrt{3}}{4}}\frac{|d\xi_k|}{(\frac{\sqrt{3}}{4}-\frac{1}{4})^2}\le 2\pi\frac{\sqrt{3}}{4}\cdot\left(\frac{4}{\sqrt{3}-1}\right)^2=4\pi(3+2\sqrt{3}).$$

Consequently,

$$\begin{aligned}|f(z)-f(w)| &\le \frac{\sqrt{n}\rho(z,w)}{2\pi}\cdot\frac{6^{\frac{n+1}{2}}\cdot 2^{p+n}\|f\|_p}{3^{p/2}(1-|w|^2)^{\frac{n+1}{2}}}\cdot 4\pi(3+2\sqrt{3})\\ &= \frac{6^{\frac{n+1}{2}}\cdot 2^{p+n+1}(3+2\sqrt{3})\sqrt{n}}{3^{p/2}}\cdot\frac{1}{(1-|w|^2)^{\frac{n+1}{2}}}\cdot\|f\|_p\cdot\rho(z,w).\end{aligned}$$

Combining the results of the two cases yields

$$|f(z)-f(w)|\le c\|f\|_p\max\left\{\frac{1}{(1-|z|^2)^{\frac{n+1}{2}}},\frac{1}{(1-|w|^2)^{\frac{n+1}{2}}}\right\}\rho(z,w),$$

where $c=\frac{6^{\frac{n+1}{2}}\cdot 2^{p+n+1}(3+2\sqrt{3})\sqrt{n}}{3^{p/2}}$. □

Inspiring by the pseudohyperbolic metric in the unit ball, for two holomorphic mappings $\varphi,\psi\colon\mathbb{B}\to\mathbb{B}$, we define

$$\rho_{\varphi,\psi}(z)=\left|\Phi_{\varphi(z)}(\psi(z))\right|,\quad z\in\mathbb{B}.$$

Evidently, $\rho_{\varphi,\psi}=\rho_{\psi,\varphi}$.

For each $w\in\mathbb{B}$, set

$$k_w(z)=\left(\frac{1-|w|^2}{(1-\langle z,w\rangle)^2}\right)^{\frac{n+1}{2}},\ z\in\mathbb{B}. \tag{2.2}$$

By ([7], Lemma 3.1), we have $k_w\in\mathcal{N}_p(\mathbb{B})$ and $\sup\limits_{w\in\mathbb{B}}\|k_w\|_p\le 1$. Also note that $k_w(w)=\left(\frac{1}{1-|w|^2}\right)^{\frac{n+1}{2}},\ \forall w\in\mathbb{B}$.

Now we are ready to formulate our main result of this paper.

Theorem 2.4. *Let $\varphi_1,\varphi_2\colon\mathbb{B}\to\mathbb{B}$ be two holomorphic mappings, $u_1,u_2:\mathbb{B}\to\mathbb{C}$ two holomorphic mappings and $p,q>0$. Let further, W_{u_1,φ_1} and W_{u_2,φ_2} be two weighted composition operators acting from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$. Then $W_{u_1,\varphi_1}-W_{u_2,\varphi_2}$ is compact if and only if the following conditions are satisfied:*

(i)

$$\lim_{r\to 1^-}\sup_{|\varphi_k(z)|>r}\left\{\frac{|u_k(z)|(1-|z|^2)^q}{(1-|\varphi_k(z)|^2)^{\frac{n+1}{2}}}\rho_{\varphi_1,\varphi_2}(z)\right\}=0\quad(k=1,2);$$

(ii)

$$\lim_{r\to 1^-}\sup_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r}\left[|u_1(z)-u_2(z)|\min\left\{\frac{(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}},\frac{(1-|z|^2)^q}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}}\right\}\right]=0.$$

Proof. **Necessity.** Suppose $W_{u_1,\varphi_1}-W_{u_2,\varphi_2}$ is a compact operator.

– *Prove* (i). It suffices to prove for $k=1$.

Since W_{u_1,φ_1} is bounded, by Theorem 1.2 and the fact that $\rho_{\varphi_1,\varphi_2}(z)\le 1, \forall z\in\mathbb{B}$, we have

$$\sup_{z\in\mathbb{B}}\left\{\frac{|u_1(z)|(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}\rho_{\varphi_1,\varphi_2}(z)\right\}\le\sup_{z\in\mathbb{B}}\frac{|u_1(z)|(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}<\infty.$$

Set

$$G(r)=\sup_{|\varphi_1(z))|>r}\left\{\frac{|u_1(z)|(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}\rho_{\varphi_1,\varphi_2}(z)\right\},\ r\in(0,1).$$

It is clear that G is bounded and decreasing on $(0,1)$, and hence, $\lim_{r\to 1^-}G(r)$ does always exist.

Assume that (i) is not true. Then there exists an $L>0$, such that $\lim_{r\to 1^-}G(r)>L$.

By the standard diagonal process, as sketched in ([7], Theorem 3.4), we can choose a sequence $\{z_m\}\subset\mathbb{B}$, such that $|\varphi_1(z_m)|\to 1$ as $m\to\infty$ and

$$\frac{|u_1(z_m)|(1-|z_m|^2)^q}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}\rho_{\varphi_1,\varphi_2}(z_m)>\frac{L}{4},\quad (m=1,2,\ldots). \tag{2.3}$$

Consider the functions

$$g_m(z)=\Phi_{\varphi_2(z_m)}(z)\cdot k_{w_m}(z),\ z\in\mathbb{B},$$

where $w_m=\varphi_1(z_m)$ and k_{w_m} is defined by (2.2), $m=1,2,\ldots$. Obviously, $g_m\in\mathcal{O}(\mathbb{B})$. Moreover, since $|\Phi_{\varphi_2(z_m)}(z)|\le 1,\forall z\in\mathbb{B}$, we have $\|g_m\|_p\le\|k_{w_m}\|_p\le 1$, which shows that $g_m(z)\in\mathcal{N}_p(\mathbb{B})$ for all $m\in\mathbb{N}$, and that the sequence $\{g_m\}$ is bounded in $\mathcal{N}_p(\mathbb{B})$. Furthermore, by the fact that $\{k_{w_m}\}_{m\in\mathbb{N}}$ converges to zero uniformly on every compact subset of $\mathbb{B}$, and $|g_m(z)|\le|k_{w_m}(z)|,\forall z\in\mathbb{B}$, we have $\{g_m\}_{m\in\mathbb{N}}$ also converges to zero uniformly on every compact subset of $\mathbb{B}$.

By Lemma 2.1,

$$|(W_{u_1,\varphi_1}-W_{u_2,\varphi_2})(g_m)|_q\to 0,\ \text{as } m\to\infty. \tag{2.4}$$

Note that for each $m\in\mathbb{N}$, $\Phi_{\varphi_2(z_m)}(\varphi_2(z_m))=0$, which implies $g_m(\varphi_2(z_m))=0$. Then,

$$
\begin{aligned}
& |(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})(g_m)|_q \\
= & \sup_{z\in\mathbb{B}} \left\{ |u_1(z) g_m(\varphi_1(z)) - u_2(z) g_m(\varphi_2(z))| \cdot (1-|z|^2)^q \right\} \\
\geq & |u_1(z_m) g_m(\varphi_1(z_m)) - u_2(z_m) g_m(\varphi_2(z_m))| \cdot (1-|z_m|^2)^q \\
= & |u_1(z_m) g_m(\varphi_1(z_m))| \cdot (1-|z_m|^2)^q = \frac{|u_1(z_m)|(1-|z_m|^2)^q}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \cdot |\Phi_{\varphi_2(z_m)}(\varphi_1(z_m))| \\
= & \frac{|u_1(z_m)|(1-|z_m|^2)^q}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \rho_{\varphi_1,\varphi_2}(z_m) > \frac{L}{4},
\end{aligned}
$$

by (2.3), which contradicts (2.4).

Thus, we must have

$$
\lim_{r\to 1^-} \sup_{|\varphi_1(z)|>r} \left\{ \frac{|u_1(z)|(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}} \rho_{\varphi_1,\varphi_2}(z) \right\} = 0,
$$

and (i) is proved.

– *Prove (ii).* Since both W_{u_1,φ_1} and W_{u_2,φ_2} are bounded, by Theorem 1.2, we have

$$
\begin{aligned}
& \sup_{z\in\mathbb{B}} \left[|u_1(z) - u_2(z)| \min\left\{ \frac{(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z|^2)^q}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}} \right\} \right] \\
\leq & \sup_{z\in\mathbb{B}} \left[|u_1(z)| \min\left\{ \frac{(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z|^2)^q}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}} \right\} \right] \\
& + \sup_{z\in\mathbb{B}} \left[|u_2(z)| \min\left\{ \frac{(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z|^2)^q}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}} \right\} \right] \\
\leq & \sup_{z\in\mathbb{B}} \frac{|u_1(z)|(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}} + \sup_{z\in\mathbb{B}} \frac{|u_2(z)|(1-|z|^2)^q}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}} < \infty.
\end{aligned}
$$

For each $r \in (0,1)$, set

$$
H(r) = \sup_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r} \left[|u_1(z) - u_2(z)| \min\left\{ \frac{(1-|z|^2)^q}{(1-|\varphi_1(z)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z|^2)^q}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}} \right\} \right].
$$

This function $H(r)$ is clearly bounded and decreasing on $(0,1)$, and hence, $\lim_{r\to 1^-} H(r)$ exists. We prove that this limit must be zero.

We follow the same scheme of proving (i), but it requires more delicate arguments. Assume that $\lim_{r\to 1^-} H(r) = L > 0$. Again by the standard diagonal process, we can choose a sequence $\{z_m\} \subset \mathbb{B}$, such that $\min\{|\varphi_1(z_m)|, |\varphi_2(z_m)|\}$

$\to 1$ as $m \to \infty$ and that for each $m \in \mathbb{N}$,

$$|u_1(z_m) - u_2(z_m)| \min\left\{\frac{(1-|z_m|^2)^q}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z_m|^2)^q}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right\} > \frac{L}{4}.$$

For each $m \in \mathbb{N}$, we have either $|\varphi_1(z_m)| \geq |\varphi_2(z_m)|$ or $|\varphi_1(z_m)| \leq |\varphi_2(z_m)|$. Choose a subsequence $\{z_{m_k}\}$ of $\{z_m\}$, such that for each $k \in \mathbb{N}$, $|\varphi_1(z_{m_k})| \geq |\varphi_2(z_{m_k})|$. Otherwise, there are only finitely many indexes m, such that $|\varphi_1(z_m)| \geq |\varphi_2(z_m)|$ and in this case, we choose a subsequence $\{z_{m_k}\}$ of $\{z_m\}$, such that for each $k \in \mathbb{N}$, $|\varphi_1(z_{m_k})| \leq |\varphi_2(z_{m_k})|$. We only consider the first case, omitting the second case (which can be proved by interchanging the role of φ_1 and φ_2), and without loss of generality, for the purpose of convenience, we write $\{z_{m_k}\}$ as $\{z_m\}$.

Since $|\varphi_1(z_m)| \geq |\varphi_2(z_m)|$ for each $m \in \mathbb{N}$, we have

$$\begin{aligned} & |u_1(z_m) - u_2(z_m)| \min\left\{\frac{(1-|z_m|^2)^q}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z_m|^2)^q}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right\} \\ = \; & |u_1(z_m) - u_2(z_m)| \frac{(1-|z_m|^2)^q}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} > \frac{L}{4}. \end{aligned}$$

Since the sequence $\{\rho_{\varphi_1,\varphi_2}(z_m)\}$ is bounded, it contains a convergent subsequence. Without loss of generality, we can assume that

$$\lim_{m\to\infty} \rho_{\varphi_1,\varphi_2}(z_m) = \ell \geq 0.$$

There are two cases for ℓ:

– Case 1: $\ell > 0$. In this case, there exists an $m_0 \in \mathbb{N}$ such that $\rho_{\varphi_1,\varphi_2}(z_m) > \frac{\ell}{2}$, $\forall m > m_0$. In this case, we have

$$\begin{aligned} & \frac{(1-|z_m|^2)^q |u_1(z_m) - u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} \\ \leq \; & \frac{(1-|z_m|^2)^q |u_1(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} + \frac{(1-|z_m|^2)^q |u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} \\ \leq \; & \frac{2}{\ell}\rho_{\varphi_1,\varphi_2}(z_m)\left[\frac{(1-|z_m|^2)^q |u_1(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} + \frac{(1-|z_m|^2)^q |u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right] \\ \leq \; & \frac{2}{\ell}\rho_{\varphi_1,\varphi_2}(z_m)\left[\frac{(1-|z_m|^2)^q |u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} + \frac{(1-|z_m|^2)^q |u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right], \end{aligned}$$

which gives

$$\rho_{\varphi_1,\varphi_2}(z_m)\left[\frac{(1-|z_m|^2)^q|u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}+\frac{(1-|z_m|^2)^q|u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right]\geq\frac{L\ell}{8},\ \forall m>m_0.$$

However, since $|\varphi_2(z_m)|\leq|\varphi_1(z_m)|\leq 1,\ \forall m\in\mathbb{N}$, from

$$\lim_{m\to\infty}\min\{|\varphi_1(z_m)|,|\varphi_2(z_m)|\}=1$$

it follows that

$$\lim_{m\to\infty}|\varphi_1(z_m)|=\lim_{m\to\infty}|\varphi_2(z_m)|=1.$$

Hence, by (i),

$$\lim_{m\to\infty}\rho_{\varphi_1,\varphi_2}(z_m)\frac{(1-|z_m|^2)^q|u_k(z_m)|}{(1-|\varphi_k(z_m)|^2)^{\frac{n+1}{2}}}=0,\ k=1,2,$$

and hence,

$$\rho_{\varphi_1,\varphi_2}(z_m)\left(\frac{(1-|z_m|^2)^q|u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}+\frac{(1-|z_m|^2)^q|u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right)\to 0,\ \text{as } m\to\infty,$$

which is impossible.

– Case 2: $\ell=0$. We claim, for the probe functions k_{w_m}, where $w_m=\varphi_1(z_m)$, that

$$\left|1-(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}k_{w_m}(\varphi_2(z_m))\right|\to 0,\ \text{as } m\to\infty. \tag{2.5}$$

Indeed, by Lemma 2.3, we have

$$\begin{aligned}&\left|1-(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}k_{w_m}(\varphi_2(z_m))\right|\\ =\ &(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}\cdot|k_{w_m}(\varphi_1(z_m))-k_{w_m}(\varphi_2(z_m))|\\ \leq\ &(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}\cdot c\|k_{w_m}\|_p\cdot\rho_{\varphi_1,\varphi_2}(z_m)\cdot\\ &\max\left\{\frac{1}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}},\frac{1}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right\}\\ \leq\ &\frac{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}c\rho_{\varphi_1,\varphi_2}(z_m)}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}=c\rho_{\varphi_1,\varphi_2}(z_m).\end{aligned}$$

The last expression converges to $\ell=0$ as $m\to\infty$, and (2.5) follows. Here c is the constant defined in Lemma 2.3.

Furthermore, since $\lim_{m\to\infty}\rho_{\varphi_1,\varphi_2}(z_m)=0$, there exists $m_1\in\mathbb{N}$, such that $\rho_{\varphi_1,\varphi_2}(z_m)<\frac{1}{2},\ \forall m\geq m_1$. Then by Lemma 2.2, $\dfrac{1-|\varphi_2(z_m)|^2}{1-|\varphi_1(z_m)|^2}\leq 6$. Also since W_{u_2,φ_2} is bounded from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$, by Theorem 1.2, there exists some positive number $K > 0$, such that $\sup_{z\in\mathbb{B}} \frac{(1-|z|^2)^q|u_2(z)|}{(1-|\varphi_2(z)|^2)^{\frac{n+1}{2}}} < K$. Again by $\lim_{m\to\infty} \rho_{\varphi_1,\varphi_2}(z_m) = 0$, there exists $m_2 \in \mathbb{N}$, such that $\rho_{\varphi_1,\varphi_2}(z_m) < \frac{L}{8\cdot 6^{\frac{n+1}{2}} cK}$, $\forall m > m_2$.

Consequently, for $m > \max\{m_1, m_2\}$,

$$\begin{aligned}
& |(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})(k_{w_m})|_q \\
= \ & \sup_{z\in\mathbb{B}}(1-|z|^2)^q \left|u_1(z)k_{w_m}(\varphi_1(z)) - u_2(z)k_{w_m}(\varphi_2(z))\right| \\
\geq \ & (1-|z_m|^2)^q \left|u_1(z_m)k_{w_m}(\varphi_1(z_m)) - u_2(z_m)k_{w_m}(\varphi_2(z_m))\right| \\
= \ & (1-|z_m|^2)^q \left|\frac{u_1(z_m)}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} - u_2(z_m)k_{w_m}(\varphi_2(z_m))\right| \\
\geq \ & (1-|z_m|^2)^q \left|\frac{u_1(z_m)}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} - \frac{u_2(z_m)}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}\right| \\
& -(1-|z_m|^2)^q \left|\frac{u_2(z_m)}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} - u_2(z_m)k_{w_m}(\varphi_2(z_m))\right| \\
= \ & (1-|z_m|^2)^q \frac{|u_1(z_m) - u_2(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \\
& -(1-|z_m|^2)^q \frac{|u_2(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \left|1 - (1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}} k_{w_m}(\varphi_2(z_m))\right|.
\end{aligned}$$

Moreover, we also have

$$(1-|z_m|^2)^q \frac{|u_1(z_m) - u_2(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \geq \frac{(1-|z_m|^2)^q|u_1(z_m) - u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} > \frac{L}{4},$$

and

$$(1-|z_m|^2)^q \frac{|u_2(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} = (1-|z_m|^2)^q \frac{|u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} \cdot \frac{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \leq K \cdot 6^{\frac{n+1}{2}}.$$

Therefore, we arrive at

$$|(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})(k_{w_m})|_q \geq \frac{L}{4} - 6^{\frac{n+1}{2}} Kc\rho_{\varphi_1,\varphi_2}(z_m) \geq \frac{L}{4} - \frac{L}{8} = \frac{L}{8}. \tag{2.6}$$

However, since $W_{u_1,\varphi_1} - W_{u_2,\varphi_2}$ is compact, and $\{k_{w_m}\}$ converges to zero uniformly on every compact subset of $\mathbb{B}$, we must have

$$\lim_{m\to\infty} |(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})(k_{w_m})|_q = 0, \tag{2.7}$$

which contradicts (2.6).

Sufficiency. Let conditions (i)–(ii) hold. Take an arbitrary bounded sequence $\{f_m\}$ in $\mathcal{N}_p(\mathbb{B})$ that converges to zero uniformly on every compact subset of $\mathbb{B}$. By Lemma 2.1, we show that

$$|(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})(f_m)|_q \to 0, \text{ as } m \to \infty.$$

Again we use a method of contradiction. Assume that there is an $\varepsilon_0 > 0$ and a subsequence $\{f_{m_k}\}$ of $\{f_m\}$ such that

$$|(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})(f_{m_k})|_q \geq \varepsilon_0, \ \forall k \in \mathbb{N}. \tag{2.8}$$

We may assume, for the sake of simplicity, that $\{f_{m_k}\}$ is $\{f_m\}$ itself. That is, $\forall m \in \mathbb{N}$,

$$\begin{aligned} &|(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})(f_m)|_q \\ &= \sup_{z\in\mathbb{B}} (1-|z|^2)^q \, |u_1(z)f_m(\varphi_1(z)) - u_2(z)f_m(\varphi_2(z))| \geq \varepsilon_0. \end{aligned}$$

From this it follows that there exists a sequence $\{z_m\} \subset \mathbb{B}$, such that $\forall m \in \mathbb{N}$,

$$H_m = (1-|z_m|^2)^q \, |u_1(z_m)f_m(\varphi_1(z_m)) - u_2(z_m)f_m(\varphi_2(z_m))| \geq \frac{\varepsilon_0}{2}. \tag{2.9}$$

Here z_m's are not necessarily distinct.

We may also, without loss of generality, assume that both sequences $\{\varphi_1(z_m)\}$ and $\{\varphi_2(z_m)\}$ converge (as otherwise, we can consider their convergent subsequences instead).

Note that since both W_{u_1,φ_1} and W_{u_2,φ_2} are bounded, by Theorem 1.2, there exists $K > 0$, such that

$$\sup_{z\in\mathbb{B}} |u_k(z)|(1-|z|^2)^q \leq \sup_{z\in\mathbb{B}} \frac{|u_k(z)|(1-|z|^2)^q}{(1-|\varphi_k(z)|^2)^{\frac{n+1}{2}}} \leq K, \ k = 1, 2. \tag{2.10}$$

Now we consider the sequence $\max\{|\varphi_1(z_m)|, |\varphi_2(z_m)|\}$. It is clear that there exists

$$\lim_{m\to\infty} \max\{|\varphi_1(z_m)|, |\varphi_2(z_m)|\} = q \leq 1.$$

• **Claim 1:** $q = 1$.

Proof. Assume

$$\max\{|\varphi_1(z_m)|, |\varphi_2(z_m)|\} \to q < 1, \text{ as } m \to \infty. \tag{2.11}$$

Then, by (2.9),

$$\frac{\varepsilon_0}{2} \leq H_m \leq (1-|z_m|^2)^q \Big(|u_1(z_m)f_m(\varphi_1(z_m))| + |u_2(z_m)f_m(\varphi_2(z_m)| \Big)$$

$$\leq \quad K\Big(|f_m(\varphi_1(z_m))| + |f_m(\varphi_2(z_m))|\Big), \ \forall m \in \mathbb{N}.$$

Furthermore, by (2.11), there exists $m_3 \in \mathbb{N}$ such that

$$\max\{|\varphi_1(z_m)|, |\varphi_2(z_m)|\} \leq \frac{1+q}{2}, \ \forall m > m_3.$$

In particular, for all $m > m_3$, $\varphi_k(z_m) \in \left\{z : |z| \leq \frac{1+q}{2}\right\}, k = 1, 2$. Since $\left\{z : |z| \leq \frac{1+q}{2}\right\}$ is a compact set of $\mathbb{B}$, the sequence $\{f_m(z)\}$ converges uniformly to zero on this set, and hence, both sequences $\{f_m(\varphi_k(z_m))\}$, $k = 1, 2$, converge to zero, as $m \to \infty$, which shows that

$$H_m \leq K\left(|f_m(\varphi_1(z_m))| + |f_m(\varphi_2(z_m))|\right) \to 0, \ \text{as } m \to \infty.$$

But this contradicts the fact that $H_m \geq \dfrac{\varepsilon_0}{2}$, $\forall m \in \mathbb{N}$. $\square$

Thus, we have

$$\max\{|\varphi_1(z_m)|, |\varphi_2(z_m)|\} \to 1, \ \text{as } m \to \infty. \tag{2.12}$$

Then at least one of the limits $\lim\limits_{m\to\infty} |\varphi_k(z_m)|$ $(k = 1, 2)$ must be 1. So we may assume that

$$\begin{cases} \lim\limits_{m\to\infty} \varphi_1(z_m) = P, \ \text{with } |P| = 1, \\ \lim\limits_{m\to\infty} \varphi_2(z_m) = Q, \ \text{with } |Q| \leq 1. \end{cases} \tag{2.13}$$

Furthermore, we may also assume that there exists the limit

$$\lim_{m\to\infty} \rho_{\varphi_1,\varphi_2}(z_m) = \ell \geq 0$$

(otherwise we consider its convergent subsequence).

- **Claim 2:** $\ell = 0$.

Proof. Assume in contrary that $\ell > 0$. Consider two cases of $|Q| \leq 1$.

– Case 1: $|Q| = 1$. In this case, from (i) and (2.13), it follows that

$$\lim_{m\to\infty} \frac{|u_k(z_m)|(1-|z_m|^2)^q}{(1-|\varphi_k(z_m)|^2)^{\frac{n+1}{2}}} = 0 \quad (k = 1, 2). \tag{2.14}$$

Then, by Theorem 1.1 and (2.9), we have

$$\begin{aligned} H_m \ &\leq \ \frac{(1-|z_m|^2)^q|u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}|f_m(\varphi_1(z_m))| \\ &+\frac{(1-|z_m|^2)^q|u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}|f_m(\varphi_2(z_m))| \end{aligned}$$

$$\begin{aligned} &\leq \left[\frac{(1-|z_m|^2)^q|u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}+\frac{(1-|z_m|^2)^q|u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right]|f_m|_{\frac{n+1}{2}} \\ &\leq \frac{2^{p+n}}{3^{p/2}}\left[\frac{(1-|z_m|^2)^q|u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}+\frac{(1-|z_m|^2)^q|u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}\right]\|f_m\|_p. \end{aligned}$$

In the last inequality, letting $m\to\infty$, by (2.14) as well as boundedness of $\{f_m\}$ in $\mathcal{N}_p(\mathbb{B})$, we get

$$\lim_{m\to\infty} H_m = 0,$$

which is impossible, because it contradicts (2.9).

– Case 2: $|Q|<1$. In this case, the second limit in (2.13) gives $\varphi_2(z_m)\in\left\{z:|z|\leq\frac{1+|Q|}{2}\right\}$, for all m large enough, say $m>m_4$. Then by (2.10) and Theorem 1.1, for all $m>m_4$, we have

$$\begin{aligned} H_m &\leq (1-|z_m|^2)^q|u_1(z_m)||f_m(\varphi_1(z_m))| \\ &\quad +(1-|z_m|^2)^q|u_2(z_m)||f_m(\varphi_2(z_m))| \\ &= \frac{(1-|z_m|^2)^q|u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}|f_m(\varphi_1(z_m))| \\ &\quad +(1-|z_m|^2)^q|u_2(z_m)||f_m(\varphi_2(z_m))| \\ &\leq \frac{(1-|z_m|^2)^q|u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}|f_m|_{\frac{n+1}{2}}+K|f_m(\varphi_2(z_m))| \\ &\leq \frac{2^{p+n}}{3^{p/2}}\cdot\frac{(1-|z_m|^2)^q|u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}\|f_m\|_p+K|f_m(\varphi_2(z_m))|. \end{aligned}$$

Letting $m\to\infty$, from (2.14) and the fact that $f_m(\varphi_2(z_m))$ converges to zero uniformly on the compact set $\left\{z:|z|\leq\frac{1+|Q|}{2}\right\}$, it follows that right-hand side of the last inequality tends to 0 as $m\to\infty$, which again contradicts (2.9). □

Thus, we have

$$\lim_{m\to\infty}\rho_{\varphi_1,\varphi_2}(z_m)=0. \tag{2.15}$$

• **Claim 3:** $|Q|=1$, that is $\lim\limits_{m\to\infty}|\varphi_2(z_m)|=1$.

Proof. Indeed, for each $m\in\mathbb{N}$, we have

$$\begin{aligned} 1-\rho^2_{\varphi_1,\varphi_2}(z_m) &= 1-|\Phi_{\varphi_2(z_m)}(\varphi_1(z_m)|^2 \\ &= 1-\frac{(1-|\varphi_1(z_m)|^2)(1-|\varphi_2(z_m)|^2)}{|1-\langle\varphi_1(z_m),\varphi_2(z_m)\rangle|^2}. \end{aligned}$$

Since $\lim\limits_{m\to\infty}|\varphi_1(z_m)|=1$, if $\lim\limits_{m\to\infty}|\varphi_2(z_m)|=|Q|<1$, then we would have for all m large enough

$$|1-\langle\varphi_1(z_m),\varphi_2(z_m)\rangle| \geq 1-|\langle\varphi_1(z_m),\varphi_2(z_m)\rangle|$$

$$\geq \quad 1-|\varphi_1(z_m)||\varphi_2(z_m)| \geq 1-|Q| > 0.$$

But this implies that $\lim_{m\to\infty} \rho_{\varphi_1,\varphi_2}(z_m) = 1$, which contradicts (2.15). □

Now by the same reason as in the necessity part, we may assume that

(2.16) $$|\varphi_1(z_m)| \geq |\varphi_2(z_m)|, \ \forall m \in \mathbb{N}.$$

Then from (2.9)–(2.10) and Lemma 2.3, it follows that for each $m \in \mathbb{N}$

$$\begin{aligned}
H_m &= (1-|z_m|^2)^q \left|u_1(z_m) f_m(\varphi_1(z_m)) - u_2(z_m) f_m(\varphi_2(z_m))\right| \\
&\leq (1-|z_m|^2)^q \left|u_1(z_m) f_m(\varphi_1(z_m)) - u_1(z_m) f_m(\varphi_2(z_m))\right| \\
&\quad + (1-|z_m|^2)^q \left|u_1(z_m) f_m(\varphi_2(z_m)) - u_2(z_m) f_m(\varphi_2(z_m))\right| \\
&= \frac{(1-|z_m|^2)^q |u_1(z_m)|}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \cdot (1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}} |f_m(\varphi_1(z_m)) - f_m(\varphi_2(z_m))| \\
&\quad + \frac{(1-|z_m|^2)^q |f_m(\varphi_2(z_m))|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} \cdot (1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}} |u_1(z_m) - u_2(z_m)| \\
&\leq cK \cdot \|f_m\|_p \max\left\{ \frac{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}, \frac{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} \right\} \rho_{\varphi_1,\varphi_2}(z_m) \\
&\quad + \frac{2^{p+n}}{3^{p/2}} \|f_m\|_p \cdot \frac{(1-|z_m|^2)^q |u_1(z_m) - u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}.
\end{aligned}$$

However, by (2.16)

$$\begin{aligned}
&\max\left\{ \frac{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}, \frac{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} \right\} \\
&= \max\left\{ 1, \frac{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} \right\} = 1,
\end{aligned}$$

and hence,

$$\begin{aligned}
H_m &\leq cK \|f_m\|_p \cdot \rho_{\varphi_1,\varphi_2}(z_m) \\
&\quad + \frac{2^{p+n}}{3^{p/2}} \|f_m\|_p \cdot \frac{(1-|z_m|^2)^q |u_1(z_m) - u_2(z_m)|}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}} \\
&= cK \|f_m\|_p \cdot \rho_{\varphi_1,\varphi_2}(z_m) \\
&\quad + \|f_m\|_p \cdot \frac{2^{p+n}}{3^{p/2}} |u_1(z_m) - u_2(z_m)| \cdot \\
&\quad \min\left\{ \frac{(1-|z_m|^2)^q}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z_m|^2)^q}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \right\}.
\end{aligned}$$

In the inequality above, since $\{\|f_m\|_p\}$ is bounded, by (2.15),

$$\lim_{m\to\infty} \|f_m\|_p \cdot \rho_{\varphi_1,\varphi_2}(z_m) = 0.$$

Furthermore, by (ii) and $\lim_{m\to\infty} |\varphi_1(z_m)| = \lim_{m\to\infty} |\varphi_2(z_m)| = 1$, we get

$$\lim_{m\to\infty} \|f_m\|_p |u_1(z_m) - u_2(z_m)| \cdot \min\left\{ \frac{(1-|z_m|^2)^q}{(1-|\varphi_2(z_m)|^2)^{\frac{n+1}{2}}}, \frac{(1-|z_m|^2)^q}{(1-|\varphi_1(z_m)|^2)^{\frac{n+1}{2}}} \right\} = 0.$$

These equalities imply that $\lim_{m\to\infty} H_m = 0$, but this contradicts (2.9).

The theorem is proved completely. □

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REFERENCES

[1] B. Choe, H. Koo and I. Park, *Compact differences of composition operators over polydisks.* Integral Equations Operator Theory **73** (2013), 57–91.

[2] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*. CRC Press, Boca Raton, 1995.

[3] P. Duren and R. Weir, *The pseudohyperbolic metrc and Bergman spaces in the ball*. Trans. Amer. Math. Soc. **359** (2007), 63–76.

[4] L. Jiang and C. Ouyang, *Compact differences of composition operators on holomorphic function spaces in the unit ball.* Acta Math. Sci. Ser B Engl. Ed. **31** (2011), 1679–1693.

[5] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*. Springer-Verlag, 2000.

[6] K. Heller, B.D. MacCluer and R.J. Weir, *Compact differences of composition operators in several variables*. Integral Equations Operator Theory **69** (2011), 419–428.

[7] B. Hu and L.H. Khoi, *Weighted-composition operators on $\mathcal{N}_p$-spaces in the ball.* C.R. Math. Acad. Sci. Paris. Ser.I **351** (2013), 719–723.

[8] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*. Springer-Verlag, New York, 1980.

[9] J.H. Shapiro, *Composition Operators and Classical Function Theory.* Springer-Verlag, New York, 1993.

[10] C. Tong, *Compact differences of weighted composition operators on $H^\infty(\mathbb{B}_N)$*. J. Comput. Anal. Appl. **14** (2012), 32–41.

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